11

Massive gauge fields

In the preceding chapter (Section 10.2), we set up a simple Lorentz invariant Lagrangian density, which we required to be also invariant under a local $U(1)$ transformation. This requirement leads to the introduction of a 'gauge field' A_μ. The system has a degenerate ground state. Breaking the local symmetry results in the appearance of a vector field carrying mass, together with a scalar Higgs field also carrying mass. The motivation for introducing mass in this way is that the subsequent quantum theory can be renormalised. In this chapter we apply the same idea to a more complicated Lagrangian, which will turn out to have remarkable physical significance.

11.1 *SU*(2) symmetry

As a further generalisation, which is basic to the Standard Model, we shall construct a Lagrangian density that is invariant under a local $SU(2)$ transformation as well as a local $U(1)$ transformation. The idea was first explored by Yang and Mills (1954). We introduce a two-component field

$$\Phi = \begin{pmatrix} \Phi_{\mathrm{A}} \\ \Phi_{\mathrm{B}} \end{pmatrix}, \tag{11.1}$$

where now Φ_{A} and Φ_{B} are both complex scalar fields,

$$\Phi_{\mathrm{A}} = \phi_1 + \mathrm{i}\phi_2, \quad \Phi_{\mathrm{B}} = \phi_3 + \mathrm{i}\phi_4,$$

giving, in total, four real fields.

If $\mathrm{e}^{-\mathrm{i}\theta}$ is any element of the group $U(1)$ and $\mathbf{U}$ is any element of the group $SU(2)$ (discussed in Appendix B), so that $\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$, we require the Lagrangian density to be invariant under the $U(1) \times SU(2)$ transformation

$$\Phi \to \Phi' = \mathrm{e}^{-\mathrm{i}\theta}\mathbf{U}\Phi. \tag{11.2}$$

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A simple Lagrangian density that has a *global* $U(1) \times SU(2)$ symmetry is

$$\mathcal{L}_\Phi = \partial_\mu \Phi^\dagger \partial^\mu \Phi - V(\Phi^\dagger \Phi). \tag{11.3}$$

In terms of the real fields,

$$\Phi^\dagger \Phi = \Phi_A^* \Phi_A + \Phi_B^* \Phi_B = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2,$$
$$\partial_\mu \Phi^\dagger \partial^\mu \Phi = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 + \partial_\mu \phi_3 \partial^\mu \phi_3 + \partial_\mu \phi_4 \partial^\mu \phi_4.$$

If $V(\Phi^\dagger \Phi) = m^2 \Phi^\dagger \Phi$, this Lagrangian density corresponds to four independent free scalar fields, all with the same mass m (cf. (3.18)).

In the Standard Model, the $U(1)$ and $SU(2)$ global symmetries are promoted to local symmetries. The $U(1)$ transformation may be written

$$\Phi \to \Phi' = \mathrm{e}^{-\mathrm{i}\theta} \Phi = \exp(-\mathrm{i}\theta \tau^0)\Phi, \tag{11.4a}$$

where in this context we write τ^0 for the unit matrix

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For this to become a local symmetry, we must introduce a vector gauge field $B_\mu(x)\tau^0$ with the transformation law

$$B_\mu(x) \to B'_\mu(x) = B_\mu(x) + (2/g_1)\partial_\mu \theta, \tag{11.4b}$$

and make the replacement

$$\mathrm{i}\partial_\mu \to \mathrm{i}\partial_\mu - (g_1/2)B_\mu,$$

as in Chapter 7. Here the constant g_1 is a dimensionless parameter of the theory, and the factor 2 follows convention.

Any element of $SU(2)$ can be written in the form

$$\mathbf{U} = \exp(-\mathrm{i}\alpha^k \tau^k) \tag{11.5}$$

where the α^k are three real numbers and the τ^k are the three generators of the group *SU(2)*. The τ^k are identical to the Pauli spin matrices:

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the global $SU(2)$ symmetry to be made into a local $SU(2)$ symmetry, with $\mathbf{U} = \mathbf{U}(x)$ dependent on space and time coordinates, we must introduce a vector gauge field ${W_\mu}^k(x)$ for each generator τ^k. The transformation law for the matrices

$$\mathbf{W}_\mu(x) = {W_\mu}^k(x)\tau^k$$

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is

$$\mathbf{W}_\mu(x) \to \mathbf{W}'_\mu(x) = \mathbf{U}(x)\mathbf{W}_\mu(x)\mathbf{U}^\dagger(x) + (2\mathrm{i}/g_2)(\partial_\mu \mathbf{U}(x))\mathbf{U}^\dagger(x), \tag{11.6}$$

which is a generalisation of (11.4). Here g_2 is another dimensionless parameter of the theory.

