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ON THE WARING–GOLDBACH PROBLEM FOR CUBES

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Abstract. We prove that almost all natural numbers satisfying certain necessary congruence conditions can be written as the sum of two cubes of primes and two cubes of P_2-numbers, where, as usual, we call a natural number a P_2-number when it is a prime or the product of two primes. From this result we also deduce that every sufficiently large integer can be written as the sum of eight cubes of P_2-numbers.

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1. Introduction. It is expected that sums of four cubes of primes represent all sufficiently large integers that satisfy some condition arising naturally from an allied congruence, while the corresponding statement for sums of three cubes of primes is obviously false, because a trivial counting argument shows that for any given large N, there are only $O(N(\log N)^{-3})$ natural numbers up to N that can be written in the latter manner. The necessary condition for representing n as a sum of four cubes of primes is that for each natural number q, there exists a solution of the congruence $n \equiv x_1^3 + x_2^3 + x_3^3 + x_4^3 \pmod q$ with each x_j coprime to the modulus q. We write $\mathcal{N}$ for the set of all the natural numbers n satisfying this condition. A modicum of elementary calculations based on the Cauchy–Davenport theorem (see Lemma 8.7 of [**5**]) reveals with little effort that $\mathcal{N}$ is the set of natural numbers n satisfying

$$n \equiv 0 \pmod 2, \quad n \not\equiv \pm 1,\ \pm 3 \pmod 9 \quad \text{and} \quad n \not\equiv \pm 1 \pmod 7.$$

Note that if a number $n \notin \mathcal{N}$ can be written as a sum of four cubes of primes, then one of the cubes in the representation must be that of 2, 3 or 7, and therefore, there are at most $O(N(\log N)^{-3})$ natural numbers $n \leq N$ with $n \notin \mathcal{N}$ that can be written in the form under consideration.

Although it may be anticipated that all but finitely many numbers in $\mathcal{N}$ can be written as the sum of four cubes of primes, we have hitherto been unable to provide any non-trivial estimate even for the density of the exceptions to this representation. However, Roth [**8**] could show that all natural numbers up to N with $o(N)$ possible exceptions can be written as the sum of three cubes of primes and one cube of a natural number. A closer approximation to the Waring–Goldbach problem was found by Brüdern [**1**] who applied a weighted sieve in combination with the circle method and showed that all but $o(N)$ numbers $n \in \mathcal{N}$ can be written in the form

$$n = p_1^3 + p_2^3 + p_3^3 + x^3, \tag{1.1}$$

where p_1, p_2 and p_3 are primes and x is a P_4. Here we recall that a natural number is called a P_r-number if it has at most r prime factors, counted with multiplicity. Kawada [**6**] replaced P_4 with P_3 in the latter result, by altering the sieve procedure employed by Brüdern [**1**], but it seems that the corresponding statement with a P_2 is out of reach of current technology. Thus, one may now ask if it is possible to replace P_3 by P_2 in a similar statement, at the cost of allowing some of the primes to be P_2-numbers, so that one may establish that almost all numbers in $\mathcal{N}$ may be written as the sum of four cubes of P_2-numbers, a conclusion not available in the literature so far. This paper gives an affirmative answer to this question.

THEOREM. *Let $E(N)$ be the number of $n \in \mathcal{N}$ not exceeding N that cannot be written in the form*

$$n = x^3 + y^3 + p_1^3 + p_2^3,$$

where p_1 and p_2 are primes and x and y are P_2-numbers. Then, for $N \geq 2$ and for any given $A > 0$, one has

$$E(N) \ll N(\log N)^{-A},$$

where the implicit constant depends on A.

