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THE HOMOTOPY SET OF THE AXES OF PAIRINGS

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Introduction. Varadarajan [**13**] named a map $f: A \longrightarrow X$ a *cyclic map* when there exists a map $F: X \times A \longrightarrow X$ such that

$$F|X \vee A \simeq \nabla_X \circ (1_X \vee f)$$

for the folding map $\nabla_X: X \vee X \longrightarrow X$. He defined the generalized Gottlieb set $G(A, X)$ of the homotopy classes of the cyclic maps $f: A \longrightarrow X$ and studied the fundamental properties of $G(A, X)$. If A is a co-Hopf space, then the Varadarajan set $G(A, X)$ has a group structure [**13**]. The group $G(A, X)$ is a generalization of $G(X)$ and $G_n(X)$ of Gottlieb [**2, 3**]. Some authors studied the properties of the Varadarajan set, its dual and related topics [**4, 5, 6, 7, 12, 15, 16, 17**].

Let us write $f \perp g$ when there exists a continuous map $\mu: X \times Y \longrightarrow Z$ (called a *pairing* [**11**]) with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ (Definition 1.1).

Let $v: X \longrightarrow Z$ be a fixed map. We define the set of the homotopy classes of the axes by

$$v^{\perp}(Y, Z) = \{ [g]: Y \longrightarrow Z | v \perp g \}.$$

This set depends only on the homotopy types of the spaces X, Y and Z and the homotopy class of v. If $X = Z$ and $v \simeq 1_X$, then $(1_X)^{\perp}(Y, X)$ is exactly the Varadarajan set $G(Y, X)$ [**13**].

The purpose of this paper is to generalize some of the results on the Varadarajan set in [**4, 5, 6, 7, 13, 15**] to the set of the homotopy classes of the axes of pairings.

Let G be a *topological group*. In this paper, a topological space with a G-action is called a *G-space*. We work in the category of G-spaces with base point $*$ which is fixed under the G-action through § § 1–3. The symbol $*$ also denotes the constant map.

In § 1 we study some properties of the axes of pairings and obtain some formulas for $f \perp g$. We prove the following theorem, which is a generalization of the result of Varadarajan which says that the set $G(A, X)$ has a group structure when A is a co-Hopf space [**13**].

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THEOREM 1.10. *Let* $f: X \longrightarrow Z$, $v: V \longrightarrow Z$, $g: Y \longrightarrow V$ *and* $w: W \longrightarrow V$ *be maps. Suppose that* $f \perp v$ *and* $g \perp w$. *Then the following results hold:*

(1) $\{\nabla_Z \circ (f \vee v \circ g)\} \perp (v \circ w)$.

(2) If $\theta: A \longrightarrow X \vee Y$ *is a copairing, then* $(f \dot{+} v \circ g) \perp (v \circ w)$.

In § 2 we prove the fundamental properties of $v^{\perp}(Y, Z)$. In particular, we show that given a copairing $\theta: Y \longrightarrow Y_1 \vee Y_2$, we can define an induced pairing

$$\dot{+}: v^{\perp}(Y_1, Z) \oplus v^{\perp}(Y_2, Z) \longrightarrow v^{\perp}(Y, Z).$$

In § 3 we define the set $u^{\top}(A, C)$ of the homotopy classes of the coaxes of copairings, which is the dual concept of $v^{\perp}(Y, Z)$. We study the dual results of the previous sections.

In § 4 we assume that $G = \{e\}$, the *trivial group*, and work in the category of CW complexes. We generalize the results in § § 2 and 3 of [**13**]. We firstly study the operation of the fundamental group $\pi_1(X)$ on $v^{\perp}(Y, X)$; one of the results of this section is the following theorem.

THEOREM 4.2. *Let* $v: X \longrightarrow Z$ *be a map.*

(1) The subgroup $v_*\pi_1(X)$ *operates trivially on* $v^{\perp}(Y, Z)$.

(2) Let σ *be an element of* $v^{\perp}(S^1, Z)$. *Then* σ *operates trivially on* $\mathrm{Im}(v_*: [A, X] \longrightarrow [A, Z])$ *for any space A.*

We also study some relations between the axes of pairings and the generalized Whitehead product [**1**]. Hoo [**4**] and Lim [**6**] studied similar results for cyclic maps. Varadarajan [**13**] defined a group $P(\Sigma Y, Z)$. We define related groups $v^{P}(\Sigma Y, Z)$; also we introduce $v^{W}(\Sigma Y, Z)$ and $v^{C}(Y, Z)$, which are generalizations of the groups $W(\Sigma A, X)$ and $C(A, X)$ of Lim [**6**].

We denote by $f: X \longrightarrow Y$ a G-map. We call a G-map $f: X \longrightarrow Y$ simply a map $f: X \longrightarrow Y$. The symbol $f \simeq g$ means that f is G-homotopic to g and $[f]: X \longrightarrow Y$ the G-homotopy class of $f: X \longrightarrow Y$.

