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MULTI-FLAG SYSTEMS AND ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We discuss the Monge problem for under-determined systems of ordinary differential equations with an arbitrary degree of freedom and give a sufficient condition, in terms of truncated multi-flag systems, for the Monge property to hold. This condition extends in a natural way the Cartan criterion valid for systems with one degree of freedom.

§1. Introduction

The study of under-determined systems of ordinary differential equations seems to have been initiated by Gaspard Monge who exhibited, on some specific examples, a very convenient parametrization of the general solution evidencing its dependence on an arbitrary function of one variable and on all its derivatives up to a certain order ([16]). Other authors have also examined this problem ([7], [8], [19]) and, in 1914, Élie Cartan gave a general criterion for the existence of such Monge parametrizations in the case of under-determined systems with one degree of freedom ([1], [3], [4], [12]). This criterion states that the canonical Pfaffian system associated to the given equations must be a flag system.

Our purpose here is to extend this criterion to systems with an arbitrary degree of indetermination and we are thus led to introduce multi-flag as well as truncated multi-flag systems that are natural extensions of the above mentioned flag systems. However, under-determined systems with just one degree of freedom behave quite differently from those with a larger degree since, for the former, the Cartan criterion provides a necessary and sufficient condition whereas, for the latter, the condition is only sufficient.

The geometric fact underlying this distinction can be retraced in the following group-theoretical argument. While the pseudogroup of all the

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local automorphisms of the single equation $dx - y\,dt = 0$ is the contact pseudogroup in 3 variables, that of the system

$$(1) \quad dx^1 - y^1\,dt = 0\,, \quad dx^2 - y^2\,dt = 0\,, \;\cdots\;, dx^k - y^k\,dt = 0\,, \quad k \geq 2\,,$$

is locally equivalent, via prolongation, to the pseudogroup of all the local diffeomorphisms of $\mathbf{R}^{k+1}$. More precisely, any finite (resp. infinitesimal) automorphism of the system (1) is locally the prolongation of a diffeomorphism (resp. a vector field) of the space $\mathbf{R}^{k+1} = \{(t, x^1, \cdots, x^k)\}$ to the projective bundle of all the tangent 1-spaces to $\mathbf{R}^{k+1}$ ([14]). Both pseudogroups are simple but have essentially different structures. Further, it can be shown ([11], [17]) that the pseudoalgebra of all the infinitesimal symmetries of a flag system is locally equivalent, via prolongation, to the pseudoalgebra of the first pseudogroup and the corresponding pseudoalgebra of a multi-flag system is locally equivalent, via prolongation, to that of the second pseudogroup.

The paper is organized as follows: We recall, in Section 2, some basic properties of Pfaffian systems and define their Martinet structure tensor, their derived systems, their automorphisms and their characteristics. Next, we define the polar spaces and the covariant systems, that become the main tool in the subsequent discussion, and introduce the multi-flag and truncated multi-flag systems. In Section 3, we study the first systems and show (Theorem 1) that under suitable conditions, called normality conditions, they become transitive and reduce locally to a normal form. We also show (Corollary 1) how the normality conditions simplify when the systems are assumed to be transitive. In Section 4, we extend the above results to the truncated systems (Theorems 2 and 3 and Corollary 2). In Section 5, we define Monge systems (also called Monge equations), examine their associated Pfaffian systems, determine the rank of the first derived systems (Proposition 4 and Corollary 3) and discuss Monge parametrizations. Finally, in Section 6, we prove the extended Cartan criterion (Theorem 4) that provides a sufficient condition for a Monge system to admit Monge parametrizations.

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§2. Definitions and notations

For simplicity, we assume that all the data is C^∞ smooth though of course, in each specific case, C^k smoothness for some k suffices and assume further that all the manifolds are connected and second countable. A Pfaffian system on a manifold M is as a locally trivial vector sub-bundle $S \subset T^*M = T^*$. We write $\operatorname{rank} S = \dim\ S_x$, this dimension being independent of $x \in M$, and say that a linear differential form ω belongs to S when $\omega_x \in S_x$ for all x in the domain of ω. The annihilator $\Sigma = S^\perp \subset TM = T$ is a distribution on the manifold M, $\Sigma^\perp = S$ and we say that a vector field η belongs to Σ when $\eta_x \in \Sigma_x$, $\forall\, x$.

A Pfaffian system S is *transitive* when the pseudogroup of all its local automorphisms operates transitively on M. It is *infinitesimally transitive* when, at each point $x \in M$, the linear sub-space induced by all its local infinitesimal automorphisms is equal to T_xM. Infinitesimal transitivity implies transitivity but the converse is not always true.

The set of all the local infinitesimal automorphisms of S that belong to $S^\perp$ is a pre-sheaf of Lie algebras known as the characteristic pseudoalgebra associated to S. The field of linear sub-spaces $x \in M \longmapsto \Delta_x \subset T_xM$ induced by the elements of this pseudoalgebra need not be of constant dimension. However, when this is the case, Δ is known as the characteristic distribution associated to S, it is integrable and the integer $q = \operatorname{rank} \Delta^\perp$ is the (Cartan) *class* of S. If we choose, in a neighborhood of an arbitrary point $x \in M$, q independent first integrals $\{y^j\}$ of Δ, we can determine a local basis $\{\omega^i\}$ of S such that each ω^i has the expression $\omega^i = \sum f^i_j(y^1, \cdots, y^q)\, dy^j$. The system S is said to be of constant class when $\dim \Delta_x$ is constant and, this being the case, $\Delta^\perp$ is generated by the set $\{\omega\,, i(\eta)d\omega\}$ where ω is an arbitrary differential form belonging to S and η an arbitrary vector field belonging to $S^\perp$.

Much in the same way, the set of all the local infinitesimal automorphisms ξ of a differential form ω that further verify $i(\xi)\omega = 0$ is a pre-sheaf of Lie algebras known as the characteristic pseudoalgebra associated to ω. When the field $x \longmapsto \bar{\Delta}_x$ induced by the above pseudoalgebra has constant dimension, $\bar{\Delta}$ is the characteristic distribution associated to ω and the integer $q = \operatorname{rank} \bar{\Delta}^\perp$ is its (Darboux) *class*. The form ω can then be expressed, locally, in terms of the first integrals of $\bar{\Delta}$ and their differentials and we say that ω has constant class. In particular, 1-forms and closed two forms of constant class have well known local canonical expressions (the

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Darboux theorems). The following proposition relates the Cartan class with the Darboux class ([9]).

Proposition 1. *The Cartan class of any Pfaffian system S of* rank 1 *is odd. It is equal to $2p+1$ if and only if S is locally generated by* 1*-forms of Darboux class equal to $2p+1$.*

Let us recall that S is integrable if, for every form ω belonging to S,

$$d\omega \equiv 0 \mod S, \quad i.e., \quad d\omega = \sum \sigma^i \wedge \omega^i, \tag{2}$$

where the forms ω^i belong to S.

The Martinet *structure tensor* ([13], [15]), is the vector bundle morphism

$$\delta : S \longrightarrow \wedge^2(T^*/S) \tag{3}$$

defined on the local sections of S by $\delta(\omega) = d\omega \mod S$. Since $d(f\omega) = f\,d\omega + df \wedge \omega \equiv f\,d\omega \mod S$, the above defined pre-sheaf morphism is linear over the functions of M and therefore induces the desired vector bundle morphism. We now assume that the rank of δ is constant and define the *derived system* $S_1 = \ker \delta$. Then $S_1 \subset S$ and the integrability condition (2) reads $S = S_1$. We define inductively the $(\nu+1)$-st derived system $S_{\nu+1} = (S_\nu)_1$ by considering, at each step, the Martinet tensor $\delta_\nu : S_\nu \longrightarrow \wedge^2(T^*/S_\nu)$ of S_ν and by assuming of course that the rank of δ_ν is constant. The previous construction yields a decreasing sequence of Pfaffian systems that necessarily becomes stationary, namely:

$$S = S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_\ell = S_{\ell+1} = \cdots . \tag{4}$$

The integer ℓ is called the *length* of the Pfaffian system S, the last term S_ℓ is integrable and is referred to it as the *terminating system* of S. Length zero characterizes the integrability of S. We denote by $[\omega]_\nu \in T^*/S_\nu$ the equivalence class of $\omega \in T^*$.

Definition 1. A Pfaffian system is called totally regular when all the successive structure tensors have constant rank.

Definition 2. A multi-flag system of width k (k-flag for short) is a totally regular Pfaffian system of length ℓ for which $\operatorname{rank} \delta_\nu = k$, $0 \le \nu \le \ell - 1$ $(\delta_0 = \delta)$.

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A k-flag is also characterized by the conditions $\operatorname{rank}(S_\nu/S_{\nu+1}) = k$. The 1-flag systems are the usual flag systems (système de Pfaff en drapeau, [13]) often referred to, in literature, as Goursat systems.

We define the *polar space* of S at the point $x \in M$ by the condition

$$\operatorname{Pol}(S)_x = \{\mathbf{w} \in T^*_x/S_x \mid \mathbf{w} \wedge \delta(\omega) = 0\,,\ \forall\, \omega \in S_x\}\,. \tag{5}$$

When $Pol\,(S)$ is regular *i.e.*, when $\dim \operatorname{Pol}(S)_x$ is constant, we also define the *covariant system* associated to S as being the Pfaffian system $\widehat{S} = q^*(\operatorname{Pol}(S))$, inverse image of $\operatorname{Pol}(S)$ relative to the quotient map $q : T^* \to T^*/S$. In other terms, $\operatorname{Pol}(S) = \widehat{S}/S$. When S is integrable, $\operatorname{Pol}(S) = T^*/S$ and $\widehat{S} = T^*$. However, the converse is not true as evidenced by the Proposition 3. The main feature of the covariant system is described by the

Proposition 2. *Let S be a Pfaffian system such that* $\operatorname{Pol}(S)$ *is regular and* $\operatorname{rank}\operatorname{Pol}(S) \geq 1$. *Then* $(\widehat{S})_1 \supset S$.

