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NONREGULAR GRAPHS WITH MINIMAL TOTAL IRREGULARITY

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Abstract

The *total irregularity* of a simple undirected graph G is defined as $\mathrm{irr}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. Obviously, $\mathrm{irr}_t(G) = 0$ if and only if G is regular. Here, we characterise the nonregular graphs with minimal total irregularity and thereby resolve the recent conjecture by Zhu *et al.* ['The minimal total irregularity of graphs', Preprint, 2014, arXiv:1404.0931v1] about the lower bound on the minimal total irregularity of nonregular connected graphs. We show that the conjectured lower bound of $2n - 4$ is attained only if nonregular connected graphs of even order are considered, while the sharp lower bound of $n - 1$ is attained by graphs of odd order. We also characterise the nonregular graphs with the second and the third smallest total irregularity.

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1. Introduction

All graphs considered here are undirected and simple (that is, loops and multiple edges are not allowed). Let G be a graph of order $n = |V(G)|$ and size $m = |E(G)|