

AN INVARIANT REGARDING WARING'S PROBLEM FOR CUBIC POLYNOMIALS

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to the memory of Michael Schneider, ten years after

Abstract. We compute the equation of the 7-secant variety to the Veronese variety $(\mathbf{P}^4, \mathcal{O}(3))$, its degree is 15. This is the last missing invariant in the Alexander-Hirschowitz classification. It gives the condition to express a homogeneous cubic polynomial in 5 variables as the sum of 7 cubes (Waring problem). The interesting side in the construction is that it comes from the determinant of a matrix of order 45 with linear entries, which is a cube. The same technique allows to express the classical Aronhold invariant of plane cubics as a pfaffian.

§1. Introduction

We work over an algebraically closed field K of characteristic zero. The Veronese variety, given by \mathbf{P}^n embedded with the linear system $|\mathcal{O}(d)|$, lives in \mathbf{P}^N where $N = \binom{n+d}{d} - 1$. It parametrizes the homogeneous polynomials f of degree d in $n + 1$ variables which are the power of a linear form g , that is $f = g^d$.

Let $\sigma_s(\mathbf{P}^n, \mathcal{O}(d))$ be the s -secant variety of the Veronese variety, that is the Zariski closure of the variety of polynomials f which are the sum of the powers of s linear forms g_i , i.e. $f = \sum_{i=1}^s g_i^d$. In particular $\sigma_1(\mathbf{P}^n, \mathcal{O}(d)) = (\mathbf{P}^n, \mathcal{O}(d))$ is the Veronese variety itself and $\sigma_2(\mathbf{P}^n, \mathcal{O}(d))$ is the usual secant variety. For generalities about the Waring's problem for polynomials see [IK] or [RS].

Our starting point is the theorem of Alexander and Hirschowitz (see [AH] or [BO] for a survey, including a self-contained proof) which states that the codimension of $\sigma_s(\mathbf{P}^n, \mathcal{O}(d)) \subseteq \mathbf{P}^N$ is the expected one, that is $\max\{N + 1 - (n + 1)s, 0\}$, with the only exceptions

- (i) $\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$, $2 \leq k \leq n$

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- (ii) $\sigma_{\frac{1}{2}n(n+3)}(\mathbf{P}^n, \mathcal{O}(4))$, $n = 2, 3, 4$
- (iii) $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$

The case (i) corresponds to the matrices of rank $\leq k$ in the variety of symmetric matrices of order $n + 1$. In the cases (ii) and (iii) the expected codimension is zero, while the codimension is one. Hence the equation of the hypersurface $\sigma_s(\mathbf{P}^n, \mathcal{O}(d))$ in these cases is an interesting $SL(n + 1)$ -invariant. In the cases (ii) it is the catalecticant invariant, that was computed by Clebsch in the 19th century, its degree is $\binom{n+2}{2}$.

The main result of this paper is the computation of the equation of $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$. This was left as an open problem in [IK, Chap. 2, Rem. 2.4].

We consider a vector space V . For any nonincreasing sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots)$ it is defined the Schur module $\Gamma^\alpha V$, which is an irreducible $SL(V)$ -module (see [FH]). For $\alpha = (p)$ we get the p -th symmetric power of V and for $\alpha = (1, \dots, 1)$ (p times) we get the p -th alternating power of V . The module $\Gamma^\alpha V$ is visualized as a Young diagram containing α_i boxes in the i -th row. In particular if $\dim V = 5$ then $\Gamma^{2,2,1,1}V$ and its dual $\Gamma^{2,1,1}V$ have both dimension 45.

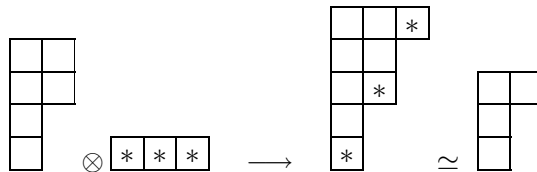
Our main result is the following

THEOREM 1.1. *Let V be a vector space of dimension 5. For any $\phi \in S^3V$, let $B_\phi: \Gamma^{2,2,1,1}V \rightarrow \Gamma^{2,1,1}V$ be the $SL(V)$ -invariant contraction operator. Then there is an irreducible homogeneous polynomial P of degree 15 on S^3V such that*

$$2P(\phi)^3 = \det B_\phi$$

The polynomial P is the equation of $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$.

The coefficient 2 is needed because we want the invariant polynomials to be defined over the rational numbers. The picture in terms of Young diagrams is



This picture means that $\Gamma^{2,1,1}V$ is a direct summand of the tensor product $\Gamma^{2,2,1,1}V \otimes S^3V$, according to the Littlewood-Richardson rule ([FH]).