As was already pointed out in the introduction of Brüdern [**1**], propositions of this kind have close connections with representation by sums of eight cubes. For a large natural number n, we put $a_n = 1$ or 2 according to n being even or odd and consider the set $\mathcal{S}_n$ of the numbers of the form $n - (a_n p_1)^3 - p_2^3 - p_3^3 - p_4^3$ with primes $p_j \leq \frac{1}{3} n^{1/3}$. Then, via a familiar argument and a careful examination of the congruence conditions, we may deduce from Hua's inequality (see Theorem 4 of [**5**]) that the cardinality of $\mathcal{S}_n \cap \mathcal{N}$ is $\gg n(\log n)^{-C}$ with some absolute constant C. Thus our theorem assures that there is an element of $\mathcal{S}_n$ that can be written as the sum of two cubes of primes and two cubes of P_2-numbers, whence every large n may be written as

$$n = (a_n p_1)^3 + p_2^3 + \cdots + p_6^3 + x^3 + y^3,$$

with primes p_j and P_2-numbers x and y. In particular, therefore, every sufficiently large integer n can be written as the sum of eight cubes of P_2-numbers, and this last conclusion also appears to be new.

Throughout the paper, we use the lowercase letter p, with or without subscript, to denote prime numbers, and the greatest common divisor of a and b is (a, b). Euler's totient function is $\varphi(q)$, and we write $e(\alpha) = \exp(2\pi i\alpha)$. Euler's constant is denoted by γ.

2. Outline of the proof. We use the same notation as in Brüdern [**1**] whenever possible. Let A be the given positive number appearing in the statement of our theorem, and fix another positive real number θ. Let N be a sufficiently large real number, and put

$$P = \frac{2}{3}N^{1/3}, \quad Q = P^{5/6}, \quad L = (\log P)^{9A+250} \quad \text{and} \quad D = P^{\theta}.$$

We denote by $\Omega(x)$ the number of prime factors of x counted with multiplicity and by $\Pi(z)$ the product of all primes less than z; $\Pi(z) = \prod_{p<z} p$.

Then we write $R(n)$ for the number of representations of n in the form

$$n = x^3 + y^3 + p_1^3 + p_2^3, \tag{2.1}$$

with integers x, y and primes p_1, p_2 satisfying

$$P < x,\ y \leq 2P, \quad (xy, \Pi(D^{1/3})) = 1, \quad \Omega(y) \leq 2, \quad Q < p_1,\ p_2 \leq 2Q. \tag{2.2}$$

Also we write $R'(n)$ for the number of representations of n in the same form (2.1) with (2.2) and the additional condition that $\Omega(x) \geq 3$.

Before we can state the results we shall show on $R(n)$ and $R'(n)$, we require the following notation and facts, which are mostly recalled from Brüdern [**1**]:

$$v(\beta) = \int_P^{2P} e(\beta t^3)dt, \quad w(\beta, \Xi) = \int_{\Xi}^{2\Xi} \frac{e(\beta t^3)}{\log t}dt,$$

$$J(n) = \int_{-LP^{-3}}^{LP^{-3}} v(\beta)w(\beta, P)w(\beta, Q)^2 e(-n\beta)d\beta,$$

$$S(q, a) = \sum_{r=1}^{q} e\left(\frac{ar^3}{q}\right), \quad S^*(q, a) = \sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{ar^3}{q}\right),$$

$$T_d(q, n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S(q, ad^3)S^*(q, a)^3}{q\varphi(q)^3} e\left(-\frac{an}{q}\right).$$

An upper bound for $T_d(q, n)$ is needed later, and we begin by deriving such an estimate when q is prime. Indeed, when p is a prime with $p \nmid m$, it follows easily from the definition that $T_d(p, n) = T_d(p, nm^3)$, so we have

$$\begin{aligned} T_d(p, n) &= (p-1)^{-1} \sum_{m=1}^{p-1} T_d(p, nm^3) \\ &= p^{-1}(p-1)^{-4} \sum_{a=1}^{p-1} S(p, ad^3)S^*(p, a)^3 S^*(p, -an). \end{aligned} \tag{2.3}$$

