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# Exponential multiple mixing for commuting automorphisms of a nilmanifold

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*Abstract.* Let $l \in \mathbb{N}_{\geq 1}$ and $\alpha : \mathbb{Z}^l \to \mathrm{Aut}(\mathscr{N})$ be an action of $\mathbb{Z}^l$ by automorphisms on a compact nilmanifold $\mathscr{N}$. We assume the action of every $\alpha(z)$ is ergodic for $z \in \mathbb{Z}^l \setminus \{0\}$ and show that $\alpha$ satisfies exponential $n$-mixing for any integer $n \geq 2$. This extends the results of Gorodnik and Spatzier [Mixing properties of commuting nilmanifold automorphisms. *Acta Math.* **215** (2015), 127–159].

Key words: exponential mixing, multiple mixing, nilmanifold automorphisms, Schmidt's subspace theorem, unit equations
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## 1. *Introduction*

Building on the work of Gorodnik and Spatzier [9, 10], the goal of this paper is to show multiple mixing with exponential rate for actions of commuting ergodic automorphisms of a compact nilmanifold. We start by recalling the basic notions involved.

A *compact nilmanifold* is a quotient $\mathscr{N} = G/\Lambda$, where $G$ is a nilpotent, connected, simply connected real Lie group and $\Lambda$ is a cocompact discrete subgroup of $G$. It carries a unique $G$-invariant probability measure that we denote by $m$ and call the *Haar measure* on $\mathscr{N}$. The *group of automorphisms* of $\mathscr{N}$ is defined by

$$\mathrm{Aut}(\mathscr{N}) = \{\psi \in \mathrm{Aut}(G),\ \psi(\Lambda) = \Lambda\}$$

and acts by measure-preserving transformations on $(\mathscr{N}, m)$.

This article deals with the mixing properties of a finite family of commuting automorphisms of $\mathscr{N}$, realized as a morphism $\alpha : \mathbb{Z}^l \to \mathrm{Aut}(\mathscr{N})$. Given $n \geq 2$, we say that $\alpha$ is *n-mixing* if for every tuple of bounded measurable functions $f_1, \ldots, f_n : \mathscr{N} \to \mathbb{R}$ and for $z_1, \ldots, z_n \in \mathbb{Z}^l$, we have

$$\int_{\mathscr{N}} \prod_{i=1}^{n} f_i(\alpha(z_i)x)\, dm(x) \longrightarrow \prod_{i=1}^{n} \int_{\mathscr{N}} f_i(x)\, dm(x)$$

as $\min_{i \neq j} \|z_i - z_j\| \to +\infty$.

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Note that the case where $n = 2$ corresponds to the usual notion of mixing, and also that we do not lose generality by considering smooth test functions.

Our aim is to estimate the rate of mixing of a finite family of ergodic commuting automorphisms of $\mathscr{N}$. To do so, we need to restrict our attention to a set of regular test functions. We fix an arbitrary Riemannian metric on $\mathscr{N}$, and for $\theta \in (0, 1]$, denote by $\mathcal{H}^\theta(\mathscr{N})$ the space of $\theta$-Hölder functions $f$ on $\mathscr{N}$ endowed with the norm

$$\|f\|_{\mathcal{H}^\theta} = \sup_{\mathscr{N}} |f| + \sup_{x \neq y \in \mathscr{N}} \frac{|f(x) - f(y)|}{d(x, y)^\theta}.$$

We state our main result.

THEOREM 1.1. (Multiple mixing with exponential rate) *Let $\alpha : \mathbb{Z}^l \to Aut(\mathscr{N})$ be an action of $\mathbb{Z}^l$ ($l \geq 1$) by automorphisms on a compact nilmanifold $\mathscr{N} = G/\Lambda$. Assume $\alpha(z)$ acts ergodically on $(\mathscr{N}, m)$ for all non-zero $z \in \mathbb{Z}^l$.*

*Then for every $n \geq 2$, $\theta \in (0, 1]$, there is an effective constant $\eta > 0$ and an ineffective constant $C > 0$ such that for all $\theta$-Hölder functions $f_1, \ldots, f_n \in \mathcal{H}^\theta(\mathscr{N})$, translation parameters $g_1, \ldots, g_n \in G$, and $\underline{z} = (z_1, \ldots, z_n) \in (\mathbb{Z}^l)^n$, one has*

$$\bigg| \int_{\mathscr{N}} \prod_{i=1}^n f_i(g_i \alpha(z_i) x)\, dm(x) - \prod_{i=1}^n \int_{\mathscr{N}} f_i(x)\, dm(x) \bigg| \leq \frac{C}{N(\underline{z})^\eta} \prod_{i=1}^n \|f_i\|_{\mathcal{H}^\theta},$$

*where $N(\underline{z}) = \exp(\min_{i \neq j} \|z_i - z_j\|)$.*

In other words, a finite family of commuting ergodic automorphisms of a compact nilmanifold satisfies multiple mixing with exponential rate in the class of $\theta$-Hölder functions (and their $G$-translates). In particular, we also obtain exponential multiple mixing for actions of commuting *affine* automorphisms.

We explain what effectivity or ineffectivity of the constants mean in §1.2 below.

1.1. *Earlier results.* Theorem 1.1 for mixing without a rate, that is, the statement that ergodicity of $\alpha(z)$ for every $z \in \mathbb{Z}^l \setminus \{0\}$ implies that $\alpha$ is $n$-mixing for all $n \geq 2$, was known before. It is due to Parry [16] for $l = 1$, Schmidt and Ward [18] in the case where $\mathscr{N}$ is a torus and $l \geq 2$, and Gorodnik and Spatzier [10] in general.