The polynomial P gives the necessary condition to express a cubic homogeneous polynomial in five variables as a sum of seven cubes. We prove in Lemma 3.2 that if ϕ is decomposable then $\text{rk}(B_\phi) = 6$. The geometrical explanation that $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$ is an exceptional case is related to the fact that given seven points in \mathbf{P}^4 there is a unique rational normal curve through them, and it was discovered independently by Richmond and Palatini in 1902, see [CH] for a modern reference. Our approach gives a different (algebraic) proof of the fact that $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$ is an exceptional case. Another argument, by using syzygies, is in [RS]. B. Reichstein found in [Re] an algorithm to check when a cubic homogeneous polynomial in five variables is the sum of seven cubes, see the Remark 3.4.

The resulting table of the Alexander-Hirschowitz classification is the following

	<i>exp. codim</i>	<i>codim</i>	<i>equation</i>
$\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$ $2 \leq k \leq n$	$\max(\frac{(n+1)(n+2-2k)}{2}, 0)$	$\binom{n-k+2}{2}$	$(k+1) - \text{minors}$
$\sigma_{\frac{1}{2}n(n+3)}(\mathbf{P}^n, \mathcal{O}(4))$ $n = 2, 3, 4$	0	1	catalecticant inv.
$\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$	0	1	see Theorem 1.1

The degree of $\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$ was computed by C. Segre, it is equal to $\prod_{i=0}^{n-k} \binom{n+1+i}{n+1-k-i} / \binom{2i+1}{i}$. We will use in the proof of Theorem 1.1 the fact that $\sigma_{k-1}(\mathbf{P}^n, \mathcal{O}(2))$ is the singular locus of $\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$ for $k \leq n$.

A general cubic polynomial in five variables can be expressed as a sum of eight cubes in ∞^5 ways, parametrized by a Fano 5-fold of index one (see [RS]). A cubic polynomial in five variables which can be expressed as a sum of seven cubes was called degenerate in [RS], hence what we have found is the locus of degenerate cubics. A degenerate cubic in five variables can be expressed as a sum of seven cubes in ∞^1 ways, parametrized by \mathbf{P}^1 (see [RS, 4.2]).

To explain our technique, we consider the Aronhold invariant of plane cubics.

The Aronhold invariant is the degree 4 equation of $\sigma_3(\mathbf{P}^2, \mathcal{O}(3))$, which can be seen as the $SL(3)$ -orbit of the Fermat cubic $x_0^3 + x_1^3 + x_2^3$ (sum of three cubes), see [St, Prop. 4.4.7] or [DK, (5.13.1)].

Let W be a vector space of dimension 3. In particular $\Gamma^{2,1}W = \text{ad } W$ is self-dual and it has dimension 8. We get

THEOREM 1.2. *For any $\phi \in S^3W$, let $A_\phi: \Gamma^{2,1}W \rightarrow \Gamma^{2,1}W$ be the $SL(W)$ -invariant contraction operator. Then A_ϕ is skew-symmetric and the pfaffian $Pf A_\phi$ is the equation of $\sigma_3(\mathbf{P}(W), \mathcal{O}(3))$, i.e. it is the Aronhold invariant.*

The corresponding picture is



The Aronhold invariant gives the necessary condition to express a cubic homogeneous polynomial in three variables as a sum of three cubes. The explicit expression of the Aronhold invariant is known since the 19th century, but we have not found in the literature its representation as a pfaffian. In the Remark 2.3 we apply this representation to the Scorza map between plane quartics.

In Section 2 we give the proof of Theorem 1.2. This is introductory to Theorem 1.1, which is proved in Section 3. In Section 4 we review, for completeness, some known facts about the catalecticant invariant of quartic hypersurfaces.

We are indebted to S. Sullivant, for his beautiful lectures at Nordfjordeid in 2006 about [SS], where a representation of the Aronhold invariant is found with combinatorial techniques.

§2. The Aronhold invariant as a pfaffian

Let e_0, e_1, e_2 be a basis of W and fix the orientation $\bigwedge^3 W \simeq K$ given by $e_0 \wedge e_1 \wedge e_2$. We have $\text{End } W = \text{ad } W \oplus K$. The $SL(W)$ -module $\text{ad } W = \Gamma^{2,1}(W)$ consists of the subspace of endomorphisms of W with zero trace. We may interpret the contraction

$$A_\phi: \Gamma^{2,1}W \longrightarrow \Gamma^{2,1}W$$

as the restriction of a linear map $A'_\phi: \text{End } W \rightarrow \text{End } W$, which is defined for $\phi = e_{i_1}e_{i_2}e_{i_3}$ as

$$A'_{e_{i_1}e_{i_2}e_{i_3}}(M)(w) = \sum_{\sigma} (M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge w)e_{i_{\sigma(3)}}$$

where $M \in \text{End } W, w \in W$ and σ covers the symmetric group Σ_3 .

Then A'_ϕ is defined for a general ϕ by linearity, and it follows from the definition that it is $SL(V)$ -invariant.

The Killing scalar product on $\text{End } W$ is defined by $\text{tr}(M \cdot N)$.

LEMMA 2.1. (i) $\text{Im}(A'_\phi) \subseteq \text{ad } W \quad K \subseteq \text{Ker}(A'_\phi)$
 (ii) A'_ϕ is skew-symmetric.