The map $1_X: X \longrightarrow X$ is the *identity map* defined by $1_X(x) = x$ for any element x of X. The map $\triangle_X: X \longrightarrow X \times X$ denotes the diagonal map defined by $\triangle_X(x) = (x, x)$ for any element x of X and $\nabla_X: X \vee X \longrightarrow X$ the folding map defined by $\nabla_X(x, *) = x = \nabla_X(*, x)$ for any element of x of X. The map $T: X \times Y \longrightarrow Y \times X$ is the *switching map* defined by $T(x, y) = (y, x)$ for any elements x of X and y of Y.

Let $f_1: X_1 \longrightarrow Y_1$ and $f_2: X_2 \longrightarrow Y_2$ be maps. We define the *product map*

$$f_1 \times f_2: X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

by $(f_1 \times f_2)(x_1, x_2) = \big(f_1(x_1), f_2(x_2)\big)$ for any elements x_1 of X_1 and x_2 of X_2. The *wedge map* is defined by

$$f_1 \vee f_2 = f_1 \times f_2 | X_1 \vee X_2: X_1 \vee X_2 \longrightarrow Y_1 \vee Y_2.$$

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1. Axes of pairings of topological spaces. We first recall the definition of a pairing [**11**].

Definition 1.1. We call a map $\mu: X \times Y \longrightarrow Z$ a *pairing* with the *axes* $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$, when it satisfies

$$\mu | X \vee Y \simeq \nabla_Z \circ (f \vee g): X \vee Y \longrightarrow Z.$$

We write $f \perp g$ when there exists a pairing $\mu: X \times Y \longrightarrow Z$ with the axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$.

Given a pairing $\mu: X \times Y \longrightarrow Z$, we define a map $\alpha \dotplus \beta: B \longrightarrow Z$ for any maps $\alpha: B \longrightarrow X$ and $\beta: B \longrightarrow Y$ by

$$\alpha \dotplus \beta = \mu \circ (\alpha \times \beta) \circ \triangle_B.$$

This defines a pairing $\dotplus: [B, X] \times [B, Y] \longrightarrow [B, Z]$.

Example 1.2. (1) A map $f: Y \longrightarrow X$ is *cyclic* if and only if $1_X \perp f$ [**13**].

(2) A G-space X is a *Hopf G-space* if and only if $1_X \perp 1_X$.

Proposition 1.3. *(1) Let $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ be maps. Then $f \perp g$ if and only if $g \perp f$.*

(2) Let $f_0, f_1: X \longrightarrow Z$ and $g_0, g_1: Y \longrightarrow Z$ be maps. Suppose that $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $f_0 \perp g_0$ if and only if $f_1 \perp g_1$.

Proof. These results are direct consequences of Definition 1.1.

Theorem 1.4. *Let $f_1: X_1 \longrightarrow Z$, $f_2: X_2 \longrightarrow X_1$, $g_1: Y_1 \longrightarrow Z$ and $g_2: Y_2 \longrightarrow Y_1$ be maps. Then $f_1 \perp g_1$ implies $(f_1 \circ f_2) \perp (g_1 \circ g_2)$.*

Proof. Let $\mu: X_1 \times Y_1 \longrightarrow Z$ be a pairing for $f_1 \perp g_1$. Then the composite map $\mu \circ (f_2 \times g_2): X_2 \times Y_2 \longrightarrow Z$ is the required pairing for $(f_1 \circ f_2) \perp (g_1 \circ g_2)$.

Theorem 1.5. *Let $f: X \longrightarrow Z$, $g: Y \longrightarrow Z$ and $w: Z \longrightarrow W$ be maps. Then $f \perp g$ implies $(w \circ f) \perp (w \circ g)$.*

Proof. Let $\mu: X \times Y \longrightarrow Z$ be a pairing with the axes f and g. Then $w \circ \mu: X \times Y \longrightarrow W$ is a pairing with the axes $w \circ f$ and $w \circ g$.

Corollary 1.6. *(1) If $f: A \longrightarrow X$ is a cyclic map, then $f \circ g: B \longrightarrow X$ is a cyclic map for any map $g: B \longrightarrow A$* [**13**].

(2) Let $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ be maps. If $f \perp 1_Z$ or if $1_Z \perp g$, then $f \perp g$.

(3) If $r: X \longrightarrow Y$ is a map with a right homotopy inverse and $f: A \longrightarrow X$ a cyclic map, then $r \circ f: A \longrightarrow Y$ is a cyclic map [**13**, **5**].

Proof. (1) Suppose that $1_X \perp f$. Then we have $1_X \perp (f \circ g)$ by Theorem 1.4.

(2) If $f \perp 1_Z$ or if $1_Z \perp g$, then we have $f \perp (1_Z \circ g)$ or $(1_Z \circ f) \perp g$ by Theorem 1.4, and hence $f \perp g$.

(3) Suppose that $1_X \perp f$. Then we have $(r \circ 1_X \circ h) \perp (r \circ f)$ by Theorems 1.4 and 1.5, where $h: Y \longrightarrow X$ is the right homotopy inverse map of $r: X \longrightarrow Y$. Since $r \circ h \simeq 1_Y$, we have $1_Y \perp (r \circ f)$ by Proposition 1.3(2).

The following result is a generalization of Proposition 4.6 of Lim [**5**].