There are also many prior results on exponential mixing. The case of Theorem 1.1 where $\mathscr{N}$ is a torus and $l = 1$ is due to Lind [13] for $n = 2$, and to Dolgopyat [3] for general $n$. Note that some difficulty arises from the fact that an ergodic automorphism of a torus is not necessarily hyperbolic: it can have eigenvalues of modulus 1. Still for a torus but now general $l$, Miles and Ward [15] prove directional uniformity in the—*a priori* non-exponential—rate of 2-mixing under some entropic conditions. For general $\mathscr{N}$ and $l \geq 1$, the case where $n \leq 3$ was established by Gorodnik and Spatzier [9, 10]. They also examined the case of higher order mixing, that is $n \geq 4$, but only obtain partial results and mention serious difficulties to reach full generality. Those difficulties consist in a fine understanding of the size of solutions of some diophantine inequalities.

Theorem 1.1 is new for $n \geq 4$, even for tori.

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1.2. *Effectivity of the constants.* The constant $\eta$ is *effective*. This means that by following the arguments in this paper and its references, one could determine an explicit value for $\eta$ such that the theorem holds (for some $C$). In contrast, the constant $C$ is *ineffective*. This means that the proof of the theorem only shows that there exists a finite (and positive) value of $C$ for which the theorem holds; however, we do not have any means of determining such a value.

This ineffectivity is caused by our reliance on W. Schmidt's subspace theorem. In the cases $n \leq 3$, this tool can be replaced by the theory of linear forms in logarithms, as it is done by Gorodnik and Spatzier in [**10**], and one can obtain the theorem with effective constants. We note that, as they are written, the arguments in [**10**] appeal to [**2**, Theorem 7.3.2] in several places (see pp. 139–140), which relies on the subspace theorem making the resulting constants ineffective. However, the application of [**2**, Theorem 7.3.2] could be replaced by [**14**, Proposition 14.13] at the expense of adjusting the exponents in [**10**].

1.3. *Motivation.* Katok and Spatzier have made a conjecture about rigidity of higher rank abelian Anosov actions on compact manifolds, which can be stated somewhat informally as follows.

*Conjecture 1.2.* When $l \geq 2$, all irreducible Anosov genuine $\mathbb{Z}^l$-actions on compact manifolds are $C^\infty$-conjugate to actions on infranilmanifolds by affine automorphisms.

We do not define all the terms appearing in the above conjecture, but we comment on them briefly. An Anosov diffeomorphism of a compact manifold is a diffeomorphism such that the tangent bundle can be split as the sum of two invariant subbundles, with one subbundle that is exponentially contracting and one that is exponentially expanding under the action. A $\mathbb{Z}^l$-action is Anosov if it contains an Anosov diffeomorphism.

The conjecture is false for rank 1 actions, see [**17**, §1.2] for a simple example or [**6**] for a more elaborate one. These can be modified to obtain some higher rank examples, e.g., actions on manifolds that have quotients on which the action factors through a rank 1 action. The adjective genuine is meant to exclude examples like this, but we do not give a precise definition.

An infranilmanifold is a compact manifold that is finitely covered by a nilmanifold.

Conjecture 1.2 motivates Theorem 1.1 for two reasons. First, if the conjecture is true, it implies that actions by affine automorphisms on (infra)nilmanifolds are the only examples of higher rank abelian Anosov actions in some sense, which demonstrates the importance of this class of dynamical systems. Second, some recent progress by Fisher, Kalinin, and Spatzier [**8**] and by Rodriguez Hertz and Wang [**17**] toward Conjecture 1.2 relies on exponential mixing of actions by automorphisms. We note, however, that these applications require results about 2-mixing only, which is already covered in [**10**].

We refer to [**7**, **17**] and their references for more information on higher rank abelian Anosov actions.

1.4. *An outline of the paper.* Our approach is inspired by the papers of Gorodnik and Spatzier [**9**, **10**]. The argument we propose is, however, simpler.

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The mixing estimate in Theorem 1.1 can be recast as a problem about quantitative equidistribution of a certain affine subnilmanifold $\mathscr{S}$ in the product nilmanifold $\mathscr{N}^n$. In §2, we use a result by Green and Tao [**11**, Theorem 1.16] to reduce quantitative equidistribution of $\mathscr{S}$ to a diophantine condition on its Lie algebra. This condition is related to [**10**, Theorem 2.3], though our formulation is simpler and easier to prove.

The diophantine condition arising in §2 requires bounding the solutions of certain generalized unit equations. This problem is studied in §3 using W. Schmidt's subspace theorem. It is related to [**10**, Proposition 3.1], but we prove a more uniform statement, which is needed to establish the exponential mixing rate. Our proof is based on that of a very closely related result of Evertse [**4**].

To apply the results of §3 to the diophantine equations governing the equidistribution of $\mathscr{S}$ in $\mathscr{N}^n$, we need to estimate the growth of the eigencharacters of $(\alpha(z))_{z\in\mathbb{Z}^l}$ acting on the abelianized Lie algebra $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. We do this in §4 using the ergodicity assumption.

Section 5 concludes the paper with the proof of Theorem 1.1 combining the ingredients worked out in the previous three sections.

