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# L. G. KOVÁCS AND VARIETIES OF GROUPS

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Communicated by D. L. Flannery

Dedicated to the memory of Laci Kovács to whom I owe very much. He was inspiring, as supervisor, as colleague and as friend

**Abstract**

This is a short account of some of the work of L. G. (Laci) Kovács on varieties of groups.

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## 1. Introduction

The study of varieties of groups may be said to have begun with Neumann [25] in 1937, but the most intense activity (so far) is included in the years 1955–1975. Laci Kovács became involved through his time at Manchester (from 1960), which then included both Bernhard and Hanna Neumann, and continued when he followed them to Canberra in 1963 and began his long and very fruitful collaboration with Mike Newman. Varieties remained a major interest until well into the 1970s. Many of the proofs, however, made substantial use of representation theory, and one can see this interest overtaking his interest in varieties during the early 1970s. (This transition, from using an area of group theory to prove results on varieties, to working directly in the same area of group theory, can be seen in many people at that time.) Canberra was a major centre of the topic (the other two major centres being Oxford and Moscow) and Laci was a major influence on those who thought about varieties of groups.

I have identified some 28 papers of Laci's which have a clear connection with the theory of varieties. I have divided a majority of these into four threads reflecting varying interests. I should stress that this is not intended as a survey of variety theory but as a discussion of Laci's work; others who made similar contributions of major interest will mostly not be referenced. There is, however, one other whose influence on Laci's work in this period is hard to underestimate. Mike (M. F.) Newman was a very frequent collaborator with Laci; the collaboration lasted for many years, but was particularly intense in the 1960s when all four of these threads were begun. It is not possible for this author to untangle the contributions of Laci and Mike to the topics we shall discuss here. Even when Mike's name is not on the paper there may be an acknowledgement from Laci that his consideration of the topic began in discussions with Mike.

A major event in the history of variety theory was the International Conference on the Theory of Groups held in Canberra in 1965. Laci was a co-editor of the proceedings (with Bernhard Neumann) and both Laci and Mike were contributors to the conference. The papers for the first thread I will discuss were published the following year, while each of the other three threads had their origin in a paper presented at the conference. The second such conference, held in 1973, marks the evening of the period of strong activity in variety theory. Many papers were still to be published on the topic (and, I hope, many still are) but this period of intense activity was coming to a close. Laci himself was developing his already strong interest in representation theory which was eventually to replace his interest in varieties.

## 2. Varieties of groups

We are writing under the assumption that many readers will not be familiar with the concepts of varieties of groups. We therefore give a very short introduction to basics and terminology. Laci has provided a superb introduction in [13], which is an expansion of a talk given to the Annual Meeting of the Australian Mathematical Society in 1967. He also gave another excellent survey in the first section of [15]. The reader wanting a more complete account should refer to [26].

Suppose that $X_\infty$ is a free group with a (countably) infinite basis $X = \{x_1, x_2, x_3, \ldots\}$. An element $w$ of $X_\infty$ will be called a *word*. Then $w$ is a *law* of a group $G$ if, for any homomorphism $f : X_\infty \longrightarrow G$, we have $f(w) = 1$. Thus, for example, $x_1^2$ is a law of the cyclic group of order two or, indeed, any elementary abelian 2-group. And the left-normed commutator $[x_1, x_2, x_3]$ is a law of any group which is nilpotent of class at most two.

If $Y$ is any subset of $X_\infty$ then the set of all groups $G$ for which every element of $Y$ is a law of $G$ is called a *variety* (of groups). It is straightforward to check that a variety of groups is closed under the operations of taking subgroups and quotient groups and forming cartesian products. Conversely, any class of groups which is closed under these three operations will be definable by a suitable set of laws.

Given a variety $\mathfrak{V}$, the set of all laws satisfied by all groups in $\mathfrak{V}$ is a subgroup of $X_\infty$ which is invariant under the action of any endomorphism; that is, a *fully invariant* subgroup. It is called the *verbal* subgroup (of $X_\infty$) corresponding to $\mathfrak{V}$. More generally, if $G$ is any group then the verbal subgroup $\mathfrak{V}(G)$ is the set of all expressions $w(g_1, \ldots, g_n)$ where $g_i \in G$ and $w$ is a law of $\mathfrak{V}$. The quotient of any free group $F$ by the verbal subgroup $\mathfrak{V}(F)$ is a *relatively free* group.