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PROPOSITION 1.7. *Let* $f_1: X_1 \longrightarrow Z_1$, $f_2: X_2 \longrightarrow Z_2$, $g_1: Y_1 \longrightarrow Z_1$, $g_2: Y_2 \longrightarrow Z_2$ *be maps. If* $f_1 \perp g_1$ *and* $f_2 \perp g_2$, *then* $(f_1 \times f_2) \perp (g_1 \times g_2)$.

Proof. Let $\mu_1: X_1 \times Y_1 \longrightarrow Z_1$ and $\mu_2: X_2 \times Y_2 \longrightarrow Z_2$ be pairings for $f_1 \perp g_1$ and $f_2 \perp g_2$ respectively. Then the composite map $(\mu_1 \times \mu_2) \circ (1_{X_1} \times T \times 1_{Y_2}): X_1 \times X_2 \times Y_1 \times Y_2 \longrightarrow X_1 \times Y_1 \times X_2 \times Y_2 \longrightarrow Z_1 \times Z_2$ is the required pairing for $(f_1 \times f_2) \perp (g_1 \times g_2)$.

Definition 1.8. ([**11**]) We call a map $\theta: A \longrightarrow B \vee C$ a *copairing* with the *coaxes* $h: A \longrightarrow B$ and $r: A \longrightarrow C$ if it satisfies the condition that

$$j \circ \theta \simeq (h \times r) \circ \triangle_A : A \longrightarrow B \times C$$

for the inclusion map $j: B \vee C \longrightarrow B \times C$.

Given a copairing $\theta: A \longrightarrow B \vee C$, we define a map $\alpha \dot{+} \beta: A \longrightarrow X$ for any maps $\alpha: B \longrightarrow X$ and $\beta: C \longrightarrow X$ by

$$\alpha \dot{+} \beta = \nabla_X \circ (\alpha \vee \beta) \circ \theta.$$

This defines a pairing $\dot{+}: [B, X] \times [C, X] \longrightarrow [A, X]$.

Concerning the pairings $\dot{+}$ and $\underset{\cdot}{+}$, we have the following results (Propositions 3.2 and 3.4 of [**11**]).

PROPOSITION 1.9. *(1) Let X be a Hopf G-space. If* $\theta: A \longrightarrow B \vee C$ *is a copairing with coaxes* $h: A \longrightarrow B$ *and* $r: A \longrightarrow C$, *then*

$$\alpha \dot{+} \beta = h^*(\alpha) \underset{\cdot}{+} r^*(\beta) \quad \text{and} \quad h^*(\alpha) \underset{\cdot}{+} r^*(\beta) = r^*(\beta) \underset{\cdot}{+} h^*(\alpha)$$

in $[A, X]$ *for any elements* α *of* $[B, X]$ *and* β *of* $[C, X]$.

(2) Let A be a co-Hopf G-space. If $\mu: X \times Y \longrightarrow Z$ *is a pairing with axes* $f: X \longrightarrow Z$ *and* $g: Y \longrightarrow Z$, *then*

$$\alpha \underset{\cdot}{+} \beta = f_*(\alpha) \dot{+} g_*(\beta) \quad \text{and} \quad f_*(\alpha) \dot{+} g_*(\beta) = g_*(\beta) \dot{+} f_*(\alpha)$$

in $[A, Z]$ *for any elements* α *of* $[A, X]$ *and* β *of* $[A, Y]$.

THEOREM 1.10. *Let* $f: X \longrightarrow Z$, $v: V \longrightarrow Z$, $g: Y \longrightarrow V$ *and* $w: W \longrightarrow V$ *be maps. Suppose that* $f \perp v$ *and* $g \perp w$. *Then the following results hold.*

(1) $\{\nabla_Z \circ (f \vee v \circ g)\} \perp (v \circ w)$.

(2) If $\theta: A \longrightarrow X \vee Y$ *is a copairing, then* $(f \dot{+} v \circ g) \perp (v \circ w)$.

Proof. Let $\mu_1: X \times V \longrightarrow Z$ be a pairing for $f \perp v$ and $\mu_2: Y \times W \longrightarrow V$ be a pairing for $g \perp w$. We define a pairing $\mu: (X \vee Y) \times W \longrightarrow Z$ by

$$\mu = \mu_1 \circ (1_X \times \mu_2) \circ (j \times 1_W): (X \vee Y) \times W \longrightarrow X \times Y \times W \longrightarrow X \times V \longrightarrow Z,$$

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where $j: X \vee Y \longrightarrow X \times Y$ is the inclusion map. Then μ is a pairing for $\{\nabla_Z \circ (f \vee v \circ g)\} \perp (v \circ w)$.

(2) By (1) and Theorem 1.4, we have $\{\nabla_Z \circ (f \vee v \circ g) \circ \theta\} \perp (v \circ w)$, namely, $(f \dot{+} v \circ g) \perp (v \circ w)$.

Let A be a *co-group like G-space* [**14**], that is, A is a homotopy associative co-Hopf G-space with a homotopy inverse $\nu: A \longrightarrow A$, namely, $1_A \dot{+} \nu \simeq * \simeq \nu \dot{+} 1_A$. As an application of Theorem 1.10 we have the following result. Related results are Theorem 1.5 of Varadarajan [**13**], Theorem 2 of Hoo [**4**] and Proposition 4.13 of Lim [**5**].