## 2. *Equidistribution of rational submanifolds*

Let $\mathscr{N}=G/\Lambda$ be a Riemannian compact nilmanifold and $m$ its Haar probability measure. An *affine rational submanifold* $\mathscr{S}\subseteq\mathscr{N}$ is a quotient $gH/(H\cap\Lambda)$, where $H$ is a connected closed subgroup of $G$ intersecting $\Lambda$ cocompactly and $g\in G$ is a translation parameter. It carries a unique $gHg^{-1}$-invariant probability measure, denoted by $m_{\mathscr{S}}$.

The goal of the section is to reduce quantitative equidistribution of $(\mathscr{S}, m_{\mathscr{S}})$ in $(\mathscr{N}, m)$ to a diophantine condition on the Lie algebra of $H$.

We call $\mathfrak{h}\subseteq\mathfrak{g}$ the respective Lie algebras of $H\subseteq G$ and write $\mathfrak{t}=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ for the largest abelian quotient of $\mathfrak{g}$. The projection map from $\mathfrak{g}$ to $\mathfrak{t}$ is denoted by $\mathfrak{g}\to\mathfrak{t}$, $w\mapsto\overline{w}$ and sends $\log\Lambda$ to a lattice $Z_\Lambda=\overline{\log\Lambda}$ in $\mathfrak{t}$. For future reference, we fix a basis of the latter, which leads to identifications $\mathfrak{t}\equiv\mathbb{R}^d$, $Z_\Lambda\equiv\mathbb{Z}^d$.

PROPOSITION 2.1. (Equidistribution of rational submanifolds) *Let $\mathscr{N}=G/\Lambda$ be a Riemannian compact nilmanifold and let $\theta\in(0,1]$. There exists $L>0$ such that for every $\delta\in(0,1/2)$, any affine rational submanifold $\mathscr{S}\subseteq\mathscr{N}$ satisfying*

$$\min\{\|q\| : q\in\mathbb{Z}^d\setminus\{0\}, \langle q,\overline{\mathfrak{h}}\rangle=0\}\geq\left(\frac{1}{\delta}\right)^L$$

*is $\delta$-equidistributed with respect to $\mathcal{H}^\theta(\mathscr{N})$, that is,*

$$\left|\int_{\mathscr{S}} f\,dm_{\mathscr{S}}-\int_{\mathscr{N}} f\,dm\right|\leq\delta\|f\|_{\mathcal{H}^\theta}$$

*for all $f\in\mathcal{H}^\theta(\mathscr{N})$.*

We deduce this proposition from Theorem 2.2, which is itself a direct corollary of a much stronger result of Green and Tao [**11**, Theorem 1.16].

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THEOREM 2.2. *Let $\mathscr{N} = G/\Lambda$ be a Riemannian compact nilmanifold. Then there is a constant $L > 0$ such that, for every $\delta \in (0, 1/2)$, $g \in G$, and $w \in \mathfrak{g}$, at least one of the following two statements is true.*

(1) *The sequence $(g \exp(kw)\Lambda)_{1\le k\le n}$ is $\delta$-equidistributed with respect to $\mathcal{H}^1(\mathscr{N})$ for all sufficiently large $n$. That is to say, for all sufficiently large $n$, for all $f \in \mathcal{H}^1(\mathscr{N})$, we have*

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(g\exp(kw)\Lambda) - \int_{\mathscr{N}} f\,dm\right| \le \delta\|f\|_{\mathcal{H}^1}.$$

(2) *There is $q \in \mathbb{Z}^d\backslash\{0\}$ with $\|q\| < \delta^{-L}$ and $\langle q, \overline{w}\rangle \in \mathbb{Z}$.*

The above result does not use the full force of [11, Theorem 1.16] in two significant ways. First, the result of Green and Tao is for general polynomial sequences in place of $g\exp(kw)$. Second, it is possible to control in a very efficient way how large $n$ needs to be in item (1) at the expense of replacing $\langle q, \overline{w}\rangle \in \mathbb{Z}$ in item (2) by an upper bound on $\operatorname{dist}(\langle q, \overline{w}\rangle, \mathbb{Z})$. These features are crucial in some other applications of [11, Theorem 1.16], but we do not need them.

*Proof of Proposition 2.1.* We suppose without loss of generality that $\theta = 1$ (see [10, proof of Theorem 2.3]). We choose $L > 0$ as in Theorem 2.2 and consider an affine rational submanifold $\mathscr{S} \subseteq \mathscr{N}$ as well as a constant $\delta \in (0, 1/2)$.

Assume that $\mathscr{S}$ is not $\delta$-equidistributed for $\mathcal{H}^1(\mathscr{N})$. Fix a vector $w \in \mathfrak{h}$ such that $\|\overline{w}\| \le \frac{1}{2}\delta^L$ and whose projection in $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ does not belong to any translate of a $H \cap \Lambda$-rational proper subspace by a $H \cap \Lambda$-rational vector. By Theorem 2.2, or in this case even by an earlier theorem of Leon Green [1, 12], we have the weak-$*$ convergence:

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{g\exp(kw)\Lambda} \longrightarrow m_{\mathscr{S}}.$$