By Lemma 4.3 of Vaughan [**9**], on noting that $S^*(p, a) = S(p, a) - 1$, we see that whenever $p \nmid a$, one has

$$S(p, ad^3) \ll p^{1/2}(p, d)^{1/2}, \qquad S^*(p, -an) \ll p^{1/2}(p, n)^{1/2}$$

and $S^*(p, a) \ll p^{1/2}$. Therefore it follows from (2.3) that

$$T_d(p, n) \ll p^{-3/2}(p, n)^{1/2}(p, d)^{1/2}. \tag{2.4}$$

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Next, provided that $p \nmid a$, Lemma 8.3 of Hua [**5**] asserts that $S^*(p^l, a) = 0$ when $p \neq 3$ and $l \geq 2$ or when $p = 3$ and $l \geq 3$, so under these circumstances, one has $T_d(p^l, n) = 0$. Moreover, one may straightforwardly confirm that $T_d(q, n)$ is multiplicative with respect to q (see Chapters 2 and 4 of [**9**], for example). Thus we may deduce from (2.4) that for any fixed $\varepsilon > 0$, one has

$$T_d(q, n) \ll q^{\varepsilon - 3/2}(q, n)^{1/2}(q, d)^{1/2}. \tag{2.5}$$

Now we define

$$\mathfrak{S}_d(n) = \sum_{q=1}^{\infty} T_d(q, n) \quad \text{and} \quad \mathfrak{S}(n) = \mathfrak{S}_1(n),$$

noting that the absolute convergence of $\mathfrak{S}_d(n)$ is assured by (2.5). We recall from Roth [**8**] that for $n \geq 3$ one has

$$(\log\log n)^{-c} \ll \mathfrak{S}(n) \ll (\log\log n)^{c}, \tag{2.6}$$

with some positive constant c, so that we can define

$$\omega_n(d) = \frac{\mathfrak{S}_d(n)}{\mathfrak{S}(n)} \quad \text{and} \quad W_n(z) = \prod_{p<z}\Big(1 - \frac{\omega_n(p)}{p}\Big).$$

As regards the function $\omega_n(d)$, moreover, Brüdern [**1**, see, in particular, Lemma 5 and the arguments on pp. 469–470] showed that

$$0 \leq \omega_n(p) < p \quad \text{and} \quad \omega_n(p) = 1 + O(p^{-1/2}), \tag{2.7}$$

for all primes p and $n \in \mathcal{N}$ (see also the allied comments on pp. 15–16 of [**6**]). The latter fact allows us to apply the linear sieve in the final section. In particular it follows that the inequality

$$W_n(z) \gg (\log z)^{-1} \tag{2.8}$$

holds uniformly in $z > 2$ and $n \in \mathcal{N}$.

Next we note, as is mentioned in Brüdern (see (3.8) of [**1**]), that one may easily confirm that

$$n^{2/9}(\log n)^{-3} \ll J(n) \ll n^{2/9}(\log n)^{-3}, \tag{2.9}$$

for all $n \in (N, 2N]$, by the routine endgame techniques in the Hardy–Littlewood method.

We are now in a position to formulate the main results on $R(n)$ and $R'(n)$.

LEMMA 1. *Provided that $\theta < 1/3$ and that N is sufficiently large, the inequality*

$$R(n) > \mathfrak{S}(n)J(n)W_n(D^{1/3})\frac{2e^{\gamma}}{3}\Big(1 + \log\Big(\frac{3}{\theta} - 1\Big)\Big)(\log 2 - 10^{-4})$$

holds for all but $O(N(\log N)^{-A})$ values of $n \in \mathcal{N} \cap (N, 2N]$.

LEMMA 2. *Provided that $\theta < 1/3$ and that N is sufficiently large, the inequality*

$$R'(n) < \mathfrak{S}(n)J(n)W_n(D^{1/3})\frac{2e^{\gamma}}{3}\Big(\frac{3}{\theta e^{\gamma}} + 10^{-4} - 1 - \log\Big(\frac{3}{\theta} - 1\Big)\Big)$$

holds for all but $O(N(\log N)^{-A})$ values of $n \in \mathcal{N} \cap (N, 2N]$.