Proof. (i) follows from

$$\begin{aligned} \text{tr} \left[A_{e_{i_1} e_{i_2} e_{i_3}}(M) \right] &= \sum_s A_{e_{i_1} e_{i_2} e_{i_3}}(M)(e_s) e_s^\vee \\ &= \sum_\sigma (M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge e_{i_{\sigma(3)}}) = 0 \end{aligned}$$

The second inclusion is evident. To prove (ii), we have to check that

$$\text{tr}(A_\phi(M) \cdot N) = -\text{tr}(A_\phi(N) \cdot M)$$

for $M, N \in \text{End } W$. Indeed let $\phi = e_{i_1} e_{i_2} e_{i_3}$. We get

$$\begin{aligned} \text{tr}(A_{e_{i_1} e_{i_2} e_{i_3}}(M) \cdot N) &= \sum_s A_{e_{i_1} e_{i_2} e_{i_3}}(M)(N(e_s)) e_s^\vee \\ &= \sum_\sigma M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge N(e_{i_{\sigma(3)}}) \end{aligned}$$

which is alternating in M and N , where we denoted by e_i^\vee the dual basis. \square

It follows from Lemma 2.1 that the restriction

$$A'_\phi|_{\text{ad } W} : \text{ad } W \longrightarrow \text{ad } W$$

coincides, up to scalar multiple, with the contraction operator A_ϕ of Theorem 1.2 and it is skew-symmetric.

LEMMA 2.2. Let $\phi = w^3$ with $w \in W$. Then $\text{rk } A_\phi = 2$. More precisely

$$\begin{aligned} \text{Im } A_{w^3} &= \{M \in \text{ad } W \mid \text{Im } M \subseteq \langle w \rangle\} \\ \text{Ker } A_{w^3} &= \{M \in \text{ad } W \mid w \text{ is an eigenvector of } M\} \end{aligned}$$

Proof. The statement follows from the equality

$$A_{w^3}(M)(v) = 6(M(w) \wedge w \wedge v)w$$

As an example, note that $\text{Im } A_{e_0^3} = \langle e_0 \otimes e_1^\vee, e_0 \otimes e_2^\vee \rangle$ and $\text{Ker } A_{e_0^3}$ is spanned by all the basis monomials, with the exception of $e_0^\vee \otimes e_1$ and $e_0^\vee \otimes e_2$. Due to the $SL(W)$ -invariance, this example proves the general case. \square

Proof of Theorem 1.2. Let $\phi \in \sigma_3(\mathbf{P}(W), \mathcal{O}(3))$. By the definition of higher secant variety, ϕ is in the closure of elements which can be written as $\phi_1 + \phi_2 + \phi_3$ with $\phi_i \in (\mathbf{P}(W), \mathcal{O}(3))$. From Lemma 2.2 it follows that

$$\text{rk } A_\phi \leq \text{rk } A_{\sum_{i=1}^3 \phi_i} = \text{rk } \sum_{i=1}^3 A_{\phi_i} \leq \sum_{i=1}^3 \text{rk } A_{\phi_i} = 2 \cdot 3 = 6$$

Hence $Pf(A_\phi)$ has to vanish on $\sigma_3(\mathbf{P}(W), \mathcal{O}(3))$.

Write a cubic polynomial as

$$\begin{aligned} \phi = & v_{000}x_0^3 + 3v_{001}x_0^2x_1 + 3v_{002}x_0^2x_2 + 3v_{011}x_0x_1^2 + 6v_{012}x_0x_1x_2 \\ & + 3v_{022}x_0x_2^2 + v_{111}x_1^3 + 3v_{112}x_1^2x_2 + 3v_{122}x_1x_2^2 + v_{222}x_2^3 \end{aligned}$$

We order the monomial basis of $\wedge^2 W \otimes W$ with the lexicographical order in the following way:

$$\begin{aligned} & (w_0 \wedge w_1)w_0, (w_0 \wedge w_1)w_1, (w_0 \wedge w_1)w_2, \\ & (w_0 \wedge w_2)w_0, (w_0 \wedge w_2)w_1, (w_0 \wedge w_2)w_2, \\ & (w_1 \wedge w_2)w_0, (w_1 \wedge w_2)w_1, (w_1 \wedge w_2)w_2 \end{aligned}$$