In particular, for large enough $n$, the sequence $(g\exp(kw)\Lambda)_{0\le k\le n}$ does not satisfy $\delta$-equidistribution for $\mathcal{H}^1(\mathscr{N})$ either. Theorem 2.2 yields some integer vector $q \in \mathbb{Z}^d \setminus \{0\}$ such that $\|q\| < \delta^{-L}$ and $\langle q, \overline{w}\rangle \in \mathbb{Z}$. This forces $\langle q, \overline{w}\rangle = 0$ as our choice of $w$ guarantees that $|\langle q, \overline{w}\rangle| \le 1/2$. As the smallest rational subspace of $\mathfrak{t}$ containing $\overline{w}$ is $\overline{\mathfrak{h}}$, we get that $\langle q, \overline{\mathfrak{h}}\rangle = 0$, which concludes the proof. □

## 3. *Diophantine estimates*

Let $K$ be a number field. We denote by $M_K$ the set of its places, by $M_{K,\infty}$ its Archimedean places, and by $M_{K,f}$ its finite places. We use the convention that we include complex places twice (one for each complex conjugate embedding), and we write $|x|_v$ for $x \in K$ and $v \in M_{K,\infty}$ to denote the usual absolute value arising from the embedding of $K$ in $\mathbb{R}$ or $\mathbb{C}$ associated to $v$. (This convention is not standard, but it is also not unprecedented, see [14, Ch. 14].) The finite places will not play much of a role in this paper, so we do not specify which convention is used for inclusion in $M_{K,f}$ or for the normalization of the corresponding absolute values. We do insist, however, that the product formula $\prod_{v\in M_K} |x|_v = 1$ holds for all $x \in K$.

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The projective height of a tuple $(a_1, \ldots, a_n) \in K^n$ is denoted by

$$H(a_1, \ldots, a_n) = \prod_{v \in M_K} \max_j (|a_j|_v^{1/[K:\mathbb{Q}]}).$$

We note that this quantity is a projective notion, that is, it is invariant under multiplication of each coordinate by the same scalar. This fact is an immediate consequence of the product formula.

The ring of integers in $K$ is denoted by $\mathcal{O}(K)$ and for $h \in \mathbb{R}_{\geq 0}$, we set

$$\mathcal{O}(K)_h = \{a \in \mathcal{O}(K) : |a|_v \leq h \text{ for all } v \in M_{K,\infty}\}.$$

We write $\mathcal{O}(K)^\times$ for the group of (multiplicative) units in $\mathcal{O}(K)$. For $n \in \mathbb{Z}_{\geq 2}$ and $u = (u_1, \ldots, u_n) \in (\mathcal{O}(K)^\times)^n$, we write

$$\alpha(u) = \min_{I \subset \{1,\ldots,n\}, |I| \geq 2} H(u_i : i \in I)^{1/(|I|-1)}.$$

In what follows, we consider a generalization of the classical unit equation $u_1 + u_2 = 1$ to be solved for $u_1, u_2 \in \mathcal{O}(K)^\times$. This subject has a rich literature, the recent book [5] is a good general reference.

THEOREM 3.1. *Let $K$ be a number field. For all $\varepsilon > 0$ and $n \geq 2$, there is an (ineffective) constant $r > 0$ such that the following holds. Let $u = (u_1, \ldots, u_n) \in (\mathcal{O}(K)^\times)^n$ and let $a_1, \ldots, a_n \in \mathcal{O}(K)_h$ not all 0 for some $h \in \mathbb{R}_{\geq 0}$. Suppose*

$$a_1 u_1 + \cdots + a_n u_n = 0.$$

*Then $h \geq r\alpha(u)^{1-\varepsilon}$.*

We comment on how this result is related to the classical unit equation. Taking $n = 3$, $a_1 = a_2 = 1$, $a_3 = -1$, $u_3 = 1$, Theorem 3.1 implies that the classical unit equation has finitely many solutions, a result that goes back to Siegel, Mahler, and Lang.

Theorem 3.1 is not new. It could be deduced (with a slightly weaker exponent) from a result of Evertse [4], see also [5, Theorem 6.1.1]. We give a short proof based on [4] for the reader's convenience.

As a simple application of Dirichlet's box principle, we also show below that the exponent in the lower bound on $h$ is almost optimal in the sense that $1 - \varepsilon$ could not be replaced by a number larger than 1.

3.1. *The subspace theorem.* At the heart of the proof of Theorem 3.1 is W. Schmidt's subspace theorem which we now recall.

THEOREM 3.2. [2, Corollary 7.2.5] *Let $K$ be a number field and let $\{|.|_v, v \in M_K\}$ be a system of absolute values on $K$ as above. Let $n \geq 2$ and let $V$ be an $n$-dimensional vector space over $K$. For each $v \in M_{K,\infty}$, let $\Lambda_1^{(v)}, \ldots, \Lambda_n^{(v)}$ be a basis of the dual space $V^*$. Furthermore, let $\Lambda_1^{(0)}, \ldots, \Lambda_n^{(0)}$ be another basis of $V^*$. Then for all $\varepsilon > 0$, the solutions of*

$$\prod_{v \in M_{K,\infty}} \prod_{j=1}^{n} |\Lambda_j^{(v)}(x)|_v \leq H(\Lambda_1^{(0)}(x), \ldots, \Lambda_n^{(0)}(x))^{-\varepsilon}$$

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*for $x \in V$ with $\Lambda_j^{(0)}(x) \in \mathcal{O}(K)$ for all $j = 1, \ldots, n$ are contained in a finite union of proper subspaces of V.*

The functionals $\Lambda_1^{(0)}, \ldots, \Lambda_n^{(0)}$ can be used to identify $V$ with $K^n$ and using this identification, $\Lambda_j^{(v)}$ can be identified with a linear form for each $j$ and $v$. Using these identifications, Theorem 3.2 is translated into the form in [**2**, Corollary 7.2.5]. In our application that follows, there is no natural choice for an identification between $V$ and $K^n$, so the above formulation will suit our purposes best.