Note that the matrices

$$\mathbf{W}_\mu(x) = \begin{pmatrix} W^3_\mu & W^1_\mu - \mathrm{i}W^2_\mu \\ W^1_\mu + \mathrm{i}W^2_\mu & -W^3_\mu \end{pmatrix} \tag{11.7}$$

are Hermitian and have zero trace. These properties are preserved by the transformation (11.6) as is clearly necessary (Problem 11.1). A global $SU(2)$ transformation $\mathbf{W}'_\mu = \mathbf{U}\mathbf{W}_\mu\mathbf{U}^\dagger$ corresponds to a rotation of the vectors $W_\mu{}^{\mathrm{k}}$ in the three-dimensional 'weak isospin' space defined by the generators τ^k. (See Appendix B.)

Finally we define

$$D_\mu\Phi = [\partial_\mu + (\mathrm{i}g_1/2)B_\mu + (\mathrm{i}g_2/2)\mathbf{W}_\mu]\Phi. \tag{11.8a}$$

It is straightforward to show

$$D'_\mu\Phi' = [\partial_\mu + (\mathrm{i}g_1/2)B'_\mu + (\mathrm{i}g_2/2)\mathbf{W}'_\mu]\Phi' = \mathrm{e}^{-\mathrm{i}\theta}\mathbf{U}D_\mu\Phi,$$

where

$$\Phi' = \mathrm{e}^{-\mathrm{i}\theta}\mathbf{U}\Phi. \tag{11.8b}$$

Hence the locally gauge invariant Lagrangian density corresponding to (11.3) is

$$\mathcal{L}_\Phi = (D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi^\dagger\Phi). \tag{11.9}$$

$\mathcal{L}_\Phi$ is also invariant under Lorentz transformations if we require B_μ and $\mathbf{W}_\mu$ to transform as covariant four-vectors.

11.2 The gauge fields

In the case of the gauge field B_μ, we define the field strength tensor $B_{\mu\nu}$ by

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{11.10}$$

and take the dynamical contribution to the Lagrangian density to be $-(1/4)\,B_{\mu\nu}B^{\mu\nu}$, as in Section 4.2.

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There are additional complications in introducing the field strength tensors for the gauge fields $\mathbf{W}_\mu$, stemming from the non-Abelian nature of the group $SU(2)$. The field strength tensor must be taken to be

$$\mathbf{W}_{\mu\nu} = [\partial_\mu + (\mathrm{i}g_2/2)\mathbf{W}_\mu]\mathbf{W}_\nu - [\partial_\nu + (\mathrm{i}g_2/2)\mathbf{W}_\nu]\mathbf{W}_\mu. \tag{11.11}$$

Under an $SU(2)$ transformation, $\mathbf{W}_\mu \to \mathbf{W}'_\mu$, given by (11.6), it is straightforward, if tedious, to show that

$$\mathbf{W}_{\mu\nu} \to \mathbf{W}'_{\mu\nu} = \mathbf{U}\mathbf{W}_{\mu\nu}\mathbf{U}^\dagger. \tag{11.12}$$

In verifying this result, note that, since $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$,

$$\mathbf{U}(\partial_\mu \mathbf{U}^\dagger) + (\partial_\mu \mathbf{U})\mathbf{U}^\dagger = 0.$$

The complicated definition of $\mathbf{W}_{\mu\nu}$ given by (11.11) is necessary in order to achieve the simple transformation property (11.12).

We then take the total dynamical contribution to the Lagrangian density associated with the gauge fields to be

$$\mathcal{L}_{\mathrm{dyn}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}\mathrm{Tr}(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}). \tag{11.13}$$

Using (11.12) and the cyclic invariance of the trace, we can see that $\mathcal{L}_{\mathrm{dyn}}$ is invariant under a local $SU(2)$ transformation.