Proof. We set $s = \operatorname{rank} S$, $r = \operatorname{rank}\widehat{S}$ and choose, in a neighborhood of an arbitrary point x, a local field of co-frames $\{\omega^1, \cdots, \omega^n\}$, $n = \dim M$, such that the first s (resp. r) forms (locally) generate S (resp. $\widehat{S}$). Then $\{[\omega^{s+1}], \cdots, [\omega^n]\}$, where $[\omega^i] = q(\omega^i)$, is a local basis of T^*/S and expressing each $\delta(\omega^i)$, $1 \leq i \leq s$, in terms of this basis, the conditions (5) imply that all the terms in $[\omega^j] \wedge [\omega^k]$, $j\,, k \geq r+1$, must vanish hence $d(\omega^i) \equiv 0 \mod \widehat{S}$.

We also mention the following result, a direct consequence of the Proposition 3.3 in [13].

Proposition 3. *The covariant system $\widehat{S}$ associated to a 1-flag S of length $\ell \geq 2$ is equal to its characteristic system $\Delta^\perp$. When $S_\ell = 0$ and* $\dim M = \ell + 2$ *then $\widehat{S} = T^*$.*

As for the 1-flags of length 1, the above result still holds when their class is equal to 3. For larger class values, $\operatorname{Pol}(S) = 0$ and $\widehat{S} = S$ is not integrable.

The definition of a truncated system is somewhat more elaborate since the only restriction on the ranks of the successive tensors is $\operatorname{rank}\delta_\nu \geq \operatorname{rank}\delta_{\nu+1}$. The definition stated below puts in evidence the main features of such a system without referring though to any transitivity property (nor to the above rank restriction). The reason for doing so lies in the fact that Monge systems need not be transitive to have the Monge property.

Definition 3. A truncated multi-flag system (truncated flag for short) is a totally regular Pfaffian system $\mathcal{S}$ of length ℓ that satisfies the following properties:

(tf_1) $\mathcal{S}$ is not a 1-flag system i.e., there exists an integer ν, $0 \leq \nu \leq \ell - 1$, such that $\operatorname{rank} \mathcal{S}_\nu / \mathcal{S}_{\nu+1} \geq 2$. Let ν_0 be the largest such integer.

(tf_2) For each ν, $0 \leq \nu \leq \nu_0$, such that $\operatorname{rank} \mathcal{S}_\nu/\mathcal{S}_{\nu+1} > \operatorname{rank} \mathcal{S}_{\nu+1}/\mathcal{S}_{\nu+2}$, the covariant system $\widehat{\mathcal{S}_\nu}$ is integrable.

(tf_3) $\mathcal{S}_{\nu-1} \cap \widehat{\mathcal{S}_\nu} = \mathcal{S}_\nu$, $1 \leq \nu \leq \nu_0$.

(tf_4) If $\nu_0 < \ell - 1$, then $(\widehat{\mathcal{S}_{\nu_0+1}})_x \not\subset (\mathcal{S}_{\nu_0})_x$, $\forall x \in M$.

We shall see, in Section 4, that k-flags ($k \geq 2$) satisfying the normality conditions of the Theorem 1 are special cases of truncated multi-flags.

§3. Multi-flag systems

We now discuss the local structure of the k-flags of length ℓ defined in Section 2 and always assume that $k \geq 2$. As mentioned in the introduction, 1-flags behave differently from k-flags and have their own specific structure that requires an individual approach. It is worthwhile to mention briefly their differences as well as their analogies. The transitive 1-flags are those admitting, in a neighborhood of each point, the transitive von Weber model ([18]) and are characterized either by the property that the ranks of all their *reduced tensors* $\kappa_{(\lambda)}$ are equal to 1 ([13]) or by the property that their *small growth vectors* are equal to the *big growth vectors* ([5]). However, these numerical invariants are not strong enough to assure the transitivity of k-flags for which conditions on the covariant systems $\widehat{S_\nu}$ are required (cf. (20)). On the other hand, these covariant systems are quite irrelevant for 1-flags as evidenced by the Proposition 3. A 1-flag of length 1 is transitive if and only if its class is constant in which case it admits the Darboux local model of the same class. A 1-flag of length 2 is always transitive (hence of constant class) and admits the Engel local model ([6], [13]). Quite to the contrary, none of these properties hold for k-flags. As for the analogies, we can say that a k-flag of length ℓ admits locally an extended von Weber model (10) if and only if it is transitive and the last non-vanishing derived system has an integrable covariant system (Corollary 1).

Let us begin our discussion by first considering k-flags of length 1 and let us assume, for simplicity, that $\mathcal{S}_1 = 0$ whereupon $\operatorname{rank} \mathcal{S} = k$ (this condition being withdrawn at the end of this section).

Lemma 1. *If* $\widehat{\mathcal{S}}$ *is integrable then* $\operatorname{rank} \widehat{\mathcal{S}} = k + 1$.

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Proof. Whatever the point $x \in M$, the system $(\widehat{\mathcal{S}})_x$ cannot be equal to $\mathcal{S}_x$ since the later is everywhere non-integrable, hence rank $\mathrm{Pol}(\mathcal{S}) \geq 1$. If this rank were greater than one, the 2-forms $\delta(\omega)$, $\omega \in \mathcal{S}$, would all be, at each point x, multiples of a fixed 2-form Ω_x whereupon rank $\delta \leq 1$ thus contradicting the assumptions $k \geq 2$.

LEMMA 2. *If $\widehat{\mathcal{S}}$ is integrable then the k-flag $\mathcal{S}$ of length 1 admits, in a neighborhood of each point $x_0 \in M$, the following normal form*

$$\omega^1 = dx^1 + x^2\,dt\,, \quad \omega^2 = dx^3 + x^4\,dt\,, \quad \cdots, \quad \omega^k = dx^{2k-1} + x^{2k}\,dt\,, \tag{6}$$

where the coordinates x^i and t vanish at x_0. The system $\mathcal{S}$ is transitive and its class is equal to $2k+1$.

Proof. We take, according to the previous lemma, a complete set of first integrals $\{y^1, y^2, \cdots, y^{k+1}\}$ of $\widehat{\mathcal{S}}$ defined in a neighborhood of a point x_0 and can assume, without loss of generality, that $y^j(x_0) = 0$ and that $\{dy^1, \cdots, dy^k\}_{x_0}$ generates $\mathcal{S}_{x_0}$. The local generators of $\mathcal{S}$ can then be expressed by

$$\varpi^i = \sum_{1 \leq j \leq k+1} a^i_j\,dy^j\,, \tag{7}$$

with $a^i_j(x_0) = \delta^i_j$, $1 \leq i, j \leq k$ and $a^i_{k+1}(x_0) = 0$, and left multiplication by the inverse of the matrix $(a^i_j)_{1 \leq i,j \leq k}$ yields the new generators

$$\omega^i = dy^i + b^i_{k+1}\,dy^{k+1} \tag{8}$$

of $\mathcal{S}$ that, together with $\omega^{k+1} = dy^{k+1}$, also generate $\widehat{\mathcal{S}}$, the coefficients b^i_{k+1} vanishing at the point x_0. The injectivity of δ then implies the non-vanishing of

$$d\,y^1 \wedge \cdots \wedge d\,y^{k+1} \wedge d\,b^1_{k+1} \wedge \cdots \wedge d\,b^k_{k+1} \tag{9}$$

at the point x_0 and consequently the functions

$$x^{2i-1} = y^i\,, \quad x^{2i} = b^i_{k+1}\,, \quad t = y^{k+1}\,,$$

are the desired coordinates for the local model (6).

Remark. Let $\mathcal{S}$ be any Pfaffian system of rank k defined on a manifold M and let $\tilde{\mathcal{S}}$ be an integrable Pfaffian system of rank $k+1$ containing $\mathcal{S}$, *i..e.,* $\tilde{\mathcal{S}} \supset \mathcal{S}$. Then necessarily $\widehat{\mathcal{S}} \supset \tilde{\mathcal{S}}$. In fact, if $\{\omega^i\}$ is a local

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basis of $\mathcal{S}$ and $\{\omega^i, du\}$ a local basis of $\tilde{\mathcal{S}}$, the integrability of $\tilde{\mathcal{S}}$ reads $d\omega^i \equiv 0 \mod \{\omega^j, du\}$ and consequently $\delta(\omega^i) = [\sigma^i] \wedge [du]$, δ being the Martinet tensor of $\mathcal{S}$. We infer that $[du] \in \mathrm{Pol}(\mathcal{S})$ hence du is a local section of $\widehat{\mathcal{S}}$. In particular, if $\mathcal{S}$ is a k-flags of length 1 and if $\widehat{\mathcal{S}}$ is integrable, then $\widehat{\mathcal{S}}$ is the unique integrable Pfaffian system of rank $k+1$ that contains $\mathcal{S}$. We also observe that systems for which $S_1 = 0$ and $\widehat{S}$ is not integrable behave quite differently ([2]).

We now turn our attention to general k-flags of length ℓ and assume, as before, that $\mathcal{S}_\ell = 0$ whereupon $\mathrm{rank}\,\mathcal{S} = k\,\ell$ and $\mathrm{rank}\,\mathcal{S}_\nu = k\,(\ell - \nu)$, $1 \le \nu \le \ell$.

Theorem 1. *Let $\mathcal{S}$ be a k-flag of length ℓ that satisfies the properties* (*normality conditions*)

(nc_1) $\widehat{\mathcal{S}_{\ell-1}}$ *is integrable,*

(nc_2) $\mathcal{S}_{\nu-1} \cap \widehat{\mathcal{S}_\nu} = \mathcal{S}_\nu$, $1 \le \nu \le \ell - 1$.