3.2. *Proof of Theorem 3.1.* The proof is by induction on $n$. Suppose first $n = 2$. Let $u_1, u_2 \in \mathcal{O}(K)^\times$ and $a_1, a_2 \in \mathcal{O}(K)_h$ for some $h \in \mathbb{R}_{\geq 0}$ such that $a_1, a_2 \neq 0$ and $a_1 u_1 + a_2 u_2 = 0$. We observe that

$$(u_1, u_2) = \left(\frac{u_1}{a_2} a_2, -\frac{u_1}{a_2} a_1\right),$$

hence

$$H(u_1, u_2) = H(a_1, a_2) \leq (h^{1/[K:\mathbb{Q}]})^{[K:\mathbb{Q}]} = h.$$

Since $\alpha(u_1, u_2) = H(u_1, u_2)$, this proves our claim with $r = 1$.

Now suppose $n \geq 3$ and that the claim holds for all smaller values of $n$. Let $V$ be the subspace of $K^n$ of those points that satisfy the equation $x_1 + \cdots + x_n = 0$. Observe that

$$y := (a_1 u_1, \ldots, a_n u_n) \in V.$$

For each place $v \in M_{K,\infty}$, let $\Lambda_1^{(v)}, \ldots, \Lambda_{n-1}^{(v)}$ be an enumeration of all but one of the functionals $(x_1, \ldots, x_n) \mapsto x_j$ so that we omit one of the indices $j$ for which $|u_j|_v$ is maximal. We also set $\Lambda_j^{(0)}(x) = x_j$, say, for $j = 1, \ldots, n-1$. Observe that, by the product formula,

$$\prod_{v \in M_{K,\infty}} \prod_{j=1}^{n-1} |\Lambda_j^{(v)}(u_1, \ldots, u_n)|_v = \prod_{v \in M_{K,\infty}} \frac{\prod_{j=1}^n |u_j|_v}{\max_j(|u_j|_v)} = H(u_1, \ldots, u_n)^{-[K:\mathbb{Q}]}.$$

Using $a_1, \ldots, a_n \in \mathcal{O}(K)_h$ and the definition of $\Lambda_j^{(v)}$, we have

$$|\Lambda_j^{(v)}(a_1 u_1, \ldots, a_n u_n)|_v \leq h |\Lambda_j^{(v)}(u_1, \ldots, u_n)|_v$$

for all $j$ and $v$. Therefore,

$$\prod_{v \in M_{K,\infty}} \prod_{j=1}^{n-1} |\Lambda_j^{(v)}(y)|_v \leq h^{(n-1)[K:\mathbb{Q}]} H(u)^{-[K:\mathbb{Q}]}.$$

We assume $h \leq \alpha(u)^{1-\varepsilon}$ as we may, for otherwise, the claim is trivial. Then $h \leq H(u)^{(1-\varepsilon)/(n-1)}$ and we observe that

$$\begin{aligned} H(\Lambda_1^{(0)}(y), \ldots, \Lambda_{n-1}^{(0)}(y)) &= H(a_1 u_1, \ldots, a_{n-1} u_{n-1}) \\ &\leq h H(u_1, \ldots, u_{n-1}) \leq h H(u) \leq H(u)^2. \end{aligned}$$

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Thus,

$$\prod_{v \in M_{K,\infty}} \prod_{j=1}^{n-1} |\Lambda_j^{(v)}(y)|_v \leq H(u)^{-\varepsilon[K:\mathbb{Q}]} \leq H(\Lambda_1^{(0)}(y), \ldots, \Lambda_{n-1}^{(0)}(y))^{-\varepsilon[K:\mathbb{Q}]/2}.$$

Therefore, the subspace theorem applies and it follows that $(a_1 u_1, \ldots, a_n u_n)$ is contained in finitely many proper subspaces of $V$. In what follows, we use the induction hypothesis to show that for any proper subspace $U$ of $V$, there is a constant $r = r(U) > 0$ such that $h \geq r\alpha(u)^{1-\varepsilon}$ holds whenever there are $a_1, \ldots, a_n$ and $u_1, \ldots, u_n$ that satisfy the assumptions in the proposition with $y \in U$. We can then take the minimum of $r(U)$ over all subspaces $U$ that arise through the application of the subspace theorem taking into account all possible choices of the functionals $\Lambda_j^{(v)}$, of which there are finitely many. Then the claim will hold with this minimum in the role of $r$.

Let now $U$ be a proper subspace of $V$ with $y \in U$, and let $b_1, \ldots, b_n \in \mathcal{O}(K)$ be such that

$$b_1 x_1 + \cdots + b_n x_n = 0$$

for all $(x_1, \ldots, x_n) \in U$ and not all $b_j$ are equal. We also suppose that $a_j \neq 0$ for all $j$, as we may, for otherwise, we may omit the 0 coordinates, and the induction hypothesis applies directly. We now observe that

$$(b_1 - b_n)a_1 u_1 + \cdots + (b_{n-1} - b_n)a_{n-1}u_{n-1} = 0$$

and that not all of the coefficients $(b_j - b_n)a_j$ vanish.