Examples of varieties abound. The class of all abelian groups, of all nilpotent groups of class at most $c$ and of all groups having finite exponent dividing $e$ all form varieties; their standard names are $\mathfrak{A}$, $\mathfrak{N}_c$ and $\mathfrak{B}_e$. Their corresponding verbal subgroups are the derived group, the $(c + 1)$th term of the lower central series and the subgroup generated by the $e$th powers, respectively. We shall use $\mathfrak{A}_e$ for $\mathfrak{A} \cap \mathfrak{B}_e$. The class of all finite groups, or of all nilpotent groups, or of all torsion (periodic) groups do not form varieties. In fact each of these classes can belong only to the variety consisting of all groups. Given varieties $\mathfrak{U}$ and $\mathfrak{B}$, the product variety $\mathfrak{U}\mathfrak{B}$ is the collection of all groups $G$ with a normal subgroup $N$ so that $N \in \mathfrak{U}$ and $G/N \in \mathfrak{B}$.

## 3. Varieties generated by a finite group

One of the first questions that arise when considering varieties is whether the set of laws that define a variety can always be deduced from a finite subset (that is, the set of laws is *finitely based*). Roger C. Lyndon showed that this is true if the variety consists entirely of nilpotent groups. The general problem looked, and proved to be, hard. A case that looked more tractable was that when the variety is generated by a finite group. Graham Higman and his student D. C. Cross made substantial progress on this problem and the problem was eventually solved affirmatively by Oates-Williams and Powell [28].

Laci and Mike Newman provided an alternative, and much shorter, proof to that of Oates and Powell. To describe this, we need some terminology. A finite group $G$ is *critical* if it does not lie in the variety generated by the quotient groups of subgroups of $G$ (apart, of course, from the group itself). It follows quickly from this that a critical group must have a unique minimal normal subgroup; that is, it is *monolithic* (sometimes known as *subdirectly irreducible*). Not all monolithic subgroups are critical, and the papers [18, 19] are concerned with various properties of critical and monolithic groups. In particular, [19] is concerned with additional properties which force a monolithic group to be critical. This holds, for example, if the unique minimal normal subgroup contains its centralizer.

But the main result of this period is the alternative proof of the Oates–Powell theorem [17]. A *Cross* variety is a variety in which (i) every finitely generated group is finite, (ii) there are only finitely many critical groups, and (iii) the laws of the variety are finitely based. It is relatively easy to establish that subvarieties of Cross varieties are again Cross varieties and so it suffices to establish that every finite group must lie in a Cross variety. They establish a more general fact. They show that the class of all groups with specific bounds on the exponent, the order of chief factors and the class of nilpotent sections is a Cross variety. Since every finite group clearly has such bounds, the Oates–Powell theorem follows.

The key point at which their argument differs from that of Oates and Powell, and therefore allows greater brevity in the overall proof, is in the proposition that, for a finite group $G$, there is an integer $k$ so that the variety defined by the laws of $G$ which require at most $k$ variables has bounds on the order of chief factors. In order to do this they introduce a word $u_n$ so that $u_n$ is a law in any group of order less than $n$ and, conversely, if $u_n$ is a law in a finite group $G$, then the centralizer of any chief factor of

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$G$ has index less than $n$. The words $u_n$ are ingeniously chosen and very effective. (It is not possible, of course, to bound the order of a finite group with a nontrivial law.)

## 4. Varieties and Burnside's problem

The next thread we shall consider concerns locally finite varieties and connections with the restricted Burnside problem. A variety is *locally finite* if all of its groups are locally finite.