Call M_i for $i = 1, \dots, 9$ this basis. The matrix of A'_ϕ , with respect to this basis, has at the entry (i, j) the value $A'_\phi(M_j)(M_i)$ and it is the following

$$\begin{bmatrix} 0 & v_{222} & -v_{122} & 0 & -v_{122} & v_{112} & 0 & v_{022} & -v_{012} \\ -v_{222} & 0 & v_{022} & v_{122} & 0 & -v_{012} & -v_{022} & 0 & v_{002} \\ v_{122} & -v_{022} & 0 & -v_{112} & v_{012} & 0 & v_{012} & -v_{002} & 0 \\ 0 & -v_{122} & v_{112} & 0 & v_{112} & -v_{111} & 0 & -v_{012} & v_{011} \\ v_{122} & 0 & -v_{012} & -v_{112} & 0 & v_{011} & v_{012} & 0 & -v_{001} \\ -v_{112} & v_{012} & 0 & v_{111} & -v_{011} & 0 & -v_{011} & v_{001} & 0 \\ 0 & v_{022} & -v_{012} & 0 & -v_{012} & v_{011} & 0 & v_{002} & -v_{001} \\ -v_{022} & 0 & v_{002} & v_{012} & 0 & -v_{001} & -v_{002} & 0 & v_{000} \\ v_{012} & -v_{002} & 0 & -v_{011} & v_{001} & 0 & v_{001} & -v_{000} & 0 \end{bmatrix}$$

Deleting one of the columns corresponding to $(w_0 \wedge w_1)w_2$, $(w_0 \wedge w_2)w_1$ or $(w_1 \wedge w_2)w_0$ (respectively the 3rd, the 5th and the 7th, indeed their alternating sum gives the trace), and the corresponding row, we get a skew-symmetric matrix of order 8 which is the matrix of A_ϕ . To conclude the proof, it is enough to check that the pfaffian is nonzero. This can be easily checked on the point corresponding to $\phi = x_0x_1x_2$, that is when $v_{012} = 1$ and all the other coordinates are equal to zero. This means that any triangle is not in the closure of the Fermat curve. we conclude that $Pf(A_\phi)$ is the Aronhold invariant. We verified that it coincides, up to a constant, with the expression given in [St, Prop. 4.4.7] or in [DK, (5.13.1)]. \square

The vanishing of the Aronhold invariant gives the necessary and sufficient condition to express a cubic polynomial in three variables as the sum of three cubes.

Remark. A'_ϕ can be thought as a map

$$A'_\phi: \Lambda^2 W \otimes W \longrightarrow \Lambda^2 W^\vee \otimes W^\vee$$

For $\phi = w^3$ we have the formula

$$A'_\phi(\omega \otimes v)(\omega' \otimes v') = (\omega \wedge w) \otimes (v \wedge w \wedge v') \otimes (\omega' \wedge w)$$

This is important for the understanding of the next section.

Remark. We have the decomposition

$$\Lambda^2(\Gamma^{2,1}W) = S^3W \oplus \Gamma^{2,2,2}W \oplus \text{ad } W$$

and it is a nice exercise to show the behaviour of the three summands. For the first one

$$S^3W \cap \{M \in \Lambda^2(\Gamma^{2,1}W) \mid \text{rk}(M) \leq 2k\}$$

is the cone over $\sigma_k(\mathbf{P}(W), \mathcal{O}(3))$, so that we have found the explicit equations for all the higher secant varieties to $(\mathbf{P}(W), \mathcal{O}(3))$. The secant variety $\sigma_2(\mathbf{P}(W), \mathcal{O}(3))$ is the closure of the orbit of plane cubics consisting of three concurrent lines, and its equations are the 6×6 subpfaffians of A_ϕ . It has degree 15. There is a dual description for $\Gamma^{2,2,2}W$.

For the third summand, we have that

$$\text{ad } W \subseteq \{M \in \Lambda^2(\Gamma^{2,1}W) \mid \text{rk}(M) \leq 6\}$$

Indeed any $M \in \text{ad } W$ induces the skew-symmetric morphism

$$[M, -]$$

whose kernel contains M . Moreover

$$\text{ad } W \cap \{M \in \Lambda^2(\Gamma^{2,1}W) \mid \text{rk}(M) \leq 4\}$$

is the 5-dimensional affine cone consisting of endomorphisms $M \in \text{ad } W$ such that their minimal polynomial has degree ≤ 2 .

Remark 2.3. We recall from [DK] the definition of the Scorza map. Let A be the Aronhold invariant. For any plane quartic F and any point $x \in \mathbf{P}(W)$ we consider the polar cubic $P_x(F)$. Then $A(P_x(F))$ is a quartic in the variable x which we denote by $S(F)$. The rational map $S: \mathbf{P}(S^4W) \dashrightarrow \mathbf{P}(S^4W)$ is called the Scorza map. Our description of the Aronhold invariant shows that $S(F)$ is defined as the degeneracy locus of a skew-symmetric morphism on $\mathbf{P}(W)$

$$\mathcal{O}(-2)^8 \xrightarrow{f} \mathcal{O}(-1)^8$$

It is easy to check (see [Be]) that $\text{Coker } f = E$ is a rank two vector bundle over $S(F)$ such that $c_1(E) = K_{S(F)}$. Likely from E it is possible to recover the even theta-characteristic θ on $S(F)$ defined in [DK, (7.7)]. The natural guess is that

$$h^0(E \otimes (-\theta)) > 0$$

for a unique even θ , but we do not know if this is true.