Then $\mathcal{S}$ admits, in a neighborhood of each point of M, the normal form

$$\omega^i_j = dx^i_j + x^{i+1}_j\,dt, \qquad 1 \le i \le \ell, \quad 1 \le j \le k, \tag{10}$$

where the derived systems $\mathcal{S}_\nu$ are generated by the forms ω^i_j, $1 \le i \le \ell - \nu$, $1 \le j \le k$. Each covariant systems $\widehat{\mathcal{S}_{\nu-1}}$, $1 \le \nu \le \ell - 1$, is an integrable system of $\mathrm{rank}\,k\,(\ell - \nu + 1) + 1$ *and is equal to $\mathcal{S}_{\nu-1} + \widehat{\mathcal{S}_{\ell-1}}$. The system $\mathcal{S}$ is transitive and its class is equal to $k\,(\ell + 1) + 1$.*

Proof. The proof is by induction on the length ℓ of the flag. The case $\ell = 1$ is considered in the Lemma 2 and the induction step is proved as follows. We take a k-flag $\mathcal{S}$ of length $\ell + 1$ that satisfies the corresponding normality conditions, denote by δ its Martinet tensor and assume that $\mathcal{S}_1$ admits the local model (10) in a neighborhood of a point x_0. The condition (tf_1) for the system $\mathcal{S}$ and for $\nu = 1$, namely the condition $\mathcal{S} \cap \widehat{\mathcal{S}_1} = \mathcal{S}_1$, is equivalent at the point x_0 to $dt_{x_0} \notin \mathcal{S}_{x_0}$ *i.e.*, $[dt]_{x_0} \neq 0$ in the space $T^*_{x_0}/\mathcal{S}_{x_0}$. Since $\delta(\omega^\ell_j) = [dx^{\ell+1}_j] \wedge [dt] \in \wedge^2(T^*/\mathcal{S})$ vanishes for all the indices j, we can write, in the space $T^*/\mathcal{S}$ and in a neighborhood of x_0, $[dx^{\ell+1}_j] = -\alpha_j\,[dt]$ and consequently the forms $\omega^{\ell+1}_j = dx^{\ell+1}_j + \alpha_j\,dt$ belong to $\mathcal{S}$. Replacing x^ℓ_j by $y^\ell_j = x^\ell_j - \frac{1}{2}\,c_j\,t^2$, $x^{\ell+1}_j$ by $y^{\ell+1}_j = x^{\ell+1}_j + c_j\,t$ and setting $y^{\ell+2}_j = \alpha_j - c_j$, with $c^i = \alpha^i(x_0)$, we find that $\omega^\ell_j = dy^\ell_j + y^{\ell+1}_j\,dt$ and $\omega^{\ell+1}_j = dy^{\ell+1}_j + y^{\ell+2}_j\,dt$. Much in the same way, replacing $x^{\ell-1}_j$ by $y^{\ell-1}_j = x^{\ell-1}_j - \frac{1}{6}\,c_j\,t^3$, we find that $\omega^{\ell-1}_j = dy^{\ell-1}_j + y^\ell_j\,dt$. Continuing this

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process and replacing $x_j^{\ell-\nu}$ by $y_j^{\ell-\nu} = x_j^{\ell-\nu} - \frac{1}{(\nu+2)!}\, c_j\, t^{\nu+2}$, we find that $\omega_j^{\ell-\nu} = dy_j^{\ell-\nu} + y_j^{\ell-\nu+1}\, dt$ hence we can express all the forms $\{\omega_j^i\,,\ 1 \leq i \leq \ell+1\}$ in terms of the functions $\{t\,, y_j^r\,,\ 1 \leq r \leq \ell+2\}$ obtaining thus the model (10) for the k-flag $\mathcal{S}$ since all the above forms are independent. We finally show that the functions $\{t\,, y_j^r\,,\ 1 \leq r \leq \ell+2\}$ are independent, this resulting from the injectivity of δ on the sub-space generated by the forms $\omega_j^{\ell+1}$ at the point x_0. To see this, we consider the sub-space $\mathcal{T}^* \subset T_{x_0}^*$ generated by the differentials $\{dt\,, dy_j^r\,,\ 1 \leq r \leq \ell+2\}_{x_0}$ and observe that

$$\delta\,((\omega_j^{\ell+1})_{x_0}) \in (\mathcal{T}^*/\mathcal{S}_{x_0}) \wedge [dt]_{x_0} \subset \wedge^2\,(T_{x_0}^*/\mathcal{S}_{x_0})\,.$$

If $\dim \mathcal{T}^* \leq k(\ell+2)$, then $\dim \mathcal{T}^*/\mathcal{S}_{x_0} \leq k$ and, since $[dt]_{x_0} \neq 0$,

$$\dim(\mathcal{T}^*/\mathcal{S}_{x_0}) \wedge [dt]_{x_0} \leq k-1\,,$$

whereafter the forms $\{\delta(\omega_j^{\ell+1})\}$ become linearly dependent at the point x_0, which is not the case. The remaining assertions are immediate consequences of the canonical expressions (10) hence the proof is complete.

Remark. The systems $\widehat{\mathcal{S}_\nu}$, $0 \leq \nu \leq \ell-1$, are the unique integrable Pfaffian systems of ranks $k\,(\ell-\nu)+1$ that contain respectively the systems $\mathcal{S}_\nu$.

COROLLARY 1. *Let $\mathcal{S}$ be a k-flag of length ℓ, terminating by the null system. Then the following assertions are equivalent*:

(i) $\mathcal{S}$ *admits locally the normal form* (10),

(ii) $\mathcal{S}$ *verifies the normality conditions of the theorem,*

(iii) $\mathcal{S}$ *is transitive and* $\widehat{\mathcal{S}_{\ell-1}}$ *is integrable.*

Proof. The normal form obviously verifies the normality conditions. Furthermore, if $\mathcal{S}$ is transitive and verifies the condition (nc_1) of the theorem then it also verifies the condition (nc_2). In fact, if this latter condition failed, taking the maximum value of the integer ν, say ν_1, for which it fails, the k-flag $\mathcal{S}_{\nu_1}$ would still verify the normality conditions and consequently the covariant system $\widehat{\mathcal{S}_{\nu_1}}$ would be integrable. On the other hand, there would be a point x_1 such that $(\widehat{\mathcal{S}_{\nu_1}})_{x_1} \subset (\mathcal{S}_{\nu_1-1})_{x_1}$, the same inclusion holding at every other point due to transitivity. Consequently, the integrable system $\widehat{\mathcal{S}_{\nu_1}}$ would be contained in $\mathcal{S}_{\nu_1-1}$ and the rank of $\mathcal{S}_{\nu_1}$ would be at least equal to $k\,(\ell-\nu)+1$ which is not the case.

Remark. In general, without any transitivity hypothesis, the condition (nc_2) can fail at most on a closed subset with void interior and it cannot be withdrawn from the normality conditions as evidenced by the non-transitive 2-flag $\mathcal{S}$ of length 2

$$\begin{aligned} &\omega^1 = dx^1 + x^2\,dt\,, && \varpi^1 = dy^1 + y^2\,dt\,, \\ &\omega^2 = dt + x^3\,dx^2\,, && \varpi^2 = dy^2 + y^3\,dx^2\,. \end{aligned} \tag{11}$$

Along the hyperplane $\{\,x^3 = 0\,\}$, $\mathcal{S} \cap \widehat{\mathcal{S}_1} = \widehat{\mathcal{S}_1}$ and $\widehat{\mathcal{S}} \neq \mathcal{S} + \widehat{\mathcal{S}_1}$. The system (11) cannot be equivalent to the corresponding transitive model (10) and it can be shown that any 2-flag of length 2 terminating by the null system and satisfying the condition (nc_1) is locally equivalent either to the above model or to the transitive model (10). The condition (nc_1) cannot be dropped either since k-flags can be transitive without admitting the normal form (10). The system

$$dx^1 + x^2\,dx^3\,, \qquad\qquad dx^4 + x^5\,dx^6\,,$$

is a transitive 2-flag of class 6 equal to its covariant system (the corresponding normal form has class 5) and

$$dx^1 + (x^3 + \frac{1}{2}\,x^4\,x^5)\,dx^4\,, \qquad dx^2 + (x^3 - \frac{1}{2}\,x^4\,x^5)\,dx^5\,,$$

is a transitive 2-flag with a non-integrable covariant system of rank 3 ([2], [10], sect. 15). The 2-flag

$$\begin{aligned} &\omega^1 = dx^1 - x^5\,dt\,, && \omega^2 = dx^2 - x^6\,dt\,, \\ &\omega^3 = dx^3 - x^2\,dt\,, && \omega^4 = dx^4 - (x^6)^2\,dt\,, \end{aligned} \tag{12}$$

fulfills none of the two normality conditions since $\mathcal{S}_1 = \{\omega^3\,, \frac{1}{2}\,\omega^4 - x^6\,\omega^2\}$ has a non-integrable covariant system $\widehat{\mathcal{S}_1} = \{\omega^2\,, \omega^3\,, \omega^4\}$ contained in $\mathcal{S}$.

Remark. The previous discussion can be extended to k-flags that terminate by integrable systems. Taking a complete set of first integrals $\{z^1, \cdots, z^{\mu_0}\}$ of $\mathcal{S}_\ell$, we can repeat all the previous arguments incorporating the parameters z^μ much in the same way as is done in the proof of the Theorem 4.2 in [13]. Essentially, this amounts to restrict all the data to the slices $z^\mu = c^\mu$. A local model for such flags is obtained by completing (10) with the first integrals dz^μ. We observe that a direct proof of the Lemma 2 and the Theorem 1, in which we carry along the first integrals dz^μ, is feasible though rather awkward.

§4. Truncated multi-flag systems

In this section we discuss the Pfaffian systems $\mathcal{S}$ that under appropriate hypotheses admit a local model equal to the normal form (10) with some of the generating forms omitted. The basic assumptions (tf_1) in the Definition 3 states that these systems are not 1-flags and the condition (tf_4) is motivated by the Proposition 3 since it implies in particular that *class* $\mathcal{S}_{\ell-1} = 3$ when $\nu_0 = \ell - 2$, this being required by the local model. The k-flags $(k \geq 2)$ that satisfy the normality conditions of the Theorem 1 are special cases of truncated multi-flags since, for such k-flags, the condition (tf_1) is always satisfied with $\nu_0 = \ell - 1$, (tf_2) and (tf_3) are the normality conditions and the last condition is void. As in the previous section, we assume that $\mathcal{S}_\ell = 0$, this restriction being removed later. We set $\operatorname{rank}\mathcal{S} = r_\ell$, $\operatorname{rank}\mathcal{S}_{\ell-\mu} = r_\mu$, $s_\mu = r_\mu - r_{\mu-1} = \operatorname{rank}(\mathcal{S}_{\ell-\mu}/\mathcal{S}_{\ell-\mu+1})$ $(\mathcal{S}_0 = \mathcal{S},\, s_1 = r_1,\, r_0 = 0)$ and first examine the case $\nu_0 = \ell - 1$ *i.e.*, $r_1 \geq 2$.