We shall prove these lemmata in the final section, based on the work of Brüdern [**1**]. Here, we close this section by observing that our theorem follows immediately from these lemmata. We take $\theta = 30/91$ for instance and confirm the numerical estimates

$$\left(1 + \log\left(\frac{3}{\theta} - 1\right)\right)(\log 2 - 10^{-4}) > 2.14$$

and

$$\frac{3}{\theta e^{\gamma}} + 10^{-4} - 1 - \log\left(\frac{3}{\theta} - 1\right) < 2.02.$$

Then, from Lemmata 1 and 2 and the lower bounds recorded in (2.6), (2.8) and (2.9), we may deduce that for all but $O(N(\log N)^{-A})$ values of $n \in \mathcal{N} \cap (N, 2N]$, one has $R(n) > R'(n)$, which obviously means that n can be written in the form (2.1) with P_2-numbers x, y and primes p_1, p_2. Hence we have

$$E(2N) - E(N) \ll N(\log N)^{-A},$$

for every large N, and by summing up the last inequality for appropriate dyadic values of N, we may establish our theorem. Thus, the remaining part of the paper is devoted to the proofs of the above lemmata.

3. Preliminaries. In this section, we quote a couple of results from previous work. Still following Brüdern [**1**], we introduce the exponential sums

$$f_d(\alpha) = \sum_{P/d < x \le 2P/d} e(d^3x^3\alpha), \qquad h(\alpha) = \sum_{Q < p \le 2Q} e(p^3\alpha),$$

when d is a natural number, and define the major and minor arcs, $\mathfrak{M}$ and $\mathfrak{m}$, as follows:

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1);\ |\alpha - a/q| \le LP^{-3}\}, \quad \mathfrak{M} = \bigcup_{q \le L} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q} \mathfrak{M}(q, a),$$

$$\mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Further, when $\mathcal{X}$ is a finite set of integers, we write

$$g(\alpha; \mathcal{X}) = \sum_{x \in \mathcal{X}} e(x^3\alpha).$$

We first quote a result on the minor-arc contribution from Brüdern [**1**], in the following form.

LEMMA 3. *Let η_d be complex numbers with $|\eta_d| \le 1$ and $\mathcal{X}$ be a set of natural numbers belonging to $(P, 2P]$, and suppose that $\theta < 1/3$. Then, in the notation introduced above,*

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one has

$$\Big|\sum_{d\le D}\eta_d\int_{\mathfrak{m}} f_d(\alpha)g(\alpha;\mathcal{X})h(\alpha)^2e(-n\alpha)\,d\alpha\Big| \le P^{-1}Q^2(\log P)^{-6},$$

for all but $O(N(\log N)^{-A})$ *values of* $n\in(N,2N]$.

Proof. The conclusion of the lemma follows immediately from the inequality

$$\sum_{N<n\le 2N}\Big|\sum_{d\le D}\eta_d\int_{\mathfrak{m}} f_d(\alpha)g(\alpha;\mathcal{X})h(\alpha)^2e(-n\alpha)\,d\alpha\Big|^2 \ll PQ^4(\log P)^{-A-12},$$

but this estimate is given by Lemma 4 of Brüdern [**1**]. In fact, in the latter lemma of [**1**], the last inequality is established with taking $\mathcal{X}$ to be the set of all primes in $(P,2P]$, and one may notice that the proof is valid for any set $\mathcal{X}$ of integers contained in $(P,2P]$ without any alteration. Thus we obtain the desired conclusion.

Next, for each natural number r, we write

$$\mathcal{P}_r(X,z)=\{x\in\mathbb{N};\ X<x\le 2X,\quad (x,\Pi(z))=1,\quad \Omega(x)=r\}$$

and define the functions $C_r(u)$ for $u>0$ inductively on r by

$$C_1(u)=\begin{cases}1 & (u\ge 1),\\ 0 & (u<1),\end{cases}\qquad C_r(u)=\int_r^{\max\{u,r\}}\frac{C_{r-1}(t-1)}{t-1}\,dt\quad (r\ge 2).$$

Note in particular that $C_2(u)=\log(u-1)$ for $u\ge 2$ and that $C_r(u)=0$ for $0<u<r$. Then, the behaviour of $g(\alpha;\mathcal{P}_r(X,z))$ on the major arcs $\mathfrak{M}$ may be described by the following lemma.