The major influence on these considerations was the seminal paper of Hall and Higman [6]. One of the main applications of the very deep results on finite groups which were proved in that paper were reduction theorems for the restricted Burnside problem. Recall that the (bounded) Burnside problem asks whether finitely generated groups with finite exponent are finite. A related problem is the restricted Burnside problem which asks whether a restriction on both exponent and number of generators of a *finite* group implies a bound on the order of the group. Laci's first published comments in this area are from 1965 [11] and it might be useful to recall the state of these problems at that time. Hall and Higman had shown in 1956 that it sufficed to prove the restricted Burnside conjecture for prime power exponent (and to establish that there are only finitely many finite simple groups of given exponent) and Kostrikin in 1959 [10] had shown that the conjecture was true for prime exponent. A claim for a proof of the existence of infinite finitely generated groups of finite exponent was made by P. S. Novikov in 1959, but details were not to appear until 1968 in a work by Novikov and Adjan [27]. In 1965 all the infinite classes of finite simple groups were known. The first sporadic simple group since those of Mathieu was announced by Z. Janko in 1965; a short paper is included in the proceedings of the 1965 conference mentioned in the Introduction.

It is clear that the Burnside conjecture for exponent $e$ asks whether the variety $\mathfrak{B}_e$ is locally finite (that is, every finitely generated group in the variety is finite). In [11] Laci observed (after a conversation with Mike Newman) that the restricted Burnside conjecture for exponent $e$ is equivalent to the claim that the locally finite groups in $\mathfrak{B}_e$ form a subvariety. (A proof appears as 15.74 of [26].) There is some discussion of the consequences of the Hall–Higman paper in [11] but little in the way of detailed proof. Most of the details of the argument appear in [13] as an appendix; this paper also includes a general discussion of varietal issues related to the Burnside problems as well as the detailed results discussed in the next paragraphs.

The main thrust of [13] is the application of the Hall–Higman reduction results to varieties of locally finite groups. Let $\mathfrak{U}$ be a locally finite variety and let $\mathfrak{W}$ be the class of all those groups of which the nilpotent sections and finitely generated simple sections all belong to $\mathfrak{U}$. Then he showed (amongst other things) that $\mathfrak{W}$ is a locally finite variety and is contained in some power of $\mathfrak{U}$. In this form, his theorem also required that there are only finitely many finite simple groups in $\mathfrak{U}$; this is a consequence of the classification theorem for finite simple groups. He also proved a similar, but somewhat weaker, form of the result which did not require the

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classification. Thus we can regard these results as a varietal form of the result of Hall and Higman quoted above.

In [13], we also find some speculation on, and some proofs of, results concerning the possibility of determining whether these locally finite varieties, or closely related ones, have a finite basis. (The question of the existence of a finite basis for the laws of an arbitrary variety was, at the time, open.) But we shall concentrate on [12] where Laci discussed the question of how many varieties there are. If every variety is finitely based then there are clearly only countably many varieties. The converse was not clear. There could, possibly, be some varieties without a finite basis for their laws but yet still only countably many varieties in total. Laci showed that if there is a variety $\mathfrak{V}$ which is not finitely based but so that, for each $k$, every subvariety of the variety generated by the $k$-generator groups is finitely based, then $\mathfrak{V}$ has uncountably many subvarieties. Recall from above, that varieties generated by finite groups (and their subvarieties) or nilpotent varieties are finitely based. These facts meant that it would suffice to find a variety which is not finitely based either locally finite or locally nilpotent in order to determine that the number of varieties is the cardinal of the continuum. Indeed, when, in 1970, Ol'šanskiĭ [29] published the first example of a variety which is not finitely based, it was locally finite and so Laci's theorem applied. (Ol'šanskiĭi, however, had produced an independent argument to establish this.)

## 5. Subvariety lattices

After making a substantial contribution (see Section 3) to the theory of varieties generated by a finite group (Cross varieties), Laci and Mike Newman turned their attention to non-Cross varieties and, in particular, to *just-non-Cross* varieties, which are not Cross but so that every proper subvariety is Cross. In [20] they raised the problem of describing all such varieties and, using work of Šmel'kin, they describe those just-non-Cross varieties which are a nontrivial product. They quote in [20] the fact that, for $p$ prime, $\mathfrak{A}_p\mathfrak{A}_p$ is just-non-Cross and refer to a paper 'in preparation' which eventually appeared as [21]. This starts a thread which we shall follow in this section. Although we make no further mention of it here, Laci and his students, in particular John Cossey and J. M. Brady, made considerable progress in the study of just-non-Cross varieties.