§3. The invariant for cubic polynomials in five variables

Let now e_0, \dots, e_4 be a basis of V , no confusion will arise with the notations of the previous section. We fix the orientation $\Lambda^5 V \simeq K$ given by $e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We construct, for $\phi \in S^3V$, the contraction operator

$$B'_\phi: \Lambda^4 V \otimes \Lambda^2 V \longrightarrow \Lambda^4 V^\vee \otimes \Lambda^2 V^\vee \simeq \Lambda^3 V \otimes V$$

If $\phi = e_{i_1}e_{i_2}e_{i_3}$, the definition is

$$\begin{aligned} & B'_\phi(v_a \wedge v_b \wedge v_c \wedge v_d) \otimes (v_e \wedge v_f) \\ &= \sum_\sigma \left(v_a \wedge v_b \wedge v_c \wedge v_d \wedge e_{i_{\sigma(1)}} \right) \otimes \left(v_e \wedge v_f \wedge e_{i_{\sigma(2)}} \right) \otimes e_{i_{\sigma(3)}} \end{aligned}$$

where σ covers the symmetric group Σ_3 and we extend this definition, to a general ϕ , by linearity.

We may interpret B'_ϕ as a morphism

$$B'_\phi: Hom(V, \wedge^2 V) \longrightarrow Hom(\wedge^2 V, V)$$

If $\phi = e_{i_1} e_{i_2} e_{i_3}$ and $M \in Hom(V, \wedge^2 V)$ we have

$$B'_{e_{i_1} e_{i_2} e_{i_3}}(M)(v_1 \wedge v_2) = \sum_{\sigma} (M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge v_1 \wedge v_2) e_{i_{\sigma(3)}}$$

We have a $SL(V)$ -decomposition

$$\wedge^4 V \otimes \wedge^2 V = \Gamma^{2,2,1,1} V \oplus V$$

Consider the contraction $c: \wedge^4 V \otimes \wedge^2 V \rightarrow V$ defined by

$$c(\omega \otimes (v_i \wedge v_j)) = (\omega \wedge v_i) v_j - (\omega \wedge v_j) v_i$$

Then the subspace $\Gamma^{2,2,1,1} V$ can be identified with

$$\{M \in \wedge^4 V \otimes \wedge^2 V \mid c(M) = 0\}$$

or with

$$\{M \in Hom(V, \wedge^2 V) \mid \sum e_i^\vee M(e_i) = 0\}$$

The subspace $V \subset Hom(V, \wedge^2 V)$ can be identified with $\{v \wedge - \mid v \in V\}$. At the same time we have a $SL(V)$ -decomposition

$$V \otimes \wedge^3 V = \Gamma^{2,1,1} V \oplus \wedge^4 V$$

and the obvious contraction $d: V \otimes \wedge^3 V \rightarrow \wedge^4 V$. The subspace $\Gamma^{2,1,1} V$ can be identified with

$$\{N \in V \otimes \wedge^3 V \mid d(N) = 0\}$$

LEMMA 3.1. (i) $Im(B'_\phi) \subseteq \Gamma^{2,1,1} V \quad V \subseteq Ker(B'_\phi)$
 (ii) B'_ϕ is symmetric.

Proof. The statement (i) follows from the formula

$$\begin{aligned} d\left(B'_{e_{i_1}e_{i_2}e_{i_3}}(v_a \wedge v_b \wedge v_c \wedge v_d) \otimes (v_e \wedge v_f)\right) \\ = \sum_{\sigma} \left(v_a \wedge v_b \wedge v_c \wedge v_d \wedge e_{i_{\sigma(1)}}\right) \otimes \left(v_e \wedge v_f \wedge e_{i_{\sigma(2)}} \wedge e_{i_{\sigma(3)}}\right) = 0 \end{aligned}$$

In order to prove the second inclusion, for any $v \in V$ consider the induced morphism $M_v(w) = v \wedge w$. We get

$$B'_{e_{i_1}e_{i_2}e_{i_3}}(M_v)(v_1 \wedge v_2) = \sum_{\sigma} \left(v \wedge e_{i_{\sigma(1)}} \wedge e_{i_{\sigma(2)}} \wedge v_1 \wedge v_2\right) e_{i_{\sigma(3)}} = 0$$

In order to prove (ii) we may assume $\phi = v^3$.

We need to prove that

$$B'_{v^3}(\omega \otimes \xi)(\omega' \otimes \xi') = B'_{v^3}(\omega' \otimes \xi')(\omega \otimes \xi)$$

for every $\omega, \omega' \in \wedge^4 V$ and $\xi, \xi' \in \wedge^2 V$. Indeed

$$B'_{v^3}(\omega \otimes \xi)(\omega' \otimes \xi') = (\omega \wedge v) \otimes (\xi \wedge v \wedge \xi') \otimes (v \wedge \omega')$$

which is symmetric in the pair (ω, ξ) . □

It follows from Lemma 3.1 that the restriction $B'_{\phi}|_{\Gamma^{2,2,1,1}} : \Gamma^{2,2,1,1} \rightarrow \Gamma^{2,1,1}V$ coincides, up to scalar multiple, with the contraction B_{ϕ} of the Theorem 1.1 and it is symmetric. Note that

$$\text{Ker}(B_{\phi}) = \text{Ker}(B'_{\phi})/V \quad \text{Im}(B_{\phi}) = \text{Im}(B'_{\phi})$$