LEMMA 4. *Suppose that* $z\le X\le P$, $\log z\gg \log P$, *and that* $\alpha=a/q+\beta$ *with coprime integers* q *and* a *satisfying* $1\le q\le L$ *and a real number* β *satisfying* $|\beta|\le LP^{-3}$. *Then for each natural number* r, *one has*

$$g(\alpha;\mathcal{P}_r(X,z))=\varphi(q)^{-1}S^*(q,a)w_r(\beta;X,z)+O(XL^{-5}),$$

where the function $w_r(\beta;X,z)$ *satisfies the formula*

$$w_r(\beta;X,z)=C_r\Big(\frac{\log X}{\log z}\Big)w(\beta;X)+O(X(\log X)^{-2}),\tag{3.1}$$

and, in particular, one has $w_1(\beta;X,z)=w(\beta;X)$.

Proof. This lemma may be deduced routinely from the Siegel–Walfisz theorem, and for the details, refer to the proof of Lemma 7.15 of Hua [**5**] for $r=1$ and that of Lemma 2.2 of Brüdern and Kawada [**2**] for $r\ge 2$. □

4. Application of the linear sieve.

In order to prove Lemma 1, we apply the linear sieve in the manner explained in Section 2 of Brüdern [**1**]. In the first phase of the

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proof, we put

$$\mathcal{Y} = \{y \in \mathbb{N};\ P < y \le 2P,\quad (y, \Pi(D^{1/3})) = 1,\quad \Omega(y) \le 2\ \},$$

write $R_d(n)$ for the number of the representations of n in the form $n = x^3 + y^3 + p_1^3 + p_2^3$, subject to

$$P < x \le 2P,\quad x \equiv 0 \pmod d,\quad y \in \mathcal{Y},\quad Q < p_1,\ p_2 \le 2Q,$$

and investigate the latter quantity via the Hardy–Littlewood method for natural numbers $d \le D$ and $n \in (N, 2N]$. To this end, we recall the notation introduced in the preamble to Lemma 3 and define

$$R_d(n; \mathcal{B}) = \int_{\mathcal{B}} f_d(\alpha) g(\alpha; \mathcal{Y}) h(\alpha)^2 e(-n\alpha) d\alpha,$$

so that we have

$$R_d(n) = R_d(n; [0, 1)) = R_d(n; \mathfrak{M}) + R_d(n; \mathfrak{m}). \tag{4.1}$$

The major arc contribution $R_d(n; \mathfrak{M})$ may be evaluated by the standard strategy. When α satisfies the conditions in the statement of Lemma 4, Theorem 4.1 of Vaughan [**9**] yields the formula

$$f_d(\alpha) = (qd)^{-1} S(q, ad^3) v(\beta) + O(L).$$

Besides this formula, we apply Lemma 4 to

$$g(\alpha; \mathcal{Y}) = \sum_{r=1}^{2} g(\alpha; \mathcal{P}_r(P, D^{1/3})) \quad \text{and} \quad h(\alpha) = g(\alpha; \mathcal{P}_1(Q, Q)),$$

and then a straightforward argument reveals that

$$R_d(n; \mathfrak{M}) = d^{-1} J_1(n) \sum_{q \le L} T_d(q, n) + O(d^{-1} P^{-1} Q^2 L^{-1}), \tag{4.2}$$

for $d \le D$, where

$$J_1(n) = \int_{-LP^{-3}}^{LP^3} v(\beta)(w(\beta; P) + w_2(\beta; P, D^{1/3})) w(\beta; Q)^2 e(-n\beta) d\beta.$$

By using (3.1), the well-known inequality $v(\beta) \ll P(1 + P^3|\beta|)^{-1}$ and the trivial bound $w(\beta; X) \ll X/\log X$ for $X \ge 2$ and also by recalling (2.9), we see here that

$$\begin{aligned} J_1(n) &= \Big(1 + \log\Big(\frac{3}{\theta} - 1\Big)\Big) J(n) + O\Big(\int_0^{LP^{-3}} \frac{P^2 Q^2 (\log P)^{-4}}{1 + P^3\beta} d\beta\Big) \\ &= \Big(1 + \log\Big(\frac{3}{\theta} - 1\Big) + O\Big(\frac{\log L}{\log P}\Big)\Big) J(n), \end{aligned} \tag{4.3}$$

for $N < n \le 2N$.