Commutator calculations of Gupta and Newman [5] provided the necessary basis to describe the subvarieties of $\mathfrak{A}_p\mathfrak{A}_p$. It turns out that the lattice of subvarieties is distributive so that every variety is uniquely the join of join-irreducible varieties. So we shall describe the join-irreducibles. We do so only in the case that $p$ is odd; the case $p = 2$ is very similar. For $1 < c < p$, let $\mathfrak{F}(c)$ denote the subvariety of $\mathfrak{A}_p\mathfrak{A}_p$ consisting of all groups of class at most $c$ and exponent $p$. For $c \geq p$, we define $\mathfrak{F}(c)$ similarly except that we no longer restrict the exponent. There is also, for each natural number $r \geq 1$, a join-irreducible variety $\mathfrak{F}(rp*)$ which satisfies $\mathfrak{F}(rp-1) < \mathfrak{F}(rp*) < \mathfrak{F}(rp)$. Setting $\mathfrak{F}(1) = \mathfrak{A}_{p^2}$ completes the list of join-irreducibles and the description of the subvarieties of $\mathfrak{A}_p\mathfrak{A}_p$.

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In fact, in [21] the lattice of subvarieties of $\mathfrak{A}_{p^n}\mathfrak{A}_p$ is described for general $n$. The description is similar to the case $n = 1$ but, naturally, with more complication. In [21] they also prove what they call an *external* result. A soluble variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ has a bound on the class of its $p$-groups, and hence also on the class of its nilpotent torsion-free groups. These two results provided a base for several other investigations on soluble varieties.

Laci and Mike were able to reduce the problem of describing the lattice of metabelian varieties in general to that for locally finite varieties. They showed that a proper subvariety of $\mathfrak{A}^2$ must be a join of a torsion-free variety, a variety of the form $\mathfrak{A}_n\mathfrak{A}$ and a locally finite variety. (A torsion-free variety is one *generated* by its torsion-free groups.) Further, the torsion-free subvariety is a join of varieties of the form $\mathfrak{N}_c\mathfrak{A}_m \cap \mathfrak{A}^2$. The variety which is not locally finite mentioned are uniquely determined by the original variety. This result has not been published in the normal way. It seems to have been well known at the time to those interested in the area and appears in Bryce [3] with a sketch of the proof. There is also a much later paper [22] which fills in the detail that is missing in [3].

The second extension of the results mentioned in [21] had the flavour of the 'external' result quoted above. In [1], Laci and Brisley proved a result which started from an understanding of the subvarieties of $\mathfrak{A}_p\mathfrak{A}_p$. It has a very simple statement without any mention of varieties. (At the end of the discursive section of [13], Laci quotes an example of how understanding varieties can lead to interesting results in 'ordinary' group theory. It is good to be able to quote one of Laci's own results in the same way.)

> *Let $H$ be any finite $p$-group for which the Frattini subgroup is elementary abelian. Let $G$ be a soluble $p$-group of finite exponent. If no section of $G$ is isomorphic to $H$ then $G$ is nilpotent and its class is bounded in terms of its solubility length, its exponent and $H$.*

As an example, this shows that if $G$ is regular, in the sense of Hall, then its class is bounded in terms of its exponent and solubility length. (Take $H$ to be the (nonregular) wreath product of two cyclic groups of order $p$.)

Another consequence of this result is that any section-closed class of groups which generates $\mathfrak{A}_p\mathfrak{A}_p$ must include all finite groups in $\mathfrak{A}_p\mathfrak{A}_p$. This leads to a paper with Bryant [2]. The *skeleton* of a variety is the intersection of all those section-closed classes of groups which generate the variety. Thus the skeleton of $\mathfrak{A}_p\mathfrak{A}_p$ is just the set of finite groups in $\mathfrak{A}_p\mathfrak{A}_p$. The authors solve two questions about possible behaviours similar to that of $\mathfrak{A}_p\mathfrak{A}_p$. Let $\mathfrak{V}$ be a locally finite variety and let $m$ be an integer greater than one. Then:

(1) the skeleton of $\mathfrak{A}_m\mathfrak{V}$ is the section-closure of the monolithic groups of $\mathfrak{A}_m\mathfrak{V}$; in particular, the skeleton of $\mathfrak{A}_m\mathfrak{V}$ generates $\mathfrak{A}_m\mathfrak{V}$; and

(2) the skeleton of $\mathfrak{A}_m\mathfrak{V}$ is exactly the set of finite groups in $\mathfrak{A}_m\mathfrak{V}$ if and only if $m$ is a power of a prime $p$ and the centre of the free group of infinite rank in $\mathfrak{V}$ is a $p$-group.