THEOREM 2. *Let $\mathcal{S}$ be a truncated flag of length ℓ and $\nu_0 = \ell - 1$. Then $\mathcal{S}$ admits, in a neighborhood of each point of M, the normal form*

$$\omega^i_j = dx^i_j + x^{i+1}_j\, dt\,, \quad 1 \leq i \leq \ell\,, \quad 1 \leq j \leq s_i\,, \tag{13}$$

where $2 \leq s_1 \leq s_2 \leq \cdots \leq s_\ell$, $\sum s_i = \operatorname{rank}\mathcal{S}$, and where $\mathcal{S}_\nu$ is generated by the forms ω^i_j, $1 \leq i \leq \ell - \nu$. The system $\mathcal{S}$ is always transitive.

Proof. By the condition (tf_2) and the Lemma 2, we can find a system of coordinates, defined in a neighborhood of any given point x_0 (and vanishing at this point), such that $\mathcal{S}_{\ell-1}$ is locally represented by the set of linear forms

$$\omega^1_j = dx^1_j + x^2_j\, dt\,, \qquad 1 \leq j \leq s_1\,. \tag{14}$$

We next consider the system $\mathcal{S}_{\ell-2}$. The initial argument in the proof of the Theorem 1 leading towards the determination of the forms $\omega^{\ell+1}_j = dx^{\ell+1}_j + \alpha_j\, dt$ shows, in the present context and on account of the condition (tf_3), that we can determine s_1 independent linear forms ω^2_j belonging to $\mathcal{S}_{\ell-2}$ which, together with the forms (14) and upon rescaling the coordinates, yield a sub-system

$$\begin{aligned} \omega^1_j &= dx^1_j + x^2_j\, dt\,, \\ \omega^2_j &= dx^2_j + x^3_j\, dt\,, \qquad 1 \leq j \leq s_1\,, \end{aligned} \tag{15}$$

of $\mathcal{S}_{\ell-2}$. Consequently, $s_2 \geq s_1$. If $s_2 = s_1$, then $\mathcal{S}_{\ell-2}$ is an s_1-flag of length 2 that verifies the normality conditions of the Theorem 1 and (15) becomes its normal form. On the other hand, if $s_2 > s_1$, we complete the forms (15) to a local basis

$$\{\omega_j^1, \omega_j^2, \Omega^{s_1+1}, \cdots, \Omega^{s_2}\}, \quad 1 \le j \le s_1, \tag{16}$$

of $\mathcal{S}_{\ell-2}$. Since $s_1 \geq 2$, the expressions of the forms ω_j^2 in (15) show that $[dt]_{\ell-2}$ generates $\mathrm{Pol}(\mathcal{S}_{\ell-2})$ hence $\widehat{\mathcal{S}_{\ell-2}}$ is locally generated by $\mathcal{S}_{\ell-2}$ and dt. Furthermore, since dt is a first integral of $\widehat{\mathcal{S}_{\ell-2}}$ then so are the forms $\{dx_j^1, dx_j^2, 1 \le j \le s_1\}$. We now complete these first integrals by additional $s_2 - s_1$ independent first integrals $\{dy^j\}$ of $\widehat{\mathcal{S}_{\ell-2}}$ and proceed as in the proof of the Lemma 2, expressing the forms (16) as linear combinations of all the above first integrals. Since the forms ω_j^i are already written in the appropriate manner, we find expressions of the type (7) for the forms (16) with a coefficient matrix displaying, in the left upper corner, the identity matrix of order $2\,s_1$, all the remaining entries in the first $2\,s_1$ lines being null except for those in the last (r_2+1)-column. Furthermore, since the inverse of the matrix $(a_j^i)_{1\le i,j\le r_2}$ also displays, in the left upper corner, the identity matrix of order $2\,s_1$, all the remaining entries in the first $2\,s_1$ lines being null, we infer that left multiplication by this inverse matrix does not change the forms ω_j^i and brings the forms $\Omega^{s_1+\mu}$ to the appropriate expressions (8) with $dy^{k+1} = dt$. Furthermore, since δ is injective on the sub-bundle generated by the forms $\{\omega_j^2, \Omega^{s_1+\mu}, 1 \le \mu \le s_2 - s_1\}$, the new coefficients are necessarily independent from the coordinates $\{x_j^i, t\}$, $1 \le i \le 3$, $1 \le j \le s_1$, and can be chosen as additional coordinates. In conclusion, the system $\mathcal{S}_{\ell-2}$ can be represented locally by the forms

$$\begin{aligned} \omega_j^1 &= dx_j^1 + x_j^2\,dt, && 1 \le j \le s_1, \\ \omega_j^2 &= dx_j^2 + x_j^3\,dt, && 1 \le j \le s_2. \end{aligned} \tag{17}$$

Using a similar argument, we obtain by induction the normal form (13).

Next, we examine the case $\nu_0 < \ell - 1$.

LEMMA 3. *Let $\mathcal{S}$ be a truncated flag of length ℓ and $\nu_0 < \ell-1$. Then $\mathcal{S}$ contains a unique 1-flag $\mathcal{T}$ of length ℓ, $\mathcal{T}_{\nu-1} = \mathcal{S}_{\nu-1} \cap \widehat{\mathcal{T}_\nu}$, $\nu \le \nu_0 + 1$, and $\mathcal{T}_{\nu_0+1} = \mathcal{S}_{\nu_0+1}$.*

Proof. The system $\mathcal{S}_{\nu_0+1}$ is a 1-flag of length $\rho = r_\rho = \ell - \nu_0 - 1$ and, as indicated in the proof of the Proposition 3.3 in [13], it is (locally) generated by $\mathcal{S}_{\nu_0+2}$ and a Darboux class 3 differential form that we write $\omega^\rho = df + g\,dh$. By the Proposition 3, $\widehat{\mathcal{S}_{\nu_0+1}}$ is generated by $\mathcal{S}_{\nu_0+1}$, dg and dh hence $\operatorname{rank}\widehat{\mathcal{S}_{\nu_0+1}} = \rho+2$. The condition (tf_4) requires accordingly that $\operatorname{rank}\widehat{\mathcal{S}_{\nu_0+1}} \cap \mathcal{S}_{\nu_0} \leq \rho+1$. Since $\delta_{\nu_0}(\omega^\rho) = [dg]_{\nu_0} \wedge [dh]_{\nu_0} = 0$ (δ_{ν_0} is the Martinet tensor associated to $\mathcal{S}_{\nu_0}$) and since $(dg)_{x_0}$ and $(dh)_{x_0}$ cannot both belong to $\mathcal{S}_{\nu_0}$, we infer that either $[dg]_{\nu_0} = a\,[dh]_{\nu_0}$ or $[dh]_{\nu_0} = b\,[dg]_{\nu_0}$. Consequently, we can find a linear differential form $\omega^{\rho+1} = \alpha\,dg + \beta\,dh$, $(\omega^{\rho+1})_{x_0} \neq 0$, that belongs to $\mathcal{S}_{\nu_0}$, does not belong to $\mathcal{S}_{\nu_0+1}$ and such that $\{\omega^1, \cdots, \omega^\rho, \omega^{\rho+1}\}$ generates $\mathcal{T} = \widehat{\mathcal{S}_{\nu_0+1}} \cap \mathcal{S}_{\nu_0}$. The system $\mathcal{T}$ is a 1-flag of length $\rho+1$ contained in $\mathcal{S}_{\nu_0}$ that extends $\mathcal{S}_{\nu_0+1}$. Furthermore, if $\bar{\mathcal{T}}$ is any 1-flag of length $\rho+1$ contained on $\mathcal{S}_{\nu_0}$ then $\bar{\mathcal{T}}_1 = \mathcal{S}_{\nu_0+1}$ and any generator $\bar{\omega}^{\rho+1} \in \bar{\mathcal{T}} - \mathcal{S}_{\nu_0+1}$ is of the form

$$\bar{\omega}^{\rho+1} \equiv A\,dg + B\,dh \mod \mathcal{S}_{\nu_0+1}\,.$$

We can therefore replace $\bar{\omega}^{\rho+1}$ by $A\,dx^{\rho+1} + B\,d\psi$. However, the forms $\bar{\omega}^{\rho+1}_{x_0}$ and $\omega^{\rho+1}_{x_0}$ are linearly dependent otherwise left multiplication by the inverse of the matrix with coefficients α, β, A and B would show that $(dg)_{x_0}, (dh)_{x_0} \in (\mathcal{S}_{\nu_0})_{x_0}$, thus contradicting (tf_4). In conclusion, the 1-flag $\mathcal{T}$ of length $\rho+1$ contained in $\mathcal{S}_{\nu_0}$ is unique and extends $\mathcal{S}_{\nu_0+1}$. Let us next assume that $\mathcal{T}$ is the unique 1-flag of length $\rho = \ell - \nu$ contained in $\mathcal{S}_\nu$, $\nu \leq \nu_0$, and let us take a class 3 generator $\omega^\rho = df + g\,dh \in \mathcal{T} - \mathcal{T}_1$. Then $\widehat{\mathcal{T}}$ is generated by $\mathcal{T}$, dg and dh and we have to show that $(dg)_{x_0}$ and $(dh)_{x_0}$ cannot both belong to $\mathcal{S}_{\nu-1}$ or, equivalently, that $\operatorname{rank}\widehat{\mathcal{T}} \cap \mathcal{S}_{\nu-1} \leq \rho+1$. If $\widehat{\mathcal{S}_\nu}$ is generated by $\mathcal{S}_\nu$ and an additional form ϖ, since $[\varpi]_\nu \wedge \delta_\nu(\omega^\rho) = [\varpi]_\nu \wedge [dg]_\nu \wedge [dh]_\nu = 0$, we infer that

$$\varpi \equiv a\,dg + b\,dh \mod \mathcal{S}_\nu\,,$$

hence we can replace ϖ by $a\,dg + b\,dh$. If both $(dg)_{x_0}$ and $(dh)_{x_0}$ belonged to $\mathcal{S}_{\nu-1}$ then $(\widehat{\mathcal{S}_\nu})_{x_0}$ would be contained in $\mathcal{S}_{\nu-1}$ thus contradicting (tf_3). Using exactly the same arguments as before, we can extent $\mathcal{T}$ to a 1-flag $\mathcal{U}$ of length $\rho+1$ contained in $\mathcal{S}_{\nu-1}$ by setting $\mathcal{U} = \widehat{\mathcal{T}} \cap \mathcal{S}_{\nu-1}$.