We also note that $(b_j - b_n)a_j \in \mathcal{O}(K)_{Ch}$ for some constant $C$ depending on the $b_j$. Therefore, we are in a position to apply the induction hypothesis and conclude

$$Ch \geq c\alpha(u_1, \ldots, u_{n-1})^{1-\varepsilon} \geq r\alpha(u)^{1-\varepsilon},$$

and this completes the proof.

3.3. *Optimality of the exponent.* We now prove the claimed optimality of Theorem 3.1.

PROPOSITION 3.3. *Let notation be as in Theorem 3.1. Then there is a constant $R > 0$ such that for all $u = (u_1, \ldots, u_n) \in (\mathcal{O}(K)^\times)^n$, there exist $a_1, \ldots, a_n \in \mathcal{O}(K)_h$ not all 0 such that*

$$a_1 u_1 + \cdots + a_n u_n = 0$$

*and $h \leq R\alpha(u)$.*

*Proof.* We assume, as we may, that $\alpha(u) = H(u)^{1/(n-1)}$, for otherwise, we can omit some coordinates from $u$ so that the identity holds.

By Dirichlet's unit theorem, there exists $\lambda \in \mathcal{O}(K)^\times$ such that

$$\max(|\lambda u_1|_v, \ldots, |\lambda u_n|_v) \leq C_0 H(u)$$

for all $v \in M_{K,\infty}$ and some $C_0 > 0$ depending only on $K$. Up to replacing $u$ by $\lambda u$, which does not affect the height, we may assume $\lambda = 1$. It follows that for all $a_1, \ldots, a_n \in \mathcal{O}(K)_h$,

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$$a_1u_1 + \cdots + a_nu_n \in \mathcal{O}(K)_{C_1hH(u)},$$

where $C_1 = nC_0$.

We observe that

$$c_2h^{[K:\mathbb{Q}]} \le |\mathcal{O}(K)_h| \le C_2h^{[K:\mathbb{Q}]}$$

for some constants $c_2, C_2 > 0$ depending only on $K$ provided $h$ is sufficiently large. This means that there are at least $c_2^n(h/2)^{n[K:\mathbb{Q}]}$ choices for $a_1, \ldots, a_n \in \mathcal{O}(K)_{h/2}$ and there are at most $C_2(C_1hH(u))^{[K:\mathbb{Q}]}$ possible values for $a_1u_1 + \cdots + a_nu_n$.

Now we take $h = R\ H(u)^{1/(n-1)}$ for a suitably large constant $R$ so that

$$c_2^n(h/2)^{n[K:\mathbb{Q}]} > C_2(C_1hH(u))^{[K:\mathbb{Q}]}.$$

Dirichlet's box principle implies that there are $b, \widetilde{b} \in (\mathcal{O}(K)_{h/2})^n$ such that $b \ne \widetilde{b}$ and

$$b_1u_1 + \cdots + b_nu_n = \widetilde{b}_1u_1 + \cdots + \widetilde{b}_nu_n.$$

The claim follows by taking $a_j = b_j - \widetilde{b}_j$. □

## 4. *Growth of eigencharacters*

Let $\alpha : \mathbb{Z}^l \to \mathrm{Aut}(\mathscr{N})$ be a morphism from $\mathbb{Z}^l$ to the group of automorphisms of a compact nilmanifold $\mathscr{N} = G/\Lambda$. We assume that $\alpha(z)$ acts ergodically on $\mathscr{N}$ for every $z \ne 0$ and estimate the growth of the eigencharacters of $\alpha$ acting on the abelianized Lie algebra of $G$.

As in §2, we set $\mathfrak{t} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and identify $\mathfrak{t}$ with some $\mathbb{R}^d$ so that the projection $Z_\Lambda$ of $\log \Lambda$ in $\mathfrak{t}$ corresponds to $\mathbb{Z}^d$. The representation $\alpha$ induces by differentiation a representation of $\mathbb{Z}^l$ on $\mathfrak{t}$ which preserves $Z_\Lambda$. We call it $d_\mathfrak{t}\alpha : \mathbb{Z}^l \to \mathrm{GL}_d^{\pm 1}(\mathbb{Z})$, where $\mathrm{GL}_d^{\pm 1}(\mathbb{Z})$ denotes the set of $d \times d$-matrices with coefficients in $\mathbb{Z}$ and determinant $\pm 1$. As $\mathbb{Z}^l$ is abelian, such a representation is triangularizable over $\overline{\mathbb{Q}}$: we can decompose $\overline{\mathbb{Q}}^d$ as a direct sum of subrepresentations

$$\overline{\mathbb{Q}}^d = \oplus_{\chi \in \mathscr{X}} L_\chi,$$

where $\mathscr{X}$ is a (finite) set of characters $\chi : \mathbb{Z}^l \to \overline{\mathbb{Q}}^\times$, and for each $\chi \in \mathscr{X}$, the generalized eigenspace

$$L_\chi := \{v \in \overline{\mathbb{Q}}^d \mid \text{for all } z \in \mathbb{Z}^l, (d_\mathfrak{t}\alpha(z) - \chi(z)\mathrm{Id})^d(v) = 0\}$$

is non-zero. Moreover, the Galois group $\mathcal{G} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting coordinate-wise on $\overline{\mathbb{Q}}^d$ permutes these eigenspaces according to the formula $\sigma.L_\chi = L_{\sigma\circ\chi}$.

The assumption of ergodicity on $\alpha$ can be reformulated as follows.