The proof involves substantial amounts of representation theory, perhaps indicating the move in the main focus of Laci's interests.

## 6. Classifying nilpotent varieties of small class

**6.1. Higman's method.** The final thread, my favourite amongst the four I have chosen to discuss, is inspired by a paper of Higman [7] which was presented at the 1965 International Conference on Group Theory in Canberra. Higman's paper is a wonderful, although at times difficult, work. I shall not attempt to do it full justice but shall only sketch enough that the reader may obtain some idea of the general theme. It seems necessary to lay out some background if one is to obtain a reasonable understanding of the very substantial contribution that Laci made.

Suppose that $F$ is a relatively free group of finite rank which is nilpotent of class $c$. Then a (relatively free) basis of $F$ is also a basis of $F/F'$ as free abelian group and vice versa. It follows quickly that any automorphism of $F/F'$ is induced by an automorphism of $F$. Thus the quotients of the lower central series will be modules for $\mathrm{Aut}(F/F')$, and any fully invariant subgroup which lies between two successive terms of the lower central series will correspond to a submodule for $\mathrm{Aut}(F/F')$. By examining the module structure we may therefore hope to gain insight into the nature of fully invariant subgroups and hence subvarieties of the variety generated by $F$.

Let us make substantial restrictions on $F$ so that we can be more precise, even though the fundamental ideas have application in a much wider context. Suppose that $F$ has prime exponent $p$ and its class $c$ satisfies $c < p$. In this case, the automorphism group of $F/F'$ is $\mathrm{GL}(n, p)$, where $n$ is the rank of $F$, and it is not too hard to see that there is a one-to-one correspondence between the submodules of the last term of the lower central series of $F$ (considered as $\mathrm{GL}(n, p)$-module) and fully invariant subgroups of this last term. If $n$ is sufficiently large then we will also have a bijective correspondence between subvarieties of $\mathrm{Var}(F)$ which contain $\mathrm{Var}(F) \cap \mathfrak{N}_{c-1}$ and submodules of the last term of the lower central series of $F$.

Higman displayed several consequences of this very powerful technique. Because we have assumed that $c < p$ we can use the classical representation theory of the general linear group which tells us that modules are completely reducible and that irreducibles are classified by partitions of $c$ into not more than $n$ parts. From this he deduces, for example, that such a variety is determined by its $c$-generator groups. Set $\mathfrak{B}(p, c) = \mathfrak{B}_p \cap \mathfrak{N}_c$. A formula for the character $\chi_c$ of the representation is available and from this one can determine very quickly several interesting facts about the varieties lying between $\mathfrak{B}(p, c-1)$ and $\mathfrak{B}(p, c)$:

(1) for $c \leq 3$, $\chi_c$ is irreducible and there are no varieties strictly between $\mathfrak{B}(p, c-1)$ and $\mathfrak{B}(p, c)$;

(2) for $c$=4, 5, $\chi_c$ is multiplicity-free and the number of varieties between $\mathfrak{B}(p, c-1)$ and $\mathfrak{B}(p, c)$ is independent of $p$;

(3) for $c \geq 6$ the number of varieties between $\mathfrak{B}(p, c-1)$ and $\mathfrak{B}(p, c)$ increases with $p$ and the lattice of such varieties is not distributive;

(4) the irreducible corresponding to the partition of $c$ into $c$ parts does not occur in $\chi_c$ and so the varieties are generated by their $(c-1)$-generator groups.

These results are all obtained quickly once the (admittedly complex) structure is set up and gave hope for further use of Higman's ideas. For more general nilpotent varieties the way forward was not clear, however, even when assumptions were made which allow the application of the classical theory of the general linear group.

**6.2. Using Higman's method as inspiration.** In [23], Laci *et al.* use the inspiration given by Higman's techniques to show that the variety of all nilpotent groups of class $c$ can be generated by its $(c-1)$-generator groups but not by its $(c-2)$-generator groups. They observe that the former fact follows from Higman's arguments described above and claim that the latter fact can also be deduced by Higman's method. (It is certainly suggested by the fact that the irreducible corresponding to the partition of $c$ into $c-1$ parts does always occur in $\chi_c$.) They give, however, arguments which avoid Higman's conceptual approach, although still motivated by it.