Using the results $[\tau^2, \tau^3] = 2\mathrm{i}\tau^1$, etc., the matrix $\mathbf{W}_{\mu\nu}$ may be written

$$\mathbf{W}_{\mu\nu} = W^i_{\mu\nu}\tau^i \tag{11.14}$$

where

$$W^1_{\mu\nu} = \partial_\mu W^1_\nu - \partial_\nu W^1_\mu - g_2\left(W^2_\mu W^3_\nu - W^2_\nu W^3_\mu\right), \tag{11.15a}$$
$$W^2_{\mu\nu} = \partial_\mu W^2_\nu - \partial_\nu W^2_\mu - g_2\left(W^3_\mu W^1_\nu - W^3_\nu W^1_\mu\right), \tag{11.15b}$$
$$W^3_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu - g_2\left(W^1_\mu W^2_\nu - W^1_\nu W^2_\mu\right). \tag{11.15c}$$

Since $\mathrm{Tr}(\tau^i)^2 = 2$, and $\mathrm{Tr}(\tau^i\tau^j) = 0, i \neq j$, we can use (11.14) to express the Lagrangian density in the more reassuring form:

$$\mathcal{L}_{dyn} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \sum_{i=1}^{3}\frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu}. \tag{11.16}$$

We shall see, later in this chapter, that the fields W^1_μ and W^2_μ are electrically charged, and it is convenient to define here the complex combinations

$$W^+_\mu = \left(W^1_\mu - \mathrm{i}W^2_\mu\right)/\sqrt{2}, \quad W^-_\mu = \left(W^1_\mu + \mathrm{i}W^2_\mu\right)/\sqrt{2}. \tag{11.17}$$

Note that the field W_μ^- is the complex conjugate of the field W_μ^+. We also define

$$\begin{aligned} W_{\mu\nu}^+ &= \left(W_{\mu\nu}^1 - \mathrm{i}W_{\mu\nu}^2\right)/\sqrt{2} \\ &= \left(\partial_\mu + \mathrm{i}g_2 W_\mu^3\right)W_\nu^+ - \left(\partial_\nu + \mathrm{i}g_2 W_\nu^3\right)W_\mu^+ \end{aligned} \tag{11.18}$$

using (11.15a) and (11.15b). $W_{\mu\nu}^-$ is defined similarly.

We can also write (11.15c) as

$$W_{\mu\nu}^3 = \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3 - \mathrm{i}g_2\left(W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+\right) \tag{11.19}$$

and (11.16) becomes

$$\mathcal{L}_{\mathrm{dyn}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W_{\mu\nu}^3 W^{3\mu\nu} - \frac{1}{2}W_{\mu\nu}^- W^{+\mu\nu}. \tag{11.20}$$

11.3 Breaking the $SU(2)$ symmetry

As in equation (10.2) we take $V(\Phi^\dagger\Phi)$ to be

$$\begin{aligned} V(\Phi^\dagger\Phi) &= \frac{m^2}{2\phi_0^2}\left[(\Phi^\dagger\Phi) - \phi_0^2\right]^2 \\ &= \frac{m^2}{2\phi_0^2}\left[\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 - \phi_0^2\right]^2 \end{aligned} \tag{11.21}$$

where ϕ_0 is a fixed parameter that is the analogue of (10.2). With this expression for V, the vacuum state of our system is degenerate in the four-dimensional space of the scalar fields. We now break the $SU(2)$ symmetry. At our disposal we have the three real parameters $\alpha^k(x)$ that specify an element of $SU(2)$. We use this freedom to adopt a gauge in which for any field configuration $\Phi_{\mathrm{A}} = 0$ (two conditions) and Φ_{B} is real (one condition). The ground state is then

$$\Phi_{\mathrm{ground}} = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}, \tag{11.22}$$

and excited states are of the form

$$\Phi = \begin{pmatrix} 0 \\ \phi_0 + h(x)/\sqrt{2} \end{pmatrix}, \tag{11.23}$$

where the field *h(x)* is real.

A local $U(1)$ symmetry remains: the fields (11.23) are unchanged by a $U(1) \times SU(2)$ transformation of the form

$$\mathrm{e}^{-\mathrm{i}\theta/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta/2} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & 1 \end{pmatrix}. \tag{11.24}$$

Such matrices give a 2×2 matrix representation of the group $U(1)$. This residual symmetry will turn out to be the $U(1)$ symmetry of electromagnetism.