Also, by appealing to (2.5), we observe that

$$\mathfrak{S}_d(n) - \sum_{q\le L} T_d(q,n) \ll \sum_{q>L} q^{-4/3}(q,nd) \ll \tau(nd)L^{-1/3},$$

where $\tau(m)$ denotes the divisor function. Combining this estimate with (2.9), (4.2) and (4.3), we deduce from (4.1) that one has

$$R_d(n) = \frac{\omega_n(d)}{d}\mathfrak{S}(n)J_1(n) + R_d(n;\mathfrak{m}) + O\Big(\frac{\tau(n)\tau(d)Q^2}{dPL^{1/3}}\Big), \tag{4.4}$$

for $d \le D$ and $N < n \le 2N$.

By (4.4) and (2.7), we may apply the linear sieve to estimate $R(n)$ for $n \in \mathcal{N}$ and actually obtain the lower bound

$$R(n) > \mathfrak{S}(n)J_1(n)W_n(D^{1/3})\Big(\frac{2e^{\gamma}}{3}\log 2 + O((\log L)^{-3/10})\Big) + \sum_{d\le D}\eta_d R_d(n;\mathfrak{m}) + O(\tau(n)P^{-1}Q^2L^{-1/3}(\log D)^2), \tag{4.5}$$

with suitable sieving weights η_d satisfying $|\eta_d| \le 1$ (see, for example, Theorem 9 of [**7**] or Corollary 1.1 of [**3**, Section 4.4.1]).

The well-known estimate $\sum_{n\le 2N}\tau(n) \ll N\log N$ implies that for all $n \in (N, 2N]$ with $O(N(\log N)^{-A})$ possible exceptions, one has $\tau(n) \le (\log N)^{A+1}$, in which case the last term on the right-hand side of (4.5) is $O(P^{-1}Q^2L^{-2/9})$. We also know, by Lemma 3, that the second term on the right-hand side of (4.5) is $O(P^{-1}Q^2(\log P)^{-6})$ for all but $O(N(\log N)^{-A})$ values of $n \in (N, 2N]$, provided that $\theta < 1/3$. Hence, Lemma 1 is now immediate from (4.5), in view of (2.6), (2.8), (2.9) and (4.3).

We turn to the proof of Lemma 2. We define the set

$$\mathcal{X} = \{x \in \mathbb{N};\ P < x \le 2P,\ \ (x, \Pi(D^{1/3})) = 1,\ \ \Omega(x) \ge 3\,\}$$

and write $R''(n)$ for the number of representations of n in the form

$$n = x^3 + y^3 + p_1^3 + p_2^3, \tag{4.6}$$

subject to

$$x \in \mathcal{X},\ \ P < y \le 2P,\ \ (y, \Pi(D^{1/3})) = 1,\ \ Q < p_1, p_2 \le 2Q.$$

We note that $R'(n)$ is then the number of representations counted by $R''(n)$ that satisfy the additional condition $\Omega(y) \le 2$, so that trivially one has

$$R'(n) \le R''(n). \tag{4.7}$$

We may obtain an upper bound for $R''(n)$ by applying the linear sieve in a manner similar to our argument on $R(n)$ above.

Now, let $R''_d(n)$ be the number of representations of n in the form (4.6) subject to

$$x \in \mathcal{X},\ \ P < y \le 2P,\ \ y \equiv 0 \ (\mathrm{mod}\ d),\ \ Q < p_1, p_2 \le 2Q.$$