LEMMA 3.2. *Let $\phi = v^3$ with $v \in V$. Then $\text{rk } B_{\phi} = 6$. More precisely*

$$\begin{aligned} \text{Im } B_{v^3} = \{N \in \text{Hom}(\wedge^2 V, V) \mid \sum e_i^{\vee} N(e_i \wedge v) = 0, \\ \forall v \in V, \text{Im}(N) \subseteq \langle v \rangle\} \end{aligned}$$

$$\text{Ker } B_{v^3} = \{M \in \text{Hom}(V, \wedge^2 V) \mid \sum e_i^{\vee} M(e_i) = 0, M(v) \subseteq v \wedge V\}$$

Proof. The statement follows from the equality

$$B_{v^3}(M)(v_1 \wedge v_2) = 6(M(v) \wedge v \wedge v_1 \wedge v_2)v$$

As an example, a basis of $\text{Im } B_{e_0^3}$ is given by $e_0 \otimes (e_i^{\vee} \wedge e_j^{\vee})$ for $1 \leq i < j \leq 4$ and a basis of $\text{Ker } B_{e_0^3}$ is given by all the basis monomials with the exceptions of $e_0^{\vee} \otimes (e_i \wedge e_j)$ for $1 \leq i < j \leq 4$. Due to the $SL(V)$ -invariance, this example proves the general case. □

We write $\phi \in S^3V$ as $\phi = v_{000}x_0^3 + 3v_{001}x_0^2x_1 + \dots + v_{444}x_4^3$.

LEMMA 3.3. *Every $SL(V)$ -invariant homogeneous polynomial of degree 15 on S^3V which contains the monomial*

$$v_{000}^2v_{012}^3v_{111}v_{223}^3v_{334}^3v_{144}^3$$

is irreducible.

Proof. Let t_0, \dots, t_4 be the canonical basis of \mathbb{Z}^5 . We denote by $t_i + t_j + t_k$ the weight of the monomial v_{ijk} , according to [St]. For example the weight of v_{000} is $(3, 0, 0, 0, 0)$. We denote the first component of the weight as the x_0 -weight, the second component as the x_1 -weight, and so on. We recall that every $SL(V)$ -invariant polynomial is isobaric, precisely every monomial of a $SL(V)$ -invariant polynomial of degree $5k$ has weight $(3k, 3k, 3k, 3k, 3k)$ (see [St, (4.4.14)]), this follows from the invariance with respect to the diagonal torus. We claim that there is no isobaric monomial of weight $(6, 6, 6, 6, 6)$ and degree 10 with variables among $v_{000}, v_{012}, v_{111}, v_{223}, v_{334}, v_{144}$. We divide into the following cases, by looking at the possibilities for the x_0 -weight:

- i) The monomial contains v_{000}^2 and does not contain v_{012} . By looking at the x_2 -weight, the monomial has to contain v_{223}^3 , which gives contribution 3 to the x_3 -weight. This gives a contradiction, because from v_{334} the possible values for the x_3 -weight are even, and we never make 6.
- ii) The monomial contains $v_{000}v_{012}^3$ and not higher powers. This monomial gives contribution 3 to the x_2 -weight. From v_{223} the possible values for the x_2 -weight are even, and we never make 6, again.
- iii) The monomial contains v_{012}^6 and does not contain v_{000} . This monomial gives contribution 6 to the x_0 -weight, and the same contribution is given to the x_1 -weight and to the x_2 -weight. Hence the only other possible monomial that we are allowed to use is v_{334} , which gives a x_3 -weight doubled with respect to the x_4 -weight, which is a contradiction.

This contradiction proves our claim. Nevertheless, if our polynomial is reducible, also its factors have to be homogeneous and $SL(V)$ -invariant, and the monomial in the statement should split into two factors of degree 5 and 10, against the claim. □

Proof of Theorem 1.1. Let $\phi \in \sigma_7(\mathbf{P}(V), \mathcal{O}(3))$. By the definition of higher secant variety, ϕ is in the closure of elements which can be written as $\sum_{i=1}^7 \phi_i$ with $\phi_i \in (\mathbf{P}(V), \mathcal{O}(3))$. From Lemma 3.2 it follows that

$$\text{rk } B_\phi \leq \text{rk } B_{\sum_{i=1}^7 \phi_i} = \text{rk } \sum_{i=1}^7 B_{\phi_i} \leq \sum_{i=1}^7 \text{rk } B_{\phi_i} = 6 \cdot 7 = 42$$

Hence $\det(B_\phi)$ has to vanish on $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$.