THEOREM 1.11. ([**13, 5, 6**]) *If A is a co-group like G-space, then $G(A, X)$ is an abelian subgroup of* $[A, X]$.

Proof. Set $v = w = 1_X$ and $X = Y = A$ in Theorem 1.10 (2), then we see that $G(A, X)$ is closed under the operation $\dot{+}$. It also contains an inverse element $\alpha \circ \nu$ for any element α of $G(A, X)$ by Theorem 1.4. Moreover, we see that $G(A, X)$ is contained in the center of $[A, X]$ by Proposition 4.3 of Lim [**6**] or Proposition 1.9(2). (Set $f = 1_X$ and $\beta = 1_Y$.)

2. The homotopy sets of the axes.. Let $v: X \longrightarrow Z$ be a map. We call a map $g: Y \longrightarrow Z$ *v-cyclic* if $v \perp g$. Then by the result of Proposition 1.3(2), we can define the following set of the homotopy classes of the v-cyclic maps $g: Y \longrightarrow Z$;

$$v^{\perp}(Y, Z) = \{[g]: Y \longrightarrow Z \mid v \perp g\} \subset [Y, Z].$$

If $v \simeq 1_X: X \longrightarrow X$, then $(1_X)^{\perp}(Y, X)$ is just the Varadarajan set $G(Y, X)$ in [**13**].

PROPOSITION 2.1. *(1) If $v \perp g$ for maps $v: X \longrightarrow Z$ and $g: Y \longrightarrow Z$, then* $\operatorname{Im}(g_*: [A, Y] \longrightarrow [A, Z]) \subset v^{\perp}(A, Z)$.

(2) For any maps $v: X \longrightarrow Z$ and $f: A \longrightarrow X$, we have $v^{\perp}(Y, Z) \subset (v \circ f)^{\perp}(Y, Z)$. Especially, for any map $v: A \longrightarrow X$, we have $G(Y, X) \subset v^{\perp}(Y, X)$.

Proof. (1) The relation $v \perp g$ implies $v \perp (g \circ h)$ for any map $h: A \longrightarrow Y$ by Theorem 1.4.

(2) is proved similarly.

PROPOSITION 2.2. *Let $v: X \longrightarrow Z$ be a map. If $w: Z \longrightarrow W$ and $a: A \longrightarrow Y$ are G-homotopy equivalences, then the following results hold.*

(1) $w_*: v^{\perp}(Y, Z) \longrightarrow (w \circ v)^{\perp}(Y, W)$ is an isomorphism.

(2) $(1_X \times a)^*: v^{\perp}(Y, Z) \longrightarrow v^{\perp}(A, Z)$ is an isomorphism.

Proof. These results are immediate consequences of Theorems 1.4 and 1.5.

Remark 2.3. By Propositions 2.2 and 1.3(2), we know that the set $v^{\perp}(Y, Z)$ depends only on the homotopy types of Y and Z and the homotopy class of v.

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THEOREM 2.4. *If Z is a Hopf G-space, then $v^{\perp}(Y,Z) = [Y,Z]$ for any map $v: X \longrightarrow Z$.*

Proof. Since Z is a Hopf G-space, we have $1_Z \perp 1_Z$ and hence $v \perp g$ for any map $g: Y \longrightarrow Z$ by Theorem 1.4.

Given a copairing $\theta: A \longrightarrow B \vee C$, a map $\alpha \dotplus \beta: A \longrightarrow X$ is defined by $\alpha \dotplus \beta = \nabla_X \circ (\alpha \vee \beta) \circ \theta$ for any maps $\alpha: B \longrightarrow X$ and $\beta: C \longrightarrow X$ (Definition 1.8). Thus a copairing $\theta: A \longrightarrow B \vee C$ in the topological spaces induces a pairing

$$\dotplus: [B,X] \times [C,X] \longrightarrow [A,X]$$

in the homotopy sets. We shall now study the induced pairings in $v^{\perp}(Y,Z)$.

In the rest of this section, we work in the category of G-CW complexes [**9, 10**], because we use the *G-homotopy extension property* (G-HEP) in the proof of Theorem 2.5. The result holds in the category of G-ANR with some conditions. See Proposition 9.3 of [**8**]. The following theorem generalizes the results of Lemma 4.8 of [**5**] and Theorem 1.5 of [**13**].

THEOREM 2.5. *Let $f: X \longrightarrow Z$, $g_1: Y_1 \longrightarrow Z$ and $g_2: Y_2 \longrightarrow Z$ be maps between G-CW complexes. Then the following results hold.*

(1) $f \perp g_1$ and $f \perp g_2$ implies $f \perp \{\nabla_Z \circ (g_1 \vee g_2)\}$.

(2) Let $\theta: Y \longrightarrow Y_1 \vee Y_2$ be a copairing. Then $f \perp g_1$ and $f \perp g_2$ implies $f \perp (g_1 \dotplus g_2)$.

Proof. (cf. proof of Theorem 1.5 of [**13**]) (1) Let $\mu_1: X \times Y_1 \longrightarrow Z$ be a pairing for $f \perp g_1$ and $\mu_2: X \times Y_2 \longrightarrow Z$ a pairing for $f \perp g_2$.