In order to apply the circle method to $R''_d(n)$ as before, we notice that

$$g(\alpha; \mathcal{X}) = \sum_{r\geq 3} g(\alpha; \mathcal{P}_r(P, D^{1/3})),$$

where the sum on the right-hand side is practically finite because $\mathcal{P}_r(P, D^{1/3})$ is empty if $D^{r/3} > 2P$. Thus we can grasp the behaviour of $g(\alpha; \mathcal{X})$ on $\mathfrak{M}$ by Lemma 4. Then, substituting $g(\alpha; \mathcal{X})$ for $g(\alpha; \mathcal{Y})$ in our previous argument leading to (4.4), we obtain a formula for $R''_d(n)$ similar to (4.4) in which $J_1(n)$ and $R_d(n; \mathfrak{m})$ are replaced respectively by

$$J_2(n) = \int_{-LP^{-3}}^{LP^3} v(\beta) \sum_{r\geq 3} w_r(\beta; P, D^{1/3}) w(\beta; Q)^2 e(-n\beta) d\beta$$

and

$$R''_d(n; \mathfrak{m}) = \int_{\mathfrak{m}} f_d(\alpha) g(\alpha; \mathcal{X}) h(\alpha)^2 e(-n\alpha) d\alpha.$$

As in (4.3), we derive by (3.1) that for $N < n \leq 2N$ one has

$$J_2(n) = \Big(\sum_{r\geq 3} C_r(3/\theta) + O((\log L)/(\log P))\Big) J(n). \tag{4.8}$$

By virtue of (2.7), the formula for $R''_d(n)$ corresponding to (4.4) allows us to apply the linear sieve to estimate $R''(n)$ for $n \in \mathcal{N}$, and consequently we obtain the inequality

$$\begin{aligned} R''(n) < {} & \mathfrak{S}(n) J_2(n) W_n(D^{1/3}) \Big(\frac{2e^\gamma}{3} + O((\log L)^{-3/10})\Big) \\ & + \sum_{d\leq D} \eta'_d R''_d(n; \mathfrak{m}) + O(\tau(n) P^{-1} Q^2 L^{-1/3} (\log D)^2), \end{aligned} \tag{4.9}$$

with suitable sieving weights η'_d satisfying $|\eta'_d| \leq 1$ (see Theorem 9 of [**7**] or Corollary 1.1 of [**3**, Section 4.4.1]). As in the final phase of the proof of Lemma 1 above, we find that the second and third terms on the right-hand side of (4.9) are $O(P^{-1}Q^2(\log P)^{-6})$ for all $n \in (N, 2N]$ with $O(N(\log N)^{-A})$ possible exceptions, provided that $\theta < 1/3$, by Lemma 3 and plain examination of the divisor function. Hence, in view of (2.6), (2.8), (2.9) and (4.8), Lemma 2 would follow immediately from (4.9) if we could show for $\theta < 1/3$ that

$$\sum_{r\geq 3} C_r(3/\theta) < \frac{3}{\theta e^\gamma} + 10^{-4} - 1 - \log\Big(\frac{3}{\theta} - 1\Big). \tag{4.10}$$

Estimates useful to confirming (4.10) are found in Grupp and Richert [**4**]. We first remark that the functions $I_r(u)$ in [**4**] and our functions $C_r(u)$ defined in the preamble to Lemma 4 satisfy the relation $uI_r(u) = C_r(u)$ for $u \geq 1$. Then, for the functions

$$f(u) = 2e^\gamma \sum_{k\geq 1} I_{2k}(u) \quad \text{and} \quad F(u) = 2e^\gamma \sum_{k\geq 1} I_{2k-1}(u),$$

Grupp and Richert [**4**] showed the inequalities

$$f(u) < 1 \qquad \text{and} \qquad F(u) < 1 + 2e^{\gamma}\Gamma(u+1)^{-1},$$

where $\Gamma(u)$ is the gamma function (see p. 212 of [**4**]). Therefore we have

$$\sum_{r\geq 1} C_r(u) = \frac{f(u)+F(u)}{2e^{\gamma}}u < e^{-\gamma}u + \Gamma(u)^{-1},$$

from which we can easily derive (4.10), by substituting $u = 3/\theta$, noting that $\Gamma(u) \geq 40320$ for $u \geq 9$ and recalling that $C_1(u) = 1$ and $C_2(u) = \log(u-1)$ for $u \geq 2$. Now we complete the proof of Lemma 2, as well as that of the theorem.

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