The canonical 1-flag $\mathcal{T}$ of length ℓ is called the *spine* of $\mathcal{S}$.

THEOREM 3. *Let $\mathcal{S}$ be a truncated flag of length ℓ, with $\nu_0 < \ell - 1$, that verifies the properties (normality conditions)*

(tnc_1) $\mathcal{T}_{\nu_0}$ is transitive, where $\mathcal{T}$ is the spine of $\mathcal{S}$,

(tnc_2) $\widehat{\mathcal{S}_{\nu_0}} \supset \widehat{\mathcal{S}_{\nu_0+1}}$.

Then, in a neighborhood of each point of M, $\mathcal{S}$ admits the normal form (13), *with $1 \leq s_1 \leq s_2 \leq \cdots \leq s_\ell$.*

Proof. Since the 1-flag $\mathcal{T}_{\nu_0}$ is transitive, it admits the standard local model

$$\omega^1 = dx^1 + x^2\,dt\,,\ \omega^2 = dx^2 + x^3\,dt\,,\ \cdots\,,\ \omega^{\rho+1} = dx^{\rho+1} + x^{\rho+2}\,dt\,, \tag{18}$$

where $\rho+1 = \ell-\nu_0$. Observing that $\widehat{\mathcal{S}_{\nu_0+1}} = \widehat{(\mathcal{T}_1)}$ contains the differential dt, the condition (tnc_2) implies that $\widehat{\mathcal{S}_{\nu_0}}$ also contains dt. Furthermore, if $(dt)_{x_1} \in (\mathcal{S}_{\nu_0})_{x_1}$, at some point x_1, then $(dx^{\rho+1})_{x_1} \in (\mathcal{S}_{\nu_0})_{x_1}$ thus contradicting the condition (tf_4). Let us now apply the condition (tf_2). The covariant system $\widehat{\mathcal{S}_{\nu_0}}$ is generated by $\mathcal{S}_{\nu_0}$ and dt hence all the differentials dx^i, $1 \le i \le \rho+1$, in the expressions (18) are also first integrals of $\widehat{\mathcal{S}_{\nu_0}}$. We can now argue as in the proof of the Theorem 2 and, setting $y^{k+1} = dt$ ($k = \rho + s_{\rho+1} = \operatorname{rank}\mathcal{S}_{\nu_0}$), complete the set (18) to a system of generators

$$\{\omega^1, \cdots, \omega^\rho, \varpi^1, \cdots, \varpi^{s_{\rho+1}}\} \tag{19}$$

of $\mathcal{S}_{\nu_0}$, where

$$\varpi^1 = dx^{\rho+1}+x^{\rho+2}\,dt\,,\ \varpi^2 = dy^3+y^4\,dt\,,\ \cdots\,,\ \varpi^{s_{\rho+1}} = dy^{2s_{\rho+1}-1}+y^{2s_{\rho+1}}\,dt\,,$$

this being a normal form for $\mathcal{S}_{\nu_0}$. Let us now assume that $\mathcal{S}_\nu$, $\nu \le \nu_0$, has locally the normal form (13) and let us consider $\mathcal{S}_{\nu-1}$. Since $\operatorname{rank}\mathcal{S}_\nu/\mathcal{S}_{\nu+1} = s_{\rho+\nu_0-\nu+1}$ and $\operatorname{rank}\mathcal{S}_{\nu-1}/\mathcal{S}_\nu = s_{\rho+\nu_0-\nu+2}$, we can use the condition (tf_3) and argue as in the case $\nu_0 = \ell-1$ (or as in the proof of the Theorem 1) so as to obtain $s_{\rho+\nu_0-\nu+1}$ linear forms having expressions of the type $dy^i+\alpha_j\,dt$. Finally, using the condition (tf_2), we can determine, as previously, $s_{\rho+\nu_0-\nu+2} - s_{\rho+\nu_0-\nu+1}$ additional forms having expressions of the type (8) in terms of the first integrals of $\widehat{\mathcal{S}_{\nu-1}}$, with $dy^{k+1} = dt$, and consequently obtain, after a suitable re-scaling of the coordinates, the normal form (13) for the system $\mathcal{S}_{\nu-1}$.

Corollary 2. *Let $\mathcal{S}$ be a truncated flag of length ℓ with $\nu_0 < \ell-1$. Then the following assertions are equivalent:*

(i) *$\mathcal{S}$ admits locally the normal form* (13) *with $1 \le s_1 \le s_2 \le \cdots \le s_\ell$,*
(ii) *$\mathcal{S}$ verifies the normality conditions of the theorem,*
(iii) *$\mathcal{S}$ is transitive and the condition (tnc_2) holds.*

Proof. If $\mathcal{S}$ is transitive then so is its spine $\mathcal{T}$ as well as $\mathcal{T}_{\nu_0}$. If $\mathcal{S}$ verifies (i) then it is transitive, the integers s_i being determined by the ranks of the derived systems $\mathcal{S}_\nu$.

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Remark. The previous discussion can be extended to truncated flags that terminate by integrable systems. Taking a complete set of first integral $\{dz^\mu\}$ of $\mathcal{S}_\ell$, the local models for such flags are obtained by adding to (13) the above first integrals.

Examples and counter-examples. The system

$$\begin{aligned} &\omega^1 = dx^1 + x^2\,dt\,, \\ &\omega^2 = dx^2 + x^3\,dt\,, \qquad\qquad \varpi^1 = dy^1 + y^2\,dx^3\,, \end{aligned} \tag{20}$$

is a transitive truncated 2-flag of length 2 that has a transitive spine and does not fulfill the condition (tnc_2) hence cannot be locally equivalent to the normal form (13) where ϖ^1 is replaced by $dy^1 + y^2\,dt$. It is noteworthy that the usual numerical invariants such as the ranks of the reduced tensors or the small growth vectors, proper to the theory of 1-flags, do not distinguish (13) from (20). Given an arbitrary point $p \in \mathbf{R}^6$ and setting $x^i = \bar{x}^i + x^i(p) = \bar{x}^i + c^i$, $y^j = \bar{y}^{\,j} + y^j(p) = \bar{y}^{\,j} + \gamma^j$ and $t = \bar{t} + t(p)$, the forms (20) re-write by

$$\begin{aligned} &\omega^1 = d(\bar{x}^{\,1} + c^2\,\bar{t} - \frac{1}{2}\,c^3\,(\bar{t}\,)^2) + (\bar{x}^{\,2} + c^3\,\bar{t}\,)d\bar{t}\,, \\ &\omega^2 = d(\bar{x}^{\,2} + c^3\,\bar{t}\,) + \bar{x}^{\,3}\,d\bar{t}\,, \qquad \varpi^1 = d(\bar{y}^1 + \gamma^2\,\bar{x}^{\,3}) + \bar{y}^{\,2}\,d\bar{x}^{\,3}\,. \end{aligned}$$

A re-scaling of the coordinates shows that the system (20) is transitive.

The transitive truncated flag (20) admits the two non-equivalent extensions

$$\begin{aligned} &\omega^1 = dx^1 + x^2\,dt\,, \\ &\omega^2 = dx^2 + x^3\,dt\,, && \varpi^1 = dy^1 + y^2\,dx^3\,, \\ &\omega^3 = dt + (x^4 + 1)\,dx^3\,, && \varpi^2 = dy^2 + y^3\,dx^3\,, \end{aligned}$$

$$\begin{aligned} &\omega^1 = dx^1 + x^2\,dt\,, \\ &\omega^2 = dx^2 + x^3\,dt\,, && \varpi^1 = dy^1 + y^2\,dx^3\,, \\ &\omega^3 = dt + x^4\,dx^3\,, && \varpi^2 = dy^2 + y^3\,dx^3\,. \end{aligned}$$

The first one is transitive ($x^4 \neq -1$) and the second is not, its spine being also non-transitive.

The system

$$\begin{aligned} &\omega^1 = dx^1 + x^2\,dt\,, \\ &\omega^2 = dx^2 + x^3\,dy^1\,, \qquad\qquad \varpi^1 = dt + y^2\,dy^1\,, \end{aligned}$$

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does not fulfill the condition (tf_4) whenever $x^3 = y^2 = 0$ and $\mathcal{S}_1 = \{\omega^1\}$ does not extend to a 1-flag of length 2 contained in S. In fact, if $\{\omega^1, \bar{\omega}^2\}$ were such a 1-flag, then the condition $d\omega^1 \wedge \omega^1 \wedge \bar{\omega}^2 = 0$ would imply that $\bar{\omega}^2 \equiv A\,dx^2 + B\,dt \mod (\omega^1)$ and therefore we could replace $\bar{\omega}^2$ by $A\,dx^2 + B\,dt$. However, the latter form belongs to $\mathcal{S}$ and a simple calculation shows that $A = B = 0$ whenever $x^3 = y^2 = 0$ hence this form cannot be a (free) generator.

The systems

$$\begin{aligned} \omega^1 &= dx^1 + x^2\,dt\,, & & \\ \omega^2 &= dx^2 + x^3\,dt\,, & \varpi^1 &= dy^1 + y^2\,dt\,, \\ \omega^3 &= dt + x^4\,dx^3\,, & \varpi^2 &= dy^2 + y^3\,dx^3\,, \end{aligned}$$

$$\begin{aligned} \omega^1 &= dx^1 + x^2\,dt\,, & & \\ \omega^2 &= dx^2 + x^3\,dt\,, & \varpi^1 &= dy^1 + y^2\,dt\,, \\ \omega^3 &= dx^3 + x^4\,dy^2\,, & \varpi^2 &= dt + y^3\,dy^2\,, \end{aligned}$$

do not fulfill the condition (tf_3). The first one has a non-transitive spine whereas the second does not have a spine since $(\widehat{\mathcal{T}})_x \subset (\mathcal{S})_x$ whenever $x^4 = y^3 = 0$, $\mathcal{T} = \{\omega^1, \omega^2\}$.