LEMMA 4.1. (Exponential growth of eigencharacters) *There exists $c > 0$ such that for every $\chi_0 \in \mathscr{X}$ and for every $z \in \mathbb{Z}^l \setminus \{0\}$,*

$$\max_{\chi \in \mathcal{G}.\chi_0} |\chi(z)| \ge e^{c\|z\|}.$$

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*Proof.* We recall the proof given in [10, Lemma 2.1]. Let $\chi_0 \in \mathscr{X}$. We claim that the morphism of $\mathbb{Z}$-modules

$$\psi : \mathbb{Z}^l \to \mathbb{R}^{|\mathcal{G}.\chi_0|}, \ z \mapsto (\log |\chi(z)|)_{\chi \in \mathcal{G}.\chi_0}$$

is injective with discrete image. Its $\mathbb{R}$-linear extension $\overline{\psi} : \mathbb{R}^l \to \mathbb{R}^{|\mathcal{G}.\chi_0|}$ is thus injective, whence satisfies $\|\overline{\psi}(z)\| \geq c_{\chi_0}\|z\|$ for some $c_{\chi_0} > 0$ and every $z \in \mathbb{Z}^l$. The lemma follows by finiteness of $\mathscr{X}$.

To prove the claim, we argue by contradiction assuming the existence of a sequence $(z_i) \in (\mathbb{Z}^l)^{\mathbb{N}}$ of non-zero vectors such that $\psi(z_i) \to 0$. This means that $|\chi(z_i)| \to 1$ for every $\chi \in \mathcal{G}.\chi_0$. In particular, the associated minimal polynomials $P_i = \prod_{\chi \in \mathcal{G}.\chi_0}(X - \chi(z_i)) \in \mathbb{Z}[X]$ have bounded coefficients, hence belong to a finite subset of $\mathbb{Z}[X]$. Necessarily, the sequence of vectors $(\chi(z_i))_{\chi \in \mathcal{G}.\chi_0}$ has a constant subsequence, yielding some $z \in \mathbb{Z}^l \setminus 0$ such that $|\chi(z)| = 1$ for every $\chi \in \mathcal{G}.\chi_0$. By Kronecker's theorem (see e.g. [2, Theorem 1.5.9]), $\chi_0(z)$ is a root of unity. This contradicts the ergodicity of the toral automorphism induced by $\alpha(z)$ on $G/[G, G]\Lambda$, whence that of $\alpha(z)$ on $G/\Lambda$. $\square$

## 5. *Proof of exponential n-mixing*

We now proceed to the proof of Theorem 1.1. Fix $n \geq 2, \theta \in (0, 1]$. Let $\underline{z} = (z_1, \ldots, z_n) \in (\mathbb{Z}^l)^n$ and consider some $\theta$-Hölder functions $f_1, \ldots, f_n \in \mathcal{H}^\theta(\mathscr{N})$ as well as translation parameters $g_1, \ldots, g_n \in G$. We begin with two observations.

- The integral at study can be rewritten as

$$\int_{\mathscr{N}} \prod_{i=1}^{n} f_i(g_i\alpha(z_i)x)\, dm(x) = \int_{\mathscr{S}} f_1 \otimes \cdots \otimes f_n \, dm_{\mathscr{S}},$$

  where $\mathscr{S}$ is an affine rational submanifold of $\mathscr{N}^n$, defined as the image of the embedding

$$\mathscr{N} \to \mathscr{N}^n, \ x \mapsto (g_1\alpha(z_1)x, \ldots, g_n\alpha(z_n)x).$$

- Equipping $\mathscr{N}^n$ with the product Riemannian metric on $\mathscr{N}$, we have

$$\|f_1 \otimes \cdots \otimes f_n\|_{\mathcal{H}^\theta} \leq \|f_1\|_{\mathcal{H}^\theta} \cdots \|f_n\|_{\mathcal{H}^\theta}.$$

These observations reduce Theorem 1.1 to showing that the submanifold $\mathscr{S}$ is $\delta$-equidistributed with respect to $\mathcal{H}^\theta(\mathscr{N}^n)$ for some $\delta < C/N(\underline{z})^\eta$, where $N(\underline{z}) = \exp(\min_{1 \leq i \neq j \leq n}\|z_i - z_j\|)$ and $C, \eta > 0$ are constants allowed to depend on the initial data $(G, \Lambda, \alpha, n, \theta)$ or possibly related structures, but not on $\underline{z}$ nor the $g_i$.

Denote by $L > 0$ the constant associated to the product nilmanifold $\mathscr{N}^n$ and $\theta$ as in Proposition 2.1. Let $\delta \in (0, 1/2)$ be such that $\mathscr{S}$ is not $\delta$-equidistributed in $\mathscr{N}^n$. By Proposition 2.1, there exists $q = (q_1, \ldots, q_n) \in (\mathbb{Z}^d)^n$ such that

$$0 < \|q\| < \left(\frac{1}{\delta}\right)^L \quad \text{and} \quad \langle q, d_{\mathfrak{t}}\alpha(\underline{z})\mathfrak{t}\rangle = 0,$$

where for $\omega \in \mathfrak{t}$, we set $d_{\mathfrak{t}}\alpha(\underline{z})\omega = (d_{\mathfrak{t}}\alpha(z_1)\omega, \ldots, d_{\mathfrak{t}}\alpha(z_n)\omega) \in \mathfrak{t}^n$, and recall that $d_{\mathfrak{t}}\alpha(z)$ is the projection of the differential of $\alpha(z)$ to $\mathfrak{t}$. Taking $\overline{\mathbb{Q}}$-linear combinations of the above equality, we get for every $\omega \in \overline{\mathbb{Q}}^d$,