**6.3. Extending the method.** Laci's next steps were begun in conversations with Mike Newman in the years after the 1965 conference. They were also inspired and substantially influenced by two papers of Klyachko [8, 9]. They formulated and then applied the 'method' of Higman in a considerably more general setting. The basic work for this is done in a very substantial paper in 1978 [14]. It was followed the next year by [15] which accomplishes the classification of torsion-free varieties of class at most five. The paper [16], presented at a major international conference on finite simple groups, gives a light outline of [15]. It observes that a substantial part of this theory involves the ordinary representation theory of the (finite) symmetric groups and that extensions of his results could possibly be made by an appropriate use of the modular representation theory. (It was, perhaps, a sign of the times that Laci pointedly avoids using the word 'variety' until the penultimate paragraph of the paper.)

**6.4. Working out the details.** In [14], Laci isolates two results that he wants to prove; in each case, Klyachko had obtained a similar result for the lattice of nilpotent varieties of $p$-power exponent and class at most $c$.

> The lattice of varieties which are torsion-free and nilpotent of class at most $c$ is distributive (and finite) if and only if $c \leq 5$.

In the case of varieties of $p$-groups, the condition $c \leq 5$ must be replaced by similar but more complicated conditions.

> The lattice of varieties which are torsion-free and nilpotent of class at most $c$ is a (finite) subdirect product of lattices each of which is isomorphic to the subspace lattice $\mathbb{A}_{l(\pi)}$ of a rational vector space of dimension $l(\pi)$.

In the $p$-group case, we must assume that $c < p$.

We attempt to outline, very briefly, the argument of [14]. We shall refer only to torsion-free varieties, but Laci's arguments apply to both these and varieties of prime

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power exponent. It is first shown that the lattice of all torsion-free varieties of class at most $c$ is a subdirect product of the sublattice of those varieties containing *only* groups of class at most $c-1$ and the sublattice $\mathbb{L}_c^0$ of those varieties which contain *all* nilpotent groups of class $c-1$. In this way, using an inductive argument, it suffices to establish the structure only of those sublattices of the second type. The arguments for this subdirect decomposition do not use the 'Higman method' but rather some more conventional techniques in group theory and lattice theory. (But 'conventional' does not mean 'easy'!)

The next, and perhaps the central, part of the argument is to establish the replacement for the role of the general linear group in the Higman theory. Let $A_c$ denote the abelian group of elements of degree $c$ in a free (noncommutative) $\mathbb{Z}$-algebra of rank $n \geq c$. Let $L_c$ denote the subgroup of elements of degree $c$ in the free Lie algebra of rank $n$. Thus we can assume that $L_c \leq A_c$. We are seeking a lattice of submodules of $L_c$ (or $\mathbb{Q}L_c$) not under the action of the general linear group but under the action of a larger endomorphism ring which is denoted $E_c$. It thus becomes necessary to develop a theory similar to the classical theory of the general linear group but for these rings $E_c$. It is shown, as a crucial step, that $\mathbb{Q}E_c$ is the ring of $\mathbb{Q}S_c$-endomorphisms of $\mathbb{Q}A_c$ where $S_c$ is the symmetric group on $c$ letters and acts on $L_c$, in the usual way, by permuting the free generators of the overlying free Lie algebra.

A special case of Morita duality is then invoked to show that the lattice $\mathbb{L}_c^0$ is isomorphic to a direct product of lattices $\mathbb{A}_{l(\pi)}$. This reflects the $S_c$-structure of $\mathbb{Q}L_c$ and the $l(\pi)$ are recoverable as the multiplicities of irreducibles in the structure of $\mathbb{Q}L_c$ as $\mathbb{Q}S_c$-module. The symbols $\pi$ refer to partitions of $c$ which index the irreducible characters of $S_c$.

The character $\chi_c$ of the symmetric group $S_c$ on $\mathbb{Q}L_c$ was determined in the 1940s (at the latest). Because all irreducible components of $\chi_c$ have multiplicity one for $c \leq 5$ and there is an irreducible component with multiplicity greater than one for $c = 6$ the first result above follows very quickly. (In [24], Laci, together with Stöhr [24], later gave an alternative proof of a theorem of Klyachko [9] which gives some of the basic properties of these multiplicities $l(\pi)$.)