We order the monomial basis of S^3V with the lexicographical order induced by $x_0 < x_1 < x_2 < x_3 < x_4$. We order also the basis of $\wedge^2 V \otimes \wedge^4 V$ with the lexicographical order. There are 50 terms, beginning with

$$(e_0 \wedge e_1) \otimes (e_0 \wedge e_1 \wedge e_2 \wedge e_3), (e_0 \wedge e_1) \otimes (e_0 \wedge e_1 \wedge e_2 \wedge e_4), \dots$$

and ending with

$$\dots, (e_3 \wedge e_4) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4)$$

These 50 terms are divided into 10 blocks, depending on the first factor $e_s \wedge e_t$. The matrix of B'_ϕ , with respect to this basis, is a 50×50 symmetric matrix with linear monomial entries from v_{ijk} .

We describe this matrix in block form. For $i = 0, \dots, 4$ let A_i be the 5×5 symmetric matrix which at the entry $(5 - s, 5 - t)$ has $(-1)^{s+t} v_{ist}$, corresponding to the monomial $x_i x_s x_t$. For example

$$A_4 = \begin{bmatrix} v_{444} & -v_{344} & v_{244} & -v_{144} & v_{044} \\ -v_{344} & v_{334} & -v_{234} & v_{134} & -v_{034} \\ v_{244} & -v_{234} & v_{224} & -v_{124} & v_{024} \\ -v_{144} & v_{134} & -v_{124} & v_{114} & -v_{014} \\ v_{044} & -v_{034} & v_{024} & -v_{014} & v_{004} \end{bmatrix}$$

Then the matrix of B'_ϕ has the following block form

$$\begin{bmatrix} & & & & & & A_4 & -A_3 & A_2 \\ & & & & -A_4 & A_3 & & & -A_1 \\ & & & A_4 & -A_2 & & & A_1 & \\ & & -A_3 & A_2 & & -A_1 & & & \\ & & & & & & & & A_0 \\ & & A_4 & -A_3 & & & & & \\ & -A_4 & & A_2 & & & & -A_0 & \\ & & A_3 & -A_2 & & & & & \\ & A_4 & & -A_1 & & & & & \\ -A_3 & & A_1 & & -A_0 & & & & \\ A_2 & -A_1 & & & & & & & \end{bmatrix}$$

Among the 50 basis elements, there are 30 tensors $(e_s \wedge e_t) \otimes (e_i \wedge e_j \wedge e_k \wedge e_l)$ such that $\{s, t\} \subseteq \{i, j, k, l\}$. The other 20 elements are divided into 5 groups, depending on the single index $\{s, t\} \cap \{i, j, k, l\}$. The contraction c maps the first group of 30 elements into 30 independent elements of $\Gamma^{2,2,1,1}V$, and each group of 4 elements has the image through c of dimension 3 in $\Gamma^{2,2,1,1}V$, indeed the images of the 4 elements satisfy a linear relation with ± 1 coefficients.

It follows that the matrix of B_ϕ can be obtained from the matrix of B'_ϕ by deleting five rows, one for each of the above groups, and the corresponding five columns. We can delete, for example, the columns and the rows corresponding to

$$\begin{aligned} &(e_0 \wedge e_1) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4), \quad (e_0 \wedge e_2) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4), \\ &(e_0 \wedge e_3) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4), \quad (e_0 \wedge e_4) \otimes (e_0 \wedge e_1 \wedge e_2 \wedge e_3), \\ &(e_0 \wedge e_4) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4) \end{aligned}$$

which have respectively number 5, 10, 15, 16, 20. Note that in the resulting matrix for B_ϕ , all entries are monomials in v_{ijk} with coefficient ± 1 .

In order to show that for general ϕ the morphism B_ϕ is invertible, the simplest way is to look at the monomial $(v_{001}v_{022}v_{113}v_{244}v_{334})^9$ which appears with nonzero coefficient in the expression of $\det B_\phi$. We prefer instead to use the monomial appearing in the statement of Lemma 3.3, which allows to prove the stronger statement that $\det B_\phi$ is the cube of an irreducible polynomial. Indeed, by substituting 0 to all the variables different from $v_{000}, v_{012}, v_{111}, v_{223}, v_{334}, v_{144}$, we get by an explicit computation that the determinant is equal to

$$-2(v_{000}^2 v_{012}^3 v_{111} v_{223}^3 v_{334}^3 v_{144}^3)^3$$

Hence for general ϕ we have $\text{rk } B_\phi = 45$. Note that this gives an alternative proof of the fact that $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ has codimension bigger than zero, and it has to appear in the Alexander-Hirschowitz classification. It follows that on the points of $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ the rank of $\text{rk } B_\phi$ drops at least by three, so that $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ is contained in the singular locus of $\det B_\phi$, and in particular $\det B_\phi$ has to vanish with multiplicity ≥ 3 on $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$. It is known that $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ is a hypersurface (see [CH]), hence its equation P has to be a factor of multiplicity ≥ 3 of $\det B_\phi$. Since every $SL(V)$ -invariant polynomial has degree $5k$, the possible values for the degree of P are 5, 10 or 15. Look at the monomials in P containing some among

the variables $v_{000}, v_{012}, v_{111}, v_{223}, v_{334}, v_{144}$, these monomials have to exist, due to the explicit computation performed before. If the degree of P is ≤ 10 , then there exists a $SL(V)$ -invariant polynomial of degree 10 with a monomial containing the above variables, but this contradicts the claim proved along the proof of the Lemma 3.3. It follows that $\deg P = \deg \sigma_7(\mathbf{P}(V), \mathcal{O}(3)) = 15$ and P^3 divides $\det B_\phi$, looking again at our explicit computation we see that we can arrange the scalar multiples in order that P is defined over the rational numbers (as all the $SL(V)$ -invariants) and the equation $2P(\phi)^3 = \det B_\phi$ holds. The Lemma 3.3 shows that P is irreducible. \square