Let us define a map

$$< \mu_1, \mu_2 >: X \times (Y_1 \vee Y_2) \longrightarrow Z$$

by the following way. We can assume that the maps μ_1 and μ_2 satisfy the following condition by G-HEP;

$$\mu_1 | X \times \{*\} = f = \mu_2 | X \times \{*\}.$$

Then we define $< \mu_1, \mu_2 > | X \times (Y_1 \times \{*\}) = \mu_1$ and $< \mu_1, \mu_2 > | X \times (\{*\} \times Y_1) = \mu_2$. This map $< \mu_1, \mu_2 >$ is well-defined since they are equal when restricted to $X \times (\{*\} \times \{*\}) = X \times \{*\}$. The map $< \mu_1, \mu_2 >$ is a pairing for $f \perp \{\nabla_Z \circ (g_1 \vee g_2)\}$.

(2) Define $\mu: X \times Y \longrightarrow Z$ by

$$\mu = < \mu_1, \mu_2 > \circ (1_X \times \theta): X \times Y \longrightarrow X \times (Y_1 \vee Y_2) \longrightarrow Z.$$

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Then we have

$$\mu|\{*\} \times Y = \nabla_Z \circ (g_1 \vee g_2) \circ \theta = g_1 \dot{+} g_2 \text{ and } \mu|X \times \{*\} = f.$$

This completes the proof.

Suppose we are given a copairing $\theta : Y \longrightarrow Y_1 \vee Y_2$. By Proposition 1.3(2) and Theorem 2.5(2), we can now define the induced pairing

$$\dot{+}: v^{\perp}(Y_1, Z) \oplus v^{\perp}(Y_2, Z) \longrightarrow v^{\perp}(Y, Z)$$

by $f_1 \dot{+} f_2 = \nabla_Z \circ (f_1 \vee f_2) \circ \theta$.

If $\theta : Y \longrightarrow Y \vee Y$ is a copairing, for example, co-Hopf structure, then θ defines a binary operation on $v^{\perp}(Y, Z)$. This is a generalization of Theorem 1.5 of [**13**]; if $v = 1_X$ in the above pairing, then we see that the Varadarajan set $G(Y, X) = (1_X)^{\perp}(Y, X)$ has a binary operation. See Corollary 4.9 of Lim [**5**] and Theorem 1.11.

PROPOSITION 2.6. *Let the spaces be G-CW complexes. Suppose that Y and B are co-Hopf G-spaces. Then the following results hold.*

(1) $w_*: v^{\perp}(Y, Z) \longrightarrow (w \circ v)^{\perp}(Y, W)$ *is a homomorphism for any map* $w: Z \longrightarrow W$.

(2) $(a \times 1_Y)^*: v^{\perp}(Y, Z) \longrightarrow (v \circ a)^{\perp}(Y, Z)$ *is a homomorphism for any map* $a: A \longrightarrow X$.

(3) $(1_X \times b)^*: v^{\perp}(Y, Z) \longrightarrow v^{\perp}(B, Z)$ *is a homomorphism for any co-Hopf map* $b: B \longrightarrow Y$.

3. The Dual Concepts.

In this section we study the duals of the results in the previous sections. We omit most of the proofs of the results in this section, since they are given by dualizing the corresponding results.

We write $h \top r$ if there exists a copairing $\theta : A \longrightarrow B \vee C$ with the coaxes $h: A \longrightarrow B$ and $r: A \longrightarrow C$ (Definition 1.8).

Example 3.1. (1) A G-map $r: A \longrightarrow C$ is *cocyclic* if and only if $1_A \top r$ [**13**].

(2) A G-space A is a co-Hopf G-space if and only if $1_A \top 1_A$.

PROPOSITION 3.2. *(1) Let* $h: A \longrightarrow B$ *and* $r: A \longrightarrow C$ *be maps. Then* $h \top r$ *if and only if* $r \top h$.

(2) Let $h_0, h_1 : A \longrightarrow B$ *and* $r_0, r_1 : A \longrightarrow C$ *be maps. Suppose that* $h_0 \simeq h_1$ *and* $r_0 \simeq r_1$. *Then* $h_0 \top r_0$ *if and only if* $h_1 \top r_1$.

THEOREM 3.3. *Let* $h_1 : A \longrightarrow B_1$, $h_2 : B_1 \longrightarrow B_2$, $r_1 : A \longrightarrow C_1$, $r_2 : C_1 \longrightarrow C_2$ *be maps. Then* $h_1 \top r_1$ *implies* $(h_2 \circ h_1) \top (r_2 \circ r_1)$.

THEOREM 3.4. *Let* $h: A \longrightarrow B$, $r: A \longrightarrow C$ *and* $d: D \longrightarrow A$ *be maps. Then* $h \top r$ *implies* $(h \circ d) \top (r \circ d)$.

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Let $u: A \longrightarrow B$ be a fixed map. We call a map $r: A \longrightarrow C$ *u-cocyclic* if $u \top r$. We can now define the following set of the homotopy classes of the u-cocyclic maps $r: A \longrightarrow C$;

$$u^{\top}(A, C) = \{[r]: A \longrightarrow C | u \top r\}.$$

If $u \simeq 1_A$, then $(1_A)^{\top}(A, C)$ is just Varadarajan's $DG(A, C)$ [**13**].

By Theorems 3.3 and 3.4, we have the following two propositions.