§5. Monge systems

A Monge system of ordinary differential equations (Monge equations for short) is an under-determined first order system of the form

$$F^i(x, y^1, \cdots, y^s, y^1_1, \cdots, y^s_1) = 0\,, \quad 1 \leq i \leq s - \sigma\,, \quad y^\lambda_1 = \frac{dy^\lambda}{dx}\,. \tag{21}$$

The integer σ indicates the degree of freedom (indetermination) of the system and of course $\sigma = 0$ means that the system is determined. We assume the following regularity condition:

$$\operatorname{rank} \frac{\partial\,(F^1, \cdots, F^{s-\sigma})}{\partial\,(y^1_1, \cdots, y^s_1)} = s - \sigma\,. \tag{22}$$

In terms of the space $J^1(\pi)$ of 1-jets of local sections of a fibration $\pi : P \longrightarrow \mathbf{R}$, a Monge system can be defined as a sub-manifold $\mathcal{R} \subset J^1(\pi)$ for which the target map $\beta : \mathcal{R} \longrightarrow P$ is a submersion. For simplicity, we assume that $\mathcal{R}$ is regularly embedded and that β is also surjective. The condition (22)

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translates precisely the rank maximality of β and σ is equal to the dimension of the β-fibre. Equivalently, $\operatorname{codim}\mathcal{R} = \dim P - (\sigma + 1)$. We further assume that $\dim \mathcal{R} < \dim J^1(\pi)$ since not much can be said about a trivial equation *i.e.*, an open subset of $J^1(\pi)$ (the set of equations (21) is empty). Taking adapted coordinates $(x, y^1, \cdots, y^s)$ in P (x is a coordinate in $\mathbf{R}$) and the corresponding coordinates $(x, y^1, \cdots, y^s, y^1_1, \cdots, y^s_1)$ in $J^1(\pi)$, the canonical contact structure $S_{s,1}$ on $J^1(\pi)$ is (locally) generated by the forms

$$\{dy^1 - y^1_1\, dx\,, \cdots, dy^s - y^s_1\, dx\}\,,$$

and the canonical Pfaffian system $\mathcal{S}$ associated to $\mathcal{R}$ *i.e.*, the restricted system $\mathcal{S} = S_{s,1}|\mathcal{R}$ is (locally) generated by the forms

$$\{\omega^1 = dy^1 - Y^1\, dx\,, \cdots, \omega^s = dy^s - Y^s\, dx\}\,, \tag{23}$$

where the coefficients Y^i are the restrictions, to $\mathcal{R}$, of the coordinate functions (first derivatives) y^i_1 and the functions x and y^i are as well the restrictions of the coordinate functions with the same label. Since the restricted target map $\beta : \mathcal{R} \longrightarrow P$ is a submersion, the above restricted functions $\{x, y^i\}$ as well as the linear forms (23) are independent and $\operatorname{rank}\mathcal{S} = s$.

We observe that $d\omega^i = dx \wedge dY^i$ hence $[dx] \wedge \delta(\omega) = 0\,, \quad \forall\, \omega \in \mathcal{S}$, and consequently $[dx] \in \operatorname{Pol}(\mathcal{S})$, with $[dx] \neq 0$ since $dx \notin \mathcal{S}$. However, this property does not hold, in general, for the derived systems $\mathcal{S}_\nu$.

If we represent, locally, the Monge equation $\mathcal{R}$ by (21), then:

Lemma 4. *The linear forms $\{\pi^1, \cdots, \pi^{s-\sigma}\}$ defined by*

$$\pi^j = \sum_{1\le i\le s} \Big(\frac{\partial F^j}{\partial y^i_1}\,|\,\mathcal{R}\Big)\, \omega^i\,, \qquad 1 \le j \le s-\sigma\,, \tag{24}$$

are independent and belong to the derived system $\mathcal{S}_1$.

Proof. Let us consider the forms

$$\bar{\pi}^j = \sum_{1\le i\le s} \frac{\partial F^j}{\partial y^i_1}\,(dy^i - y^i_1\, dx)\,, \qquad 1 \le j \le s-\sigma\,,$$

defined on an open set of $J^1(\pi)$. Then

$$d\bar{\pi}^j \equiv \sum_{1\le i\le s} \frac{\partial F^j}{\partial y^i_1}\, d(dy^i - y^i_1\, dx) \qquad \bmod\ S_{s,1}$$

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and the right hand side re-writes by

$$\sum_{1\le i\le s} -\frac{\partial F^j}{\partial y_1^i}\, dy_1^i \wedge dx = -dF^j \wedge dx + \sum_{1\le i\le s} \frac{\partial F^j}{\partial y^i}\, dy^i \wedge dx$$
$$= -dF^j \wedge dx + \sum_{1\le i\le s} \frac{\partial F^j}{\partial y^i}\,(dy^i - y_1^i\, dx) \wedge dx\,.$$

Since $\bar{\pi}^j|\mathcal{R} = \pi^j$, $(dy^i - y_1^i)|\mathcal{R} = \omega^i$ and $dF^j|\mathcal{R} = 0$, we infer that $d\pi^j \equiv 0 \mod \mathcal{S}$ as desired. The forms $\{\pi^j\}$ are independent due to (22).

For Monge equations with one degree of indetermination $\operatorname{rank} \mathcal{S}_1 = \operatorname{rank} \mathcal{S} - 1$ (cf. [12]). In general, the following proposition holds, where $\mathcal{S}$ is the Pfaffian system associated to a Monge equation $\mathcal{R}$ with s dependent variables and σ degrees of freedom.

Proposition 4. *$\operatorname{rank} \mathcal{S}_1 = s - \sigma$ and $\widehat{\mathcal{S}}$ is integrable.*

Proof. When $s = 1$, an arbitrary system $\mathcal{R}$ is either *trivial* (*i.e.*, no equation and $\sigma = 1$) or *determined* (one equation and $\sigma = 0$) and the case $\dim \mathcal{R} = 1$ (two equations) violates the requirement that $\beta : \mathcal{R} \longrightarrow P$ be a submersion. We can therefore assume that $s \geq 2$. This being so, the sub-manifold $\mathcal{R}$ admits local coordinates $(x, y^1, \cdots, y^s, Y^{i_1}, \cdots, Y^{i_\sigma})$ obtained by restricting, to $\mathcal{R}$, some of the jet coordinates (x, y^j, y_1^j) of $J^1(\pi)$, $1 \le j \le s$. Permuting the indices of the dependent variables y^j, we can assume that $(x, y^1, \cdots, y^s, Y^1, \cdots, Y^\sigma)$ is a system of coordinates on the manifold $\mathcal{R}$ and then consider the sub-bundle $\mathcal{B} \subset T^*(\mathcal{R})$ generated by the contact forms

$$\omega^j = dy^j - Y^j\, dx\,, \quad 1 \le j \le \sigma\,. \tag{25}$$

We claim that the restriction of δ to $\mathcal{B}$ is injective and consequently that $\operatorname{rank} \mathcal{S}_1 \le s - \sigma$. In fact, let A_j be an array of coefficients such that

$$\delta\Big(\sum_{1\le j\le \sigma} A_j\, \omega^j\Big) = 0\,.$$

Then,

$$0 = d\Big(\sum_{1\le j\le\sigma} A_j\, \omega^j\Big) \wedge \omega^1 \wedge \cdots \wedge \omega^s = \Big(\sum_{1\le j\le\sigma} A_j\, d\omega^j\Big) \wedge \omega^1 \wedge \cdots \wedge \omega^s$$
$$= \Big(\sum_{1\le j\le\sigma} A_j\, dx \wedge dY^j\Big) \wedge dy^1 \wedge \cdots \wedge dy^s$$
$$= \sum_{1\le j\le\sigma} (-1)^s A_j\, dx \wedge dy^1 \wedge \cdots \wedge dy^s \wedge dY^j\,,$$

hence $A_j = 0\,,\ \forall\, j$. The equality holds due to the previous lemma. Applying δ to the forms (25), we infer that $\mathrm{Pol}(\mathcal{S})$ is (locally) generated by $[dx]$ and consequently that $\widehat{\mathcal{S}}$ is (locally) generated by $\{dx\,, dy^1\,, \cdots, dy^s\}$ (cf. (23)).

COROLLARY 3. *The forms* (24) *are local generators of* $\mathcal{S}_1$.

The system $\mathcal{S}_1$ can be integrable, as evidenced by the Monge system

$$\mathcal{R} = \{\, y_1^k = 0\,,\ \sigma + 1 \le k \le s\}\,.$$

Let us next inquire about conditions enabling us to express, at least locally (*i.e.*, in a neighborhood of each point $X_0 \in \mathcal{R}$), the *general solution* of a Monge system $\mathcal{R}$ with σ degrees of freedom by means of parametrized formulas of the form:

$$\tag{26} \begin{aligned} x(t) &= \varphi(t,\, f_1(t),\, f_1'(t),\, \cdots,\, f_1^{(\mu_1)}(t),\, \cdots,\, f_\sigma(t),\, f_\sigma'(t),\, \cdots, \\ &\qquad f_\sigma^{(\mu_\sigma)}(t),\, c_1,\, \cdots,\, c_{\mu_0})\,, \\ y^1(t) &= \psi^1(t,\, f_1(t),\, f_1'(t),\, \cdots,\, f_1^{(\mu_1)}(t),\, \cdots,\, f_\sigma(t),\, f_\sigma'(t),\, \cdots, \\ &\qquad f_\sigma^{(\mu_\sigma)}(t),\, c_1,\, \cdots,\, c_{\mu_0})\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y^s(t) &= \psi^s(t,\, f_1(t),\, f_1'(t),\, \cdots,\, f_1^{(\mu_1)}(t),\, \cdots,\, f_\sigma(t),\, f_\sigma'(t),\, \cdots, \\ &\qquad f_\sigma^{(\mu_\sigma)}(t),\, c_1,\, \cdots,\, c_{\mu_0})\,, \end{aligned}$$

where the functions $f_1\,, \cdots, f_\sigma$ are arbitrary functions of the parameter t, the integers $\mu_0\,, \mu_1\,, \cdots, \mu_\sigma$ are fixed though arbitrary, the functions φ and ψ^j are given functions of $\sum \mu_i + \sigma + 1$ variables and the $f_\lambda^{(h)}$ are the h-th order derivative of the functions f_λ.

DEFINITION 4. Parametrizations of the form (26) are called Monge parametrizations and equations admitting such Monge parametrizations are said to have the Monge property.