We wish to express $\mathcal{L}_\Phi$ (equation (11.9)) in terms of the field $h(x)$. We have from (11.21)

$$V(\Phi^\dagger\Phi) = m^2h^2 + \frac{m^2h^3}{\sqrt{2}\phi_0} + \frac{m^2h^4}{8\phi_0^2} = V(h),$$

and from (11.8a) and (11.7)

$$D^\mu\Phi = \begin{pmatrix} 0 \\ \partial^\mu h/\sqrt{2} \end{pmatrix} + \frac{\mathrm{i}g_1}{2}\begin{pmatrix} 0 \\ B^\mu(\phi_0 + h/\sqrt{2}) \end{pmatrix} + \frac{\mathrm{i}g_2}{2}\begin{pmatrix} \sqrt{2}W_\mu^+(\phi_0 + h/\sqrt{2}) \\ -W_\mu^3(\phi_0 + h/\sqrt{2}) \end{pmatrix}.$$

Multiplying $(D_\mu\Phi)^\dagger$ by $D^\mu\Phi$, we find

$$\begin{aligned}
\mathcal{L}_\Phi = {} & \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{g_2^2}{2}W_\mu^- W^{+\mu}(\phi_0 + h/\sqrt{2})^2 \\
& + \left[\frac{g_2^2}{4}W_\mu^3 W^{3\mu} - \frac{g_1 g_2}{2}W_\mu^3 B^\mu + \frac{g_1^2}{4}B_\mu B^\mu\right](\phi_0 + h/\sqrt{2})^2 - V(h) \\
= {} & \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{g_2^2}{2}W_\mu^- W^{+\mu}(\phi_0 + h/\sqrt{2})^2 \\
& + \frac{1}{4}\left(g_1^2 + g_2^2\right)Z_\mu Z^\mu(\phi_0 + h/\sqrt{2})^2 - V(h).
\end{aligned} \tag{11.25}$$

We have written

$$Z_\mu = W_\mu^3\cos\theta_\mathrm{w} - B_\mu\sin\theta_\mathrm{w}, \tag{11.26}$$

where

$$\cos\theta_\mathrm{w} = \frac{g_2}{\left(g_1^2 + g_2^2\right)^{1/2}}, \quad \sin\theta_\mathrm{w} = \frac{g_1}{\left(g_1^2 + g_2^2\right)^{1/2}}. \tag{11.27}$$

θ_w is called the *Weinberg angle*.

Along with the field Z_μ, we define the orthogonal combination

$$A_\mu = W_\mu^3\sin\theta_\mathrm{w} + B_\mu\cos\theta_\mathrm{w}. \tag{11.28}$$

Equations (11.26) and (11.28) correspond to a rotation of axes in (B_μ, W_μ^3) space. The rotation can be inverted to give

$$\begin{aligned}
B_\mu &= A_\mu\cos\theta_\mathrm{w} - Z_\mu\sin\theta_\mathrm{w}, \\
W_\mu^3 &= A_\mu\sin\theta_\mathrm{w} + Z_\mu\cos\theta_\mathrm{w}.
\end{aligned} \tag{11.29}$$

Substituting in (11.10) and (11.19) gives

$$\begin{aligned}
B_{\mu\nu} &= A_{\mu\nu}\cos\theta_\mathrm{w} - Z_{\mu\nu}\sin\theta_\mathrm{w}, \\
W_{\mu\nu}^3 &= A_{\mu\nu}\sin\theta_\mathrm{w} + Z_{\mu\nu}\cos\theta_\mathrm{w} - \mathrm{i}g_2\left(W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+\right),
\end{aligned}$$

where

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (A_{\mu\nu} \text{ is the } F_{\mu\nu} \text{ of Chapter 4})$$

and

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu. \tag{11.30}$$

11.4 Identification of the fields

We are now in a position to rearrange the terms in the full Lagrangian density $\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_{\text{dyn}}$ to reveal its physical content. In $\mathcal{L}_{\text{dyn}}$ (equation (11.20)) we use (11.29) and (11.30) to express the field B_μ and W_μ^3 in terms of the fields A_μ and Z_μ, and then we may write

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2,$$

where

$$\begin{aligned}
\mathcal{L}_1 = \; & \frac{1}{2}\partial_\mu h \partial^\mu h - m^2 h^2 \\
& - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{4}\phi_0^2 \big(g_1^2 + g_2^2\big) Z_\mu Z^\mu \\
& - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} \\
& - \frac{1}{2}\big[\big(D_\mu W_\nu^+\big)^* - \big(D_\nu W_\mu^+\big)^*\big]\big[D^\mu W^{+\nu} - D^\nu W^{+\mu}\big] + \frac{1}{2} g_2^2 \phi_0^2 W_\mu^- W^{+\mu},
\end{aligned} \tag{11.31}$$

and $D_\mu W_\nu^+ = (\partial_\mu + \mathrm{i} g_2 \sin\theta_w A_\mu) W_\nu^+$.