PROPOSITION 3.5. *If $u \top r$ for maps $u: A \longrightarrow B$ and $r: A \longrightarrow C$ then* $\mathrm{Im}(r^*: [C, X] \longrightarrow [A, X]) \subset u^{\top}(A, X)$.

PROPOSITION 3.6. *Let $u: A \longrightarrow B$ be a map. If $a: V \longrightarrow A$ and $d: C \longrightarrow D$ are G-homotopy equivalences, then the following results hold.*

(1) $a^*: u^{\top}(A, C) \longrightarrow (u \circ a)^{\top}(V, C)$ is an isomorphism.

(2) $(1_B \vee d)_*: u^{\top}(A, C) \longrightarrow u^{\top}(A, D)$ is an isomorphism.

The following result is a generalization of Lemma 3.4 of Lim [**7**].

PROPOSITION 3.7. *Let $h_1: A_1 \longrightarrow B_1$, $h_2: A_2 \longrightarrow B_2$, $r_1: A_1 \longrightarrow C_1$, $r_2: A_2 \longrightarrow C_2$ be maps. If $h_1 \top r_1$ and $h_2 \top r_2$, then $(h_1 \vee h_2) \top (r_1 \vee r_2)$.*

Proof. Let $\theta_1: A_1 \longrightarrow B_1 \vee C_1$ be a copairing for $h_1 \top r_1$ and $\theta_2: A_2 \longrightarrow B_2 \vee C_2$ for $h_2 \top r_2$. Define $\theta: A_1 \vee A_2 \longrightarrow (B_1 \vee B_2) \vee (C_1 \vee C_2)$ by

$$\begin{aligned} \theta &= (1_{B_1} \vee T \vee 1_{C_2}) \circ (\theta_1 \vee \theta_2): \\ &\quad A_1 \vee A_2 \longrightarrow (B_1 \vee C_1) \vee (B_2 \vee C_2) \longrightarrow (B_1 \vee B_2) \vee (C_1 \vee C_2). \end{aligned}$$

Then θ is a copairing for $(h_1 \vee h_2) \top (r_1 \vee r_2)$.

THEOREM 3.8. *If A is a co-Hopf G-space, then $u^{\top}(A, C) = [A, C]$ for any map $u: A \longrightarrow B$.*

THEOREM 3.9. *Let $h: A \longrightarrow B$, $r: A \longrightarrow C$, $u: B \longrightarrow U$, $d: B \longrightarrow D$ be maps. Suppose that $u \top d$ and $h \top r$. Then the following results hold.*

(1) $(u \circ h) \top \{(d \circ h \times r) \circ \triangle_A\}$.

(2) If $\mu: D \times C \longrightarrow Z$ is a pairing, then $(u \circ h) \top (d \circ h \dotplus r)$.

Proof. (1) Let $\theta_1: A \longrightarrow B \vee C$ be a copairing for $h \top r$ and $\theta_2: B \longrightarrow U \vee D$ a copairing for $u \top d$. We define a copairing $\theta: A \longrightarrow U \vee (D \times C)$ by

$$\theta = (1_U \vee j) \circ (\theta_2 \vee 1_C) \circ \theta_1: A \longrightarrow B \vee C \longrightarrow U \vee D \vee C \longrightarrow U \vee (D \times C),$$

where $j: D \vee C \longrightarrow D \times C$ is the inclusion map. Then θ is a copairing for $(u \circ h) \top \{(d \circ h \times r) \circ \triangle_A\}$.

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(2) By (1) and Theorem 3.3, we have $(u \circ h) \top \{\mu \circ (d \circ h \times r) \circ \triangle_A\}$, namely, $(u \circ h) \top (d \circ h \dotplus r)$.

Let Z be a *group like G-space* [**14**], that is, Z is a homotopy associative Hopf G-space with a homotopy inverse $\nu: Z \longrightarrow Z$, namely, $1_Z \dotplus \nu \simeq * \simeq \nu \dotplus 1_Z$. Then as an application of Theorem 3.9, we have the following result of Theorem 4.2 of Lim [**7**].

THEOREM 3.10. ([**7**]) *Let Z be a group like G-space. Then $DG(A, Z)$ is an abelian subgroup of $[A, Z]$.*

Proof. Set $u = h = 1_A$ in Theorem 3.9, then we see that $DG(A, Z)$ is closed under the operation $\dotplus$. The inverse element $\nu \circ \alpha$ is contained in $DG(A, Z)$ for any element α of $DG(A, Z)$ by Theorem 3.3. Moreover, $DG(A, Z)$ is contained in the center of $[A, Z]$ by Proposition 1.9(1) (Set $h = 1_A$ and $\beta = 1_Z$). (cf. Corollary 3.10 of [**7**].)

4. Operations and Whitehead Products. Throughout this section we assume that $G = \{e\}$, the group of the identity alone. We assume furthermore that the spaces are *CW* complexes so that we can use the *homotopy extension property*. Then the fundamental group $\pi_1(X)$ operates on the homotopy set $[A, X]$. We find the definition of the operation in [**13, 14**]. If σ is an element of $\pi_1(X)$, then we have an isomorphism

$$\sigma_\#: [A, X] \longrightarrow [A, X],$$

which is induced by the operation of σ on the homotopy set $[A, X]$.