Let $\mu = max\,\{\,\mu_i\,,\, 1 \le i \le \sigma\,\}$. Then, much in the same way as in [12], we can consider the functions (26) as the scalar components of a map

$$\tag{27} G : J^\mu(\pi_1) \times \mathbf{R}^{\mu_0} \longrightarrow U \subset \mathbf{R}^{s+1}\,,$$

where $\pi_1 : \mathbf{R}^{\sigma+1} \to \mathbf{R}$ is the projection onto the first factor and $(U,\, x\,,\, y^j)$ is a coordinate patch on the manifold P adapted to the fibration $\pi : P \longrightarrow$

$\mathbf{R}$, *i.e.*, x is a coordinate in $\mathbf{R}$. Since for each choice of the functions f_λ and of the constants c_η the resulting map

$$t \longmapsto x(t) = \varphi(t, \cdots, f_j(t), \cdots, f_j^{(\mu)}(t), \cdots, c_\eta, \cdots) \tag{28}$$

represents a change of parameter on the corresponding integral curve (solution) of $\mathcal{R}$ and consequently has a non-vanishing derivative, we can extend (27) to a map

$$\mathbf{G} : J^{\mu+1}(\pi_1) \times \mathbf{R}^{\mu_0} \longrightarrow \mathcal{R} \subset J^1(\pi) \tag{29}$$

by setting

$$\mathbf{G}\,(X,\, c_\eta) = (G\,(Y,\, c_\eta),\, \partial_t\, \psi^j\,(X,\, c_\eta)/\partial_t\, \varphi\,(X,\, c_\eta))\,, \quad Y = \rho^{\mu+1,\mu} X\,, \tag{30}$$

where ∂_t denotes the total derivative with respect to the variable t. Furthermore, since any $Z \in \mathcal{R}$ is the 1-jet of a solution of $\mathcal{R}$ and since the expressions (26) provide, locally, the general solution of this equation, $\mathbf{G}$ is locally surjective (in order to avoid the rather painful explanation of what the "general solution" is, we could just assume that $\mathbf{G}$ is locally surjective. Inasmuch, we could assume that $\partial_t\, \varphi\,(X,\, c_\eta) \neq 0$). For simplicity, we assume that $\mathbf{G}$ is surjective and set $\Psi^j = \partial_t\, \psi^j / \partial_t\, \varphi$. When $\mathbf{G}$ is a submersion, the Pfaffian system $\tilde{\mathcal{S}} = \mathbf{G}^* \mathcal{S}$ inherits all the properties of $\mathcal{S}$ and, for each fixed array of constants $c = (c_\eta)$, the system $\tilde{\mathcal{S}}_c = \mathbf{G}_c^* \mathcal{S}$ is a sub-system of the canonical contact structure of $J^{\mu+1}(\pi_1)$. In fact, since the image, by $\mathbf{G}_c$, of a holonomic section of $J^{\mu+1}(\pi_1)$ is a solution of $\mathcal{R}$ and since the system $\mathcal{S}$ vanishes on these solutions, we infer that $\tilde{\mathcal{S}}_c$ vanishes on the image of every holonomic section. When $\mathcal{R}$ is represented locally by the equations (21), the components of a Monge parametrization must fulfill the equations $F^i(\varphi,\, \psi^1,\, \cdots,\, \psi^s,\, \Psi^1,\, \cdots,\, \Psi^s) = 0$, hence they fulfill a closed condition.

§6. The Cartan criterion

When the canonical Pfaffian system $\mathcal{S}$ associated to a Monge system $\mathcal{R}$ is a transitive truncated flag terminating eventually by an integrable ℓ-th derived system $\mathcal{S}_\ell$, it admits the normal form (13) to which we add a complete set of first integrals of $\mathcal{S}_\ell$ and consequently admits a parametrization of the solutions, where we can take the coordinates x_j^i as the arbitrary functions of the parameter t, subject to the restrictions indicated below:

$$
(31)\qquad
\begin{array}{ll}
x^1_j = f^1_j\,, \quad x^{1+k}_j = (-1)^k \,\dfrac{\partial^k x^1_j}{\partial t^k}\,, & 1 \le j \le s_1\,,\ 1 \le k \le \ell\,,\\[2ex]
x^2_j = f^2_j\,, \quad x^{2+k}_j = (-1)^k \,\dfrac{\partial^k x^2_j}{\partial t^k}\,, & s_1 < j \le s_2\,,\ 1 \le k \le \ell - 1\,,\\
\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
x^i_j = f^i_j\,, \quad x^{i+k}_j = (-1)^k \,\dfrac{\partial^k x^i_j}{\partial t^k}\,, & s_{i-1} < j \le s_i\,,\ 1 \le k \le \ell - i + 1\,,\\
\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
x^\ell_j = f^\ell_j, \quad x^{\ell+1}_j = (-1)^k \,\dfrac{\partial x^\ell_j}{\partial t}\,, & s_{\ell-1} < j \le s_\ell\,,\\
z^\mu = c^\mu\,, & 1 \le \mu \le \mu_0\,.
\end{array}
$$

Let us denote by $\mathcal{F}_j$, $1 \le j \le s_\ell$, each 1-flag branch of the normal form and let ℓ_j be its length. Apart the canonical 1-flag $\mathcal{T}$ of maximum length, when $\nu_0 < \ell - 1$, the 1-flag sub-systems $\mathcal{F}_j \subset \mathcal{S}$ do not have an intrinsic meaning since they are tied up with the coordinate expressions of the model. However, the integers ℓ_j are invariants of $\mathcal{S}$ and are determined by the ranks of the successive derived systems $\mathcal{S}_\nu$. Each flag $\mathcal{F}_j$ displays, besides the variable t, $\ell_j + 1$ coordinates x^i_j and, according to the Proposition 4, $s_\ell = \sigma$. Consequently, the total number of coordinates in the normal form expressions completed by the first integrals of $\mathcal{S}_\ell$ is equal to

$$
\sum_{1\le j\le\sigma} \ell_j + \mu_0 + \sigma + 1 = s + \sigma + 1 = \dim \mathcal{R}\,,
$$

hence the parametrization (31) does take care of all the coordinates of $\mathcal{R}$. The number of arbitrary functions is equal to

$$
s_1 + (s_2 - s_1) + \cdots + (s_i - s_{i-1}) + \cdots + (s_\ell - s_{\ell-1}) = s_\ell = \sigma\,,
$$

as desired. Returning to the original coordinates, we find a Monge parametrization of the form (26) and so $\mathcal{R}$ has the Monge property.

Definition 5. A Monge equation $\mathcal{R}$ is said to be transitive when the pseudogroup of all the local diffeomorphism of P that (after prolongation to $J^1(\pi)$) leave invariant the sub-manifold $\mathcal{R}$, operates transitively on $\mathcal{R}$.

We need not consider first order contact transformations of $J^1(\pi)$ since ($s \ge 2$) any such transformation is, locally, the prolongation of a base space local diffeomorphism ([14]). The transitivity of $\mathcal{R}$ is equivalent to the transitivity of the associated Pfaffian system $\mathcal{S}$.

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THEOREM 4. *Let* $\mathcal{R}$ *be a Monge equation whose associated Pfaffian system* $\mathcal{S}$ *is a truncated flag of length* ℓ.

(i) *If* $\nu_0 = \ell - 1$, *then* $\mathcal{R}$ *has always the Monge property.*

(ii) *If* $\nu_0 < \ell - 1$ *and if* $\mathcal{S}$ *verifies* (tnc_1) *and* (tnc_2), *then* $\mathcal{R}$ *has also the Monge property.*

In both cases $\mathcal{R}$ *is transitive.*

In particular, if $\mathcal{R}$ is transitive and if $\mathcal{S}$ verifies (tnc_2) then $\mathcal{R}$ has the Monge property since $\mathcal{S}$ and its spine also become transitive.

COROLLARY 4. *Let* $\mathcal{R}$ *be a transitive Monge equation whose associated Pfaffian system* $\mathcal{S}$ *is a k-flag of length* ℓ. *If* $\widehat{\mathcal{S}_{\ell-1}}$ *is integrable then* $\mathcal{R}$ *has the Monge property.*

Remark. According to the standard definitions, a solution of any differential equation is always transversal to the fibres of the source map, which is not the case above since we are dealing with the set of all the integral curves of the Pfaffian system $\mathcal{S}$. We could straighten up this imprecision by restricting the arbitrary functions $f_1, \cdots, f_\sigma$ to those yielding transversal curves but this is a rather inconvenient though calculable option. The best attitude seems to simply accept this slight misdemeanor.

SOME MORE EXAMPLES AND COUNTER-EXAMPLES. We now take the fibration $\pi : \mathbf{R}^4 \longrightarrow \mathbf{R}$ $(s = 3)$ and consider Monge equations with two degrees of freedom $(\sigma = 2)$. The 7-dimensional jet space $J^1(\pi)$ has local coordinates $\{x, y^1, y^2, y^3, y_1^1, y_1^2, y_1^3\}$, $\dim \mathcal{R} = 6$ and we assume (as in the proof of the Proposition 4) that the restrictions to $\mathcal{R}$ of the jet space coordinates $\{x, y^1, y^2, y^3, y_1^1, y_1^2\}$ are local coordinates denoted by $\{x, y^1, y^2, y^3, Y^1, Y^2\}$, the restriction of y_1^3 being indicated by U. In this context, $\operatorname{rank} \mathcal{S} = 3$, a local basis is given by

$$(32) \quad \{\omega^1 = dy^1 - Y^1\,dx\,, \quad \omega^2 = dy^2 - Y^2\,dx\,, \quad \omega^3 = dy^3 - U\,dx\}\,,$$

$\operatorname{rank} \mathcal{S}_1 = 1$ and $\operatorname{rank} T^*(\mathcal{R})/\mathcal{S} = \operatorname{rank} \wedge^2(T^*(\mathcal{R})/\mathcal{S}) = 3$. If $\mathcal{S}_1$ is integrable then $\mathcal{S}$ is a 2-flag of length 1 terminating by the integrable system $\mathcal{S}_1$ and, since $\widehat{\mathcal{S}}$ is integrable, it admits the normal form (6) completed by a first integral dz of $\mathcal{S}_1$ namely

$$\begin{aligned} &\varpi^1 = dz\,, \\ &\omega^1 = dy^1 - Y^1\,dx\,, \qquad \omega^2 = dy^2 - Y^2\,dx\,. \end{aligned}$$

The equation $\mathcal{R}$ has the Monge property, its general integral only depending upon the two arbitrary functions $y^1 = f(x)$, $y^2 = g(x)$ and an arbitrary constant c corresponding to the integral sub-manifold $z = c$. If $\mathcal{S}_1$ is not integrable ($\mathcal{S}_2 = 0$), the Corollary 3 provides the generator

$$\pi = -\frac{\partial U}{\partial Y^1}\,\omega^1 - \frac{\partial U}{\partial Y^2}\,\omega^2 + \omega^3 \tag{33}$$

of $\mathcal{S}_1$ hence we can represent the system $\mathcal{S}$ by

$$\begin{aligned} &\pi \\ &\omega^1 = dy^1 - Y^1\,dx\,, \qquad \omega^2 = dy^2 - Y^2\,dx\,, \end{aligned} \tag{34}$$

this representation not being, in general, a normal form. Nevertheless, since $\operatorname{rank}\mathcal{S}_1 = 1$, the nature of $\mathcal{S}_1$ is determined by its (Cartan) class that is equal to the (Darboux) class of π when the latter is odd. This fact enables us to study, in more detail, Monge equations of the above special type. For example, it can be shown that the equation $\mathcal{R}$ has the Monge property whenever $\pi = c_1\,\omega^1 + c_2\,\omega^2 + \omega^3$, the c_i being constant. However, a full classification of such equations is still beyond reach.