$\mathcal{L}_1$ is relatively simple: you will recognise it as the Lagrangian density for a free massive neutral scalar boson field *h(x)*, a free massive neutral vector boson field $Z_\mu(x)$, and a pair of massive charged vector boson fields $W_\mu^+(x)$ and $W_\mu^-(x)$, interacting with the electromagnetic field $A_\mu(x)$.

$\mathcal{L}_2$ is the sum of the remaining interaction terms. As the patient reader may verify,

$$\begin{aligned}
\mathcal{L}_2 = \; & \left(\frac{1}{4}h^2 + \frac{1}{\sqrt{2}} h\phi_0\right)\left(g_2^2 W_\mu^- W^{+\mu} + \frac{1}{2}\big(g_1^2 + g_2^2\big) Z_\mu Z^\mu\right) \\
& - \frac{m^2 h^3}{\sqrt{2}\phi_0} - \frac{m^2 h^4}{8\phi_0^2} + \frac{g_2^2}{4}\big(W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+\big)\big(W^{-\mu} W^{+\nu} - W^{-\nu} W^{+\mu}\big) \\
& + \frac{\mathrm{i} g_2}{2}\big(A_{\mu\nu}\sin\theta_w + Z_{\mu\nu}\cos\theta_w\big)\big(W^{-\mu} W^{+\nu} - W^{-\nu} W^{+\mu}\big) \\
& - g_2^2 \cos^2\theta_w \big(Z_\mu Z^\mu W_\nu^- W^{+\nu} - Z_\mu Z^\nu W_\nu^- W^{+\mu}\big)
\end{aligned}$$

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$$+\frac{ig_2}{2}\cos\theta_w[(Z_\mu W_\nu^- - Z_\nu W_\mu^-)(D^\mu W^{+\nu} - D^\nu W^{+\mu})$$
$$-\left(Z_\mu W_\nu^+ - Z_\nu W_\mu^+\right)(D^\mu W^{+\nu})^* - (D^\nu W^{+\mu})^*)]. \tag{11.32}$$

Most of the $U(1) \times SU(2)$ symmetry with which we began has been lost on symmetry breaking. In particular, no trace of the original $SU(2)$ symmetry is to be seen in the interactions described by $\mathcal{L}_2$. Nevertheless it is precisely this complicated set of interactions that makes the theory renormalisable, as it would be if the symmetry were not broken.

We identify the three vector fields, W_μ^+, W_μ^-, Z_μ, with the mediators of the weak interaction, the W^+, W^-, Z particles, which, subsequent to the theory, were discovered experimentally. The masses are (Particle Data Group, 2004)

$$M_{\rm w} = 80.425 \pm 0.038 \text{ GeV}, \tag{11.33}$$

$$M_{\rm z} = 91.1876 \pm 0.0021\text{GeV}. \tag{11.34}$$

From (11.31) and Section 4.9, we identify

$$\phi_0 g_2/\sqrt{2} = M_{\rm w,} \tag{11.35}$$

$$\phi_0\left(g_1^2 + g_2^2\right)^{1/2}/\sqrt{2} = M_{\rm z}. \tag{11.36}$$

Then, from (11.27), and neglecting quantum corrections to the mass ratio,

$$\cos\theta_{\rm w} = M_{\rm w}/M_{\rm z} = 0.8810 \pm 0.0016. \tag{11.37a}$$

It is usual to quote the value of $\sin^2\theta_{\rm w}$, which will appear in later calculations. The estimate above would suggest

$$\sin^2\theta_{\rm w} = 0.23120 \pm 0.00015.$$

The uncertainty arises mainly from uncertainty in $M_{\rm w}$. Other ways of estimating $\sin^2\theta_{\rm w}$ exist and the accepted value (in 1996) was

$$\sin^2\theta_{\rm w} = 0.2315 \pm 0.0004. \tag{11.37b}$$

We shall adopt this value in subsequent calculations.