THEOREM 4.1. *Let $\theta: A \longrightarrow B \vee C$ be a copairing. Let β be an element of $[B, X]$ and γ of $[C, X]$. Then*

$$\sigma_\#(\beta \dot{+} \gamma) = \sigma_\#(\beta) \dot{+} \sigma_\#(\gamma)$$

for any σ of $\pi_1(X)$.

Proof. Let f, g and s be maps representing β, γ and σ. We can choose homotopies $F: B \times I \longrightarrow X$ and $G: C \times I \longrightarrow X$ so that

$$F|B \times \{0\} = f, \; G|C \times \{0\} = g \text{ and } F|\{*\} \times I = s = G|\{*\} \times I.$$

We see that $F|B \times \{1\} = \sigma_\#(\beta)$ and $G|C \times \{1\} = \sigma_\#(\gamma)$ by the definition of the operation of σ. We define $K: (B \vee C) \times I \longrightarrow X$ by

$$K|(B \times \{*\}) \times I = F \text{ and } K|(\{*\} \times C) \times I = G.$$

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K is well-defined by the properties of F and G. Define a homotopy $H: A \times I \longrightarrow X$ by a composition

$$H = K \circ (\theta \times 1_I): A \times I \longrightarrow (B \vee C) \times I \longrightarrow X.$$

Then

$$H|A \times \{0\} = \nabla_X \circ (f \vee g) \circ \theta = \beta \dot{+} \gamma \text{ and } H|A \times \{1\} = \sigma_\#(\beta \dot{+} \gamma).$$

Here we see that $H|A \times \{1\} = \sigma_\#(\beta) \dot{+} \sigma_\#(\gamma)$ by the definition of H, and hence we have $\sigma_\#(\beta \dot{+} \gamma) = \sigma_\#(\beta) \dot{+} \sigma_\#(\gamma)$.

THEOREM 4.2. *Let $v: X \longrightarrow Z$ be a map.*
(1) The subgroup $v_*\pi_1(X)$ operates trivially on $v^\perp(Y, Z)$.
(2) Let σ be an element of $v^\perp(S^1, Z)$. Then σ operates trivially on Im $(v_*: [A, X] \longrightarrow [A, Z])$ *for any space A.*

Proof. (1) Let $[f]$ be an element of $v^\perp(Y, Z)$ and $\mu: X \times Y \longrightarrow Z$ be a pairing for $v \perp f$. For any element $\sigma = [s]: (I, \{0, 1\}) \longrightarrow (X, *)$ of $\pi_1(X)$, we define a homotopy $H: Y \times I \longrightarrow Z$ by

$$H = \mu \circ (s \times 1_Y) \circ T: Y \times I \longrightarrow I \times Y \longrightarrow X \times Y \longrightarrow Z.$$

Then $H|Y \times \{0\} \simeq f \simeq H|Y \times \{1\}$ and $H|\{*\} \times I \simeq v \circ s = v_*(s)$. Hence we have $v_*(\sigma)_\#(f) = H|Y \times \{1\} \simeq f$.

(2) Suppose that $\sigma = [s]$ and $\mu: X \times S^1 \longrightarrow Z$ is a pairing for $v \perp s$ and $[f]$ is an element of $[A, X]$. We define a homotopy $H: A \times I \longrightarrow Z$ by

$$H = \mu \circ (f \times p): A \times I \longrightarrow X \times S^1 \longrightarrow Z,$$

where $p: (I, \{0, 1\}) \longrightarrow (S^1, *)$ is the projection. Then we have

$$H|A \times \{0\} \simeq v \circ f = v_*(f) \simeq H|A \times \{1\} \text{ and } H|\{*\} \times I \simeq s \circ p.$$

It follows that $\sigma_\#(v_*(f)) \simeq H|A \times \{1\} \simeq v_*(f)$. This completes the proof.

We denote by ΣA the *reduced suspension* of a space A, and $A \wedge B$ the *smash product* $A \times B / A \vee B$ of A and B.

Arkowitz [**1**] defined the *generalized Whitehead product*

$$[\alpha, \beta]: \Sigma(A \wedge B) \longrightarrow X$$

for any elements $\alpha: \Sigma A \longrightarrow X$ and $\beta: \Sigma B \longrightarrow X$.

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THEOREM 4.3. *Let* $f: \Sigma X \longrightarrow Z$ *and* $g: \Sigma Y \longrightarrow Z$. *Then* $f \perp g$ *if and only if* $[f, g] = 0$.

Proof. By Definition 1.1, we see that $f \perp g$ if and only if there exists a pairing $\mu: \Sigma X \times \Sigma Y \longrightarrow Z$ such that $\mu | \Sigma X \vee \Sigma Y \simeq \nabla_Z \circ (f \vee g)$. The latter condition is equivalent to $[f, g] = 0$ by Proposition 5.1 of Arkowitz [**1**].

PROPOSITION 4.4. *Let* $v: \Sigma X \longrightarrow Z$ *be a map. Then*

$$v^{\perp}(\Sigma Y, Z) = \Big\{ [f]: \Sigma Y \longrightarrow Z \big| [v, f] = 0 \Big\}.$$

Proof. By Theorem 4.3, we have the result.