**6.5. Torsion-free varieties of small class.** In [15] the theme is continued. The paper starts with an initial discussion of torsion-free varieties in general and various problems concerning them. But the main emphasis is on filling in the detail of one major case from [14]. The aim is to give a complete description of the lattice of all nilpotent torsion-free varieties of class at most five. The pattern is set by the results of [14]. From there we can easily establish that the number of torsion-free varieties $\mathfrak{V}$ satisfying $\mathfrak{N}_{c-1} < \mathfrak{V} < \mathfrak{N}_c$ is 0 for $c \leq 3$, $2 = 2^2 - 2$ for $c = 4$ and $30 = 2^5 - 2$ for $c = 5$; adding the varieties $\mathfrak{N}_c$ themselves gives a total of 38. For example, the fact that the character of $S_5$ on $\mathbb{Q}L_5$ is the sum of five distinct irreducibles shows that the lattice of varieties between $\mathfrak{N}_4$ and $\mathfrak{N}_5$ (including these two named varieties) is the direct product of five two-element chains and so of order 32.

To complete the lattice picture, we must consider the possibility that, for some $c \leq 5$, there are torsion-free varieties which lie in $\mathfrak{N}_c$ but do not contain $\mathfrak{N}_{c-1}$. Laci shows that

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there is one such; the variety of all metabelian groups which are nilpotent of class at most five. Hence there are 39 torsion-free varieties of class at most five. (But more than 39 steps were required for this, the 39th of Laci's publications.) The fact that the character of $S_6$ on $\mathbb{Q}L_6$ has components of multiplicity greater than one shows that there are infinitely many torsion-free nilpotent varieties of class six.

The remaining task is to identify more clearly the individual varieties. Given what has already been done, it suffices to describe the five maximal varieties containing $\mathfrak{N}_4$. This is done by using detail available from [14]. The work is detailed, hard and, at least to this reader, very impressive. It yields a set of laws for each of the maximal subvarieties from which one can easily deduce laws for all 30 torsion-free varieties which lie strictly between $\mathfrak{N}_4$ and $\mathfrak{N}_5$. Thus the description of the lattice of all torsion-free subvarieties of $\mathfrak{N}_5$ is complete.

Laci poses a number of questions in [15] which I shall not repeat here. But I would like to turn a remark made in the paper into an extra question. It seems to me a natural follow-up to Laci's investigations and it has the advantage that the question can be posed essentially independently of the language of varieties. Recall that there are five maximal torsion-free subvarieties of $\mathfrak{N}_5$ corresponding to the five irreducible submodules of $L_5$ considered as $S_5$-module. For $c \geq 6$ there are infinitely many maximal submodules of $L_c$ but there are only finitely many homogeneous components (because $S_c$ has only finitely many irreducible representations). For $c > 6$ there will be one component for each partition of $c$ except for the two 'linear' partitions, one into $c$ parts and one into one part. To each of these components there is a corresponding subvariety. Hence we ask for a general version of what Laci has done for $c = 5$:

> For each 'nonlinear' partition $\pi$ of $c$ (with $c > 6$) identify the set (variety) of nilpotent groups of class $c$ which corresponds to the homogeneous component corresponding to $\pi$.

The word 'identify' is, of course, vague. One might hope for a description which made clear the role of the partition. Given, however, the substantial efforts needed by Laci to solve the case when $c = 5$, this may turn out to be exceedingly difficult.

**6.6. The fine detail for class four.** The final paper in this thread is a paper of Laci with his student Fitzpatrick [4]. It is the first of three papers, the other two are by Fitzpatrick alone, and between them they completely classify nilpotent varieties of class at most four. This paper deals with the case where the free groups of the varieties have no elements of order two so that the 'small class techniques' discussed above are immediately applicable, although some extra effort must be made for varieties of 3-power exponent. It is shown that the lattice of such varieties is distributive so that it suffices for many purposes to describe the join-irreducibles. It is shown that there are six such torsion-free irreducibles and, for each prime power $p^k$, five irreducibles of exponent $p^k$ when $p > 3$ and six when $p = 3$. This leads to a full classification.

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