Remark 3.4. The results obtained by Reichstein with his algorithm developed in [Re] can be verified with the Theorem 1.1. For example when w is like in the Example 1 at page 48 of [Re], a computer check shows that $\text{rk}(B_w) = 42$, confirming that $w \in \sigma_7(\mathbf{P}(V), \mathcal{O}(3))$, while when w is like in the Example 2 at page 57 of [Re] then $\text{rk}(B_w) = 45$, so that $w \notin \sigma_7(\mathbf{P}(V), \mathcal{O}(3))$.

The simplest example of a cubic which is not the sum of seven cubes is probably

$$\phi = x_0^2x_1 + x_0x_2^2 + x_1^2x_3 + x_2x_4^2 + x_3^2x_4$$

where $\det(B_\phi) = -2$, which can be checked even without a computer, but with a good amount of patience. The polynomial ϕ defines a smooth cubic 3-fold.

§4. The catalecticant invariant for Clebsch quartics

Let U be any vector space of dimension $n + 1$.

Every quartic $f \in S^4U$ induces the contraction $C_f: S^2U^\vee \rightarrow S^2U$. Clebsch realized in 1861 that if $f \in (\mathbf{P}^n, \mathcal{O}(4))$ then $\text{rk } A_f = 1$. Indeed, with the notations of the previous sections,

$$C_{v^4}(u_1u_2) = 24u_1(v)u_2(v)v^2$$

is always a scalar multiple of v^2 . Clebsch worked in the case $n = 2$ but the same result holds for every n . If $f \in \sigma_k(\mathbf{P}^n, \mathcal{O}(4))$, we get that C_f is the limit of a sum of k matrices of rank one, then $\text{rk } C_f \leq k$. The quartic f is called a Clebsch quartic if and only if $\det C_f = 0$, and this equation gives the catalecticant invariant (see [IK] or [DK]). A matrix description is the following. Let D_i for $i = 1, \dots, \binom{n+2}{2}$ be a basis of differential operators of second order on U . Then $\det(D_iD_jf)$ is the catalecticant invariant.

The picture in terms of Young diagrams for $n = 2$ is



If $n = 2$, we write

$$f = f_{0000}x_0^4 + 4f_{0001}x_0^3x_1 + 6f_{0011}x_0^2x_1^2 + \dots + 12f_{0012}x_0^2x_1x_2 + \dots + f_{2222}x_2^4$$

Then the well known expression for the degree 6 equation of $\sigma_5(\mathbf{P}^2, \mathcal{O}(4))$ is the following (we choosed the basis $\partial_{00}, \partial_{01}, \partial_{11}, \partial_{02}, \partial_{12}, \partial_{22}$)

$$\det \begin{bmatrix} f_{0000} & f_{0001} & f_{0011} & f_{0002} & f_{0012} & f_{0022} \\ f_{0001} & f_{0011} & f_{0111} & f_{0012} & f_{0112} & f_{0122} \\ f_{0011} & f_{0111} & f_{1111} & f_{0112} & f_{1112} & f_{1122} \\ f_{0002} & f_{0012} & f_{0112} & f_{0022} & f_{0122} & f_{0222} \\ f_{0012} & f_{0112} & f_{1112} & f_{0122} & f_{1122} & f_{1222} \\ f_{0022} & f_{0122} & f_{1122} & f_{0222} & f_{1222} & f_{2222} \end{bmatrix} = 0$$

The above equation gives the necessary condition to express a quartic homogeneous polynomial in 3 variables as the sum of 5 fourth powers. Mukai proves in [Mu] that a general plane quartic is a sum of 6 fourth powers in ∞^3 ways, parametrized by the Fano 3-fold V_{22} .

The Clebsch quartics give a hypersurface of degree $\binom{n+2}{2}$ in the space of all quartics.

It follows that this hypersurface contains the variety of k -secants to $(\mathbf{P}^n, \mathcal{O}(4))$ for $k = \left[\binom{n+2}{2} - 1 \right] = n(n+3)/2$, and it is equal to this secant variety for $1 \leq n \leq 4$, which turns out to be defective for $2 \leq n \leq 4$. Indeed it is a hypersurface while it is expected that it fills the ambient space. This explains why this example appears in the Alexander-Hirschowitz classification.

Added in proof: F. Schreyer communicated to us that B_ϕ of the Theorem 1.1 appears also in the apolar ring of ϕ .

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