The following theorem is a generalization of Theorem 3.2 of [**13**].

THEOREM 4.5. *Let* $v: X \longrightarrow Z$ *be a map and* β *an element of* $v^{\perp}(\Sigma Y, Z)$. *Then* $[v_*(\alpha), \beta] = 0$ *for any element* α *of* $[\Sigma A, X]$.

Proof. Suppose that $\alpha = [f]$ and $\beta = [g]$. Since $v \perp g$, we have $(v \circ f) \perp g$ for any map $f: \Sigma A \longrightarrow X$ by Theorem 1.4. It follows that $[v_*(\alpha), \beta] = [v \circ f, g] = 0$ by Theorem 4.3.

Varadarajan [**13**] defined the following subgroup of $[\Sigma Y, Z]$;

$$P(\Sigma Y, Z) = \{ \gamma \in [\Sigma Y, Z] \mid [\beta, \gamma] = 0 \text{ for all } \beta \in [\Sigma^n Y, Z] \\ \text{and all } n \geq 1 \}.$$

Let $v: X \longrightarrow Z$ be a map. We define

$$v^P(\Sigma Y, Z) = \{ \gamma \in [\Sigma Y, Z] \mid [v_*(\beta), \gamma] = 0 \text{ for all } \beta \in [\Sigma^n Y, X] \\ \text{and all } n \geq 1 \}.$$

$$v^W(\Sigma Y, Z) = \{ \gamma \in [\Sigma Y, Z] \mid [v_*(\beta), \gamma] = 0 \text{ for all } \beta \in [\Sigma B, X] \\ \text{and all space } B \}.$$

$$v^C(Y, Z) = \{ [g] \in [Y, Z] \mid v_*(\alpha) \dotplus g_*(\beta) = g_*(\beta) \dotplus v_*(\alpha) \\ \text{for all } \alpha \in [\Sigma A, X] \text{ and all } \beta \in [\Sigma A, Y] \text{ and all space } A \}.$$

The above definitions are generalizations of Definitions 4.1 and 4.5 of [**6**].

THEOREM 4.6. *Let* $v: X \longrightarrow Z$ *be a map. Then the following results hold.*
(1) $P(\Sigma Y, Z) \subset v^W(\Sigma Y, Z) \subset v^P(\Sigma Y, Z)$.
(2) $v^{\perp}(\Sigma Y, Z) \subset v^W(\Sigma Y, Z) \subset v^P(\Sigma Y, Z)$.
(3) $v^{\perp}(Y, Z) \subset v^C(Y, Z)$.

Proof. (1) is a direct consequence of the definitions.

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(2) Let γ be an element of $v^{\perp}(\Sigma Y, Z)$. Then $[v_*(\beta), \gamma] = 0$ for any element β of $[\Sigma B, X]$ by Theorem 4.5 and hence γ is an element of $v^W(\Sigma Y, Z)$.

(3) Choose an element $[g]$ of $v^{\perp}(Y, Z)$, Let $\mu: X \times Y \longrightarrow Z$ be a pairing with the axes v and g. Then we have

$$v_*(\alpha) \dot{+} g_*(\beta) = g_*(\beta) \dot{+} v_*(\alpha)$$

for any elements α of $[\Sigma A, X]$, β of $[\Sigma A, Y]$ and any space A by Proposition 1.9(2). Hence we have the result.

THEOREM 4.7. *Let $v: X \longrightarrow Z$ be a map. The homotopy sets $v^W(\Sigma Y, Z)$ and $v^P(\Sigma Y, Z)$ are subgroups of* $[\Sigma Y, Z]$.

Proof. (a) Let $\beta = [f], \gamma = [g] \in [\Sigma Y, Z]$ be the elements of $v^W(\sum Y, Z)$. Then we have $[v_*(\alpha), \beta] = [v_*(\alpha), \gamma] = 0$ for any element $\alpha = [h] \in [\Sigma B, X]$ and hence $v_*(h) \perp f$ and $v_*(h) \perp g$ by Theorem 4.3. Then we have $v_*(h) \perp (f + g)$ by Theorem 2.5(2). It follows that $[v_*(\alpha), \beta + \gamma] = 0$ by Theorem 4.3 and hence $\beta + \gamma \in v^W(\Sigma Y, Z)$.

(b) Let $\gamma = [g]$ be an element of $v^W(\Sigma Y, Z)$. Then we have $[v_*(\alpha), \gamma] = 0$ and hence $v_*(h) \perp g$ for any element $\alpha = [h] \in [\Sigma B, X]$ by Theorem 4.5. It follows that $v_*(h) \perp \{g \circ (-1_{\Sigma Y})\}$ by Theorem 1.4 and hence $[v_*(\alpha), -\gamma] = 0$ by Theorem 4.3, since $\gamma \circ (-1_{\Sigma Y}) = -\gamma$. Thus we have $-\gamma \in v^W(\Sigma Y, Z)$.

Moreover the set $v^W(\Sigma Y, Z)$ contains the zero element (the constant map). Thus we have proved that the set $v^W(\Sigma Y, Z)$ is a subgroup of $[\Sigma Y, Z]$.

Similarly we have the result for $v^P(\Sigma Y, Z)$.

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