The $W^\pm$ bosons are found experimentally to carry charge $\pm e$. In (11.31) the gauge derivative is

$$D_\mu W_\nu{}^+ = (\partial_\mu + ig_2\sin\theta_{\rm w} A_\mu)W_\nu{}^+,$$

so that from the coupling to the electromagnetic field A_μ and (11.27) we can identify

$$e = g_2\sin\theta_{\rm w} = g_1\cos\theta_{\rm w.} \tag{11.38}$$

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The fields W_μ^1, W_μ^2, and Z_μ have free field expansions similar to (4.15) but with three polarisation states (see Section 4.9). As a quantum field W_μ^+ destroys W^+ bosons and creates W^- bosons; W_μ^- destroys W^- bosons and creates W^+ bosons.

There remains the scalar Higgs field *h(x)*. The vacuum state expectation value ϕ_0 of the Higgs field is, from (11.35),

$$\phi_0 = \frac{\sqrt{2}M_\mathrm{w}}{g_2} = \frac{\sqrt{2}M_\mathrm{w}\sin\theta_\mathrm{w}}{e} = 180\,\mathrm{GeV}. \tag{11.39}$$

The only parameter not fixed from experiment is the mass $M_\mathrm{H} = \sqrt{2}m$ of the Higgs boson. No Higgs boson has yet been identified experimentally, though its existence is, apparently, an essential part of the Standard Model. The failure so far of experimental searches to find the Higgs boson suggests $M_\mathrm{H} >$ 64 GeV. Recent experimental and theoretical studies suggest an M_H close to this limit.

The requirements of $U(1)$ and $SU(2)$ symmetry, followed by $SU(2)$ symmetry breaking, have generated the electromagnetic field, the massive vector $\mathrm{W}^\pm$ and Z boson fields, and the scalar Higgs field, in a remarkably economical way. In the next chapter, we add lepton fermion fields to these boson fields, to obtain the richness of the Weinberg–Salam electroweak theory.

Problems

11.1 Show that the W'_μ defined by (11.6) are Hermitian and have zero trace. (Use the expression (B.9) of Appendix B: $\mathbf{U}= \cos\alpha\mathbf{I}+\mathrm{i}\,\sin\alpha(\hat{\alpha}\cdot\tau)$.)

11.2 Verify that the expressions (11.13) and (11.16) for $\mathcal{L}_\mathrm{dyn}$ are equivalent.

11.3 Verify that the last two terms on the right-hand side of (11.31) correspond to a pair of massive charged vector boson fields.

11.4 Show that the Higgs boson can decay to two photons, in the third order of perturbation theory. Draw the appropriate Feynman graph.

11.5 Under an $SU(2)$ transformation, $\Phi \to \Phi'$ where

$$\begin{pmatrix} \Phi'_A \\ \Phi'_B \end{pmatrix} = \mathbf{U}\begin{pmatrix} \Phi_A \\ \Phi_B \end{pmatrix}.$$

Using (B.9), show that $\tau^2 U^* = U\tau^2$. Hence show that

$$\begin{pmatrix} \Phi'^*_\mathrm{B} \\ -\Phi'^*_\mathrm{A} \end{pmatrix} = \mathbf{U}\begin{pmatrix} \Phi^*_\mathrm{B} \\ -\Phi^*_\mathrm{A} \end{pmatrix}.$$

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11.6 Show that the $SU(2)$ matrix $\mathbf{U} = \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = \alpha(\sin\phi, \cos\phi, 0)$ is

$$\mathbf{U} = \begin{pmatrix} \cos\alpha & \mathrm{e}^{\mathrm{i}\phi}\sin\alpha \\ -\mathrm{e}^{-\mathrm{i}\phi}\sin\alpha & \cos\alpha \end{pmatrix}.$$

Show that under the $SU(2)$ transformation $\Phi' = \mathbf{U}\Phi$, the two-component complex field

$$\Phi = \begin{pmatrix} \Phi_A \\ \Phi_B \end{pmatrix} = \begin{pmatrix} a\mathrm{e}^{\mathrm{i}\delta} \\ b\mathrm{e}^{\mathrm{i}\gamma} \end{pmatrix}$$

can be put in the form

$$\Phi' = \begin{pmatrix} \Phi'_{\mathrm{A}} \\ \Phi'_{\mathrm{B}} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\gamma}\sqrt{a^2 + b^2} \end{pmatrix},$$

taking $\phi = (\delta - \gamma)$ and $\alpha = -\tan^{-1}(a/b)$. Show that Φ' can then be put in the standard form (11.23) by a further $SU(2)$ transformation with $\boldsymbol{\alpha} = \gamma(0, 0, 1)$.

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