The first remark is that a Monge equation can have several apparently unrelated Monge parametrizations as evidenced by the equation $\mathcal{R}$ defined by $y_1^3 - y^2 = 0$. For this equation, $U = y^2$ and $\mathcal{S}$ is represented by

$$\begin{aligned} \pi &= dy^3 - y^2\,dx\,, \\ \omega^2 &= dy^2 - Y^2\,dx\,, \qquad \omega^1 = dy^1 - Y^1\,dx\,, \end{aligned}$$

this being a transitive truncated 2-flag of length 2. An obvious parametrization is given by G_1:

$$x = t\,, \qquad y^1 = z^2\,, \qquad y^2 = z_1^1\,, \qquad y^3 = z^1\,,$$

defined on $J^1(\pi_1)$, in which case

$$\mathbf{G}_1^*(\pi) = dz^1 - z_1^1\,dt\,, \quad \mathbf{G}_1^*(\omega^2) = dz_1^1 - z_2^1\,dt\,, \quad \mathbf{G}_1^*(\omega^1) = dz^2 - z_1^2\,dt\,,$$

where $(t,\, z^1,\, z^2,\, z_1^1,\, z_1^2,\, z_2^1,\, z_2^2)$ are the coordinates of $J^2(\pi_1)$. A less obvious parametrization is given by G_2:

$$x = z^2\,, \qquad y^1 = z^1\,, \qquad y^2 = z_2^2 \quad \text{and} \quad y^3 = \frac{1}{2}\,(z_1^2)^2 \quad (z_1^2 \neq 0)\,,$$

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defined on $J^2(\pi_1)$ and mapping into the open subset of $\mathbf{R}^4$ where $y^3 \neq 0$. In this case,

$$\mathbf{G}_2^*(\pi) = z_1^2\, dz_1^2 - z_2^2\, dz^2\,, \quad \mathbf{G}_2^*(\omega^1) = dz^1 - \frac{z_1^1}{z_1^2}\, dz^2\,, \quad \mathbf{G}_2^*(\omega^2) = dz_2^2 - \frac{z_3^2}{z_1^2}\, dz^2\,,$$

the above contact forms being defined on $J^3(\pi_1)$. We observe that these or any other generators of $\tilde{S} = \mathbf{G}_2^* \mathcal{S}$ cannot be written with dt acting as the *independent variable* differential (*i.e.*, written under the form $df - g\, dt$) since $[dt] \notin \mathrm{Pol}(\tilde{S})$ and $[dt]_1 \notin \mathrm{Pol}(\tilde{S}_1)$.

The second remark is that a Monge equation $\mathcal{R}$ can have the Monge property even if $\mathcal{S}$ does not fulfill the condition (tnc_2), as evidenced by the transitive equation $y_1^3 - \frac{1}{2}(y_1^1)^2 = 0$. In this case, $U = \frac{1}{2}(Y^1)^2$ and $\pi = d(y^3 - y^1 Y^1 + \frac{1}{2} x (Y^1)^2) - (x Y^1 - y^1)\, dY^1$ hence *class* $\mathcal{S}_1 = 3$. Setting $Z^1 = x Y^1 - y^1$ and observing that $-\omega^1 = dZ^1 - x\, dY^1$, we obtain the local model

$$\begin{aligned} \pi^1 &= dW^1 - Z^1\, dY^1\,, \\ -\omega^1 &= dZ^1 - x\, dY^1\,, \qquad \omega^2 = dy^2 - Y^2\, dx\,, \end{aligned}$$

where $W^1 = y^3 - y^1 Y^1 + \frac{1}{2} x (Y^1)^2$. The system $\mathcal{S}$ is a transitive truncated 2-flag of length 2 that does not fulfill the condition (tnc_2) but the equation $\mathcal{R}$ admits nevertheless a Monge parametrization. In fact, taking two arbitrary functions f and g of the parameter t, with the sole restriction that $f''' \neq 0$, we can write

$$\begin{aligned} &Y^1 = t\,, && W^1 = f(t)\,, && Z^1 = f'(t)\,, \\ &x = f''(t)\,, && y^2 = g(t)\,, && Y^2 = \frac{g'(t)}{f'''(t)}\,, \end{aligned} \tag{35}$$

since then $dx = f'''(t)\, dt$ and consequently $\omega^2 = dy^2 - Y^2 f'''(t)\, dt$. Let us check the above calculations.

$$\begin{aligned} &\mathbf{G}^*(\pi^1) = dz^1 - z_1^1\, dt, \qquad\qquad \mathbf{G}^*(-\omega^1) = dz_1^1 - z_2^1\, dt\,, \\ &\mathbf{G}^*(\omega^2) = dz^2 - \frac{z_1^2}{z_3^1}\, dz_2^1 = (dz^2 - z_1^2\, dt) - \frac{z_1^2}{z_3^1}(dz_2^1 - z_3^1\, dt)\,, \end{aligned} \tag{36}$$

where the $\{t,\, z^1,\, z^2,\, z_1^1,\, z_1^2,\, z_2^1,\, z_2^2,\, z_3^1,\, z_3^2\}$ are the coordinates of $J^3(\pi_1)$ and $\pi_1 : \mathbf{R}^3 \longrightarrow \mathbf{R}$.

The third remark is that a Monge equation can have the Monge property even though its associated Pfaffian system is not a truncated flag. This is evidenced by the equation $y^3_1 - y^2 - ln\,(Y^1 + 1) = 0$ in which case $\pi = \omega^1 + (Y^1 + 1)\,\omega^3$ is a form of class 5 and consequently $\mathrm{Pol}(\mathcal{S}_1) = 0$. The Monge parametrization

$$\begin{aligned} &x = t\,, \\ &y^1 = z^1\,, \qquad y^2 = ln\,(z^1_1 + 1) + z^2_2\,, \qquad y^3 = z^2_1\,, \\ &y^1_1 = z^1_1\,, \qquad y^2_1 = (z^1_1 + 1)^{-1}\,z^1_2 + z^2_3\,, \qquad y^3_1 = z^2_2\,, \end{aligned}$$

yields

$$\mathbf{G}^*\,(\omega^1) = \varpi^1\,, \quad \mathbf{G}^*\,(\omega^2) = (z^1_1 + 1)^{-1}\,\varpi^1_1 + \varpi^2_2\,, \quad \mathbf{G}^*\,(\omega^3) = \varpi^2_1\,,$$

where the $\{t,\, z^1,\, z^2,\, z^1_1,\, z^2_1,\, z^1_2,\, z^2_2,\, z^1_3,\, z^2_3\}$ are still the coordinates of $J^3(\pi_1)$ and where the forms

$$\begin{aligned} \varpi^1 &= dz^1 - z^1_1\,dt\,, & \varpi^2 &= dz^2 - z^2_1\,dt\,, \\ \varpi^1_1 &= dz^1_1 - z^1_2\,dt\,, & \varpi^2_1 &= dz^2_1 - z^2_2\,dt\,, \\ \varpi^1_2 &= dz^1_2 - z^1_3\,dt\,, & \varpi^2_2 &= dz^2_2 - z^2_3\,dt\,, \end{aligned}$$

generate the canonical contact structure on this jet bundle. The sub-system $\tilde{S} = \mathbf{G}^*\,\mathcal{S}$ is then generated by

$$\varpi^1\,, \qquad \varpi^2_1\,, \qquad \varpi^1_1 + (z^1_1 + 1)\,\varpi^2_2 \tag{37}$$

and its derived system by $\mathbf{G}^*\,(\pi) = \varpi^1 + (z^1_1 + 1)\,\varpi^2_1$.

The last remark is that a Monge equation need not have the Monge property. This is evidenced by the equation $y^3_1 - y^1_1\,y^2_1 = 0$ in which case

$$\pi = d(y^3 - y^1\,Y^2 - y^2\,Y^1 + x\,Y^1\,Y^2) + (y^2 - x\,Y^2)\,dY^1 + (y^1 - x\,Y^1)\,dY^2$$

is a form of class 5. Setting $Z^1 = x\,Y^1 - y^1$ and $Z^2 = x\,Y^2 - y^2$, we can re-write the generators of $\mathcal{S}$ by

$$\begin{aligned} \pi &= d(y^3 - y^1\,Y^2 - y^2\,Y^1 + x\,Y^1\,Y^2) - Z^2\,dY^1 - Z^1\,dY^2\,, \\ -\omega^1 &= dZ^1 - x\,dY^1\,, \qquad\qquad -\omega^2 = dZ^2 - x\,dY^2\,. \end{aligned}$$

Replacing $(x,\, y^1,\, y^2,\, y^3,\, Y^1,\, Y^2)$ by $(\phi,\, \psi^1,\, \psi^2,\, \psi^3,\, \Psi^1,\, \Psi^2)$ and ignoring for a moment the term dy^3, a (rather long) calculation will show that the differential form $\mathbf{G}^*\,(\pi - dy^3)$ must be corrected by a non-exact term (*i.e.*, a term of the form $f\,dt$ with $df \wedge dt \neq 0$) in order to become a contact form on some jet bundle $J^{\mu+1}(\pi_1)$. However, this cannot be achieved since the only possible correction consists in adding the exact term $d\psi^3$.

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