$$\langle q, d_{\mathrm{t}}\alpha(\underline{z})\omega\rangle = 0. \tag{1}$$

Let us choose once and for all a basis $\beta_\chi$ of each generalized eigenspace $L_\chi$ for $\chi \in \mathscr{X}$ whose vectors have algebraic integer coefficients and such that $d_{\mathrm{t}}\alpha(\mathbb{Z}^l)$ is represented by upper-triangular matrices. As the $L_\chi$ span $\overline{\mathbb{Q}}^d$ and $q$ is non-zero, there must exist $\chi_0 \in \mathscr{X}$ such that $\langle q_i, L_{\chi_0}\rangle \neq \{0\}$ for some $i \in \{1, \ldots, n\}$. We let $\omega_0$ be the first element of the basis $\beta_{\chi_0}$ such that $\langle q_i, \omega_0\rangle \neq 0$ for some $i$. We can then write for every $i$,

$$\langle q_i, d_{\mathrm{t}}\alpha(z_i)\omega_0\rangle = \chi_0(z_i)\langle q_i, \omega_0\rangle$$

and equation (1) with $\omega = \omega_0$ yields

$$\sum_{i=1}^{n} \chi_0(z_i)\langle q_i, \omega_0\rangle = 0.$$

We let $K$ be the number field generated by the coefficients of the vectors belonging to the basis $(\beta_\chi)_{\chi\in\mathscr{X}}$. For every $i$, we then have $\langle q_i, \omega_0\rangle \in \mathcal{O}(K)$ and $\chi_0(z_i) \in \mathcal{O}(K)^\times$ (eigenvalues of the matrix $d_{\mathrm{t}}\alpha(z_i) \in \mathrm{GL}_d^{\pm 1}(\mathbb{Z})$ are algebraic units, and the relation $d_{\mathrm{t}}\alpha(z_i)\omega_0 = \chi_0(z_i)w_0$ forces $\chi_0(z_i)$ to be in $K$). Fix $\varepsilon \in (0, 1)$ arbitrarily. Theorem 3.1 yields a constant $r > 0$ depending only on $(K, n, \varepsilon)$ such that, for some $i_0 \in \{1, \ldots, n\}$ and some Galois automorphism $\sigma \in \mathcal{G}$,

$$|\langle q_{i_0}, \sigma(\omega_0)\rangle| \geq r\alpha(u)^{1-\varepsilon}, \tag{2}$$

where $u = (\chi_0(z_1), \ldots, \chi_0(z_n))$. The exponential growth of eigencharacters presented in Lemma 4.1 yields a lower bound on $\alpha(u)$:

$$\begin{aligned}
\alpha(u) &\geq \min_{1\leq i\neq j\leq n} (H(\chi_0(z_i), \chi_0(z_j)))^{1/(n-1)} \\
&= \min_{1\leq i\neq j\leq n} \prod_{v\in M_{K,\infty}} \max(|\chi_0(z_i - z_j)|_v, 1)^{1/((n-1)[K:\mathbb{Q}])} \\
&\geq \min_{1\leq i\neq j\leq n} \exp\left(\frac{c\|z_i - z_j\|}{(n-1)[K:\mathbb{Q}]}\right).
\end{aligned}$$

Plugging this lower bound into equation (2) and using the Cauchy–Schwarz inequality, we get

$$\|q\| \geq rc_1 \exp\left(\frac{c(1-\varepsilon)}{(n-1)[K:\mathbb{Q}]} \min_{1\leq i\neq j\leq n} \|z_i - z_j\|\right),$$

where $c_1 = \min_{\sigma\in\mathcal{G}} \|\sigma(\omega_0)\|^{-1}$ only depends on our choice of $(\beta_\chi)_{\chi\in\mathscr{X}}$. Recalling that $\delta^{-L} > \|q\|$, we obtain that

$$\delta < rc_1 N(\underline{z})^{-\eta}, \quad \text{where} \quad \eta := \frac{c(1-\varepsilon)}{(n-1)[K:\mathbb{Q}]L}.$$

To sum up the above discussion, we have proven that for any $\underline{z} \in (\mathbb{Z}^l)^n$ satisfying $rc_1 N(\underline{z})^{-\eta} < 1/2$, the submanifold $\mathscr{S}$ of $\mathscr{N}^n$ is $rc_1 N(\underline{z})^{-\eta}$-equidistributed for $\mathcal{H}^\theta(\mathscr{N}^n)$.

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It follows that for all $\underline{z} \in (\mathbb{Z}^l)^n$ and all $f_1, \ldots, f_n \in \mathcal{H}^\theta(\mathcal{N})$, $g_1, \ldots, g_n \in G$, we have

$$\left| \int_{\mathcal{N}} \prod_{i=1}^{n} f_i(g_i \alpha(z_i) x) \, dm(x) - \prod_{i=1}^{n} \int_{\mathcal{N}} f_i(x) \, dm(x) \right| \leq \frac{4 r c_1}{N(\underline{z})^\eta} \prod_{i=1}^{n} \|f_i\|_{\mathcal{H}^\theta}$$

and this concludes the proof. (The constant 4 on the right is needed to get a valid statement when $r c_1 N(\underline{z})^{-\eta} \geq 1/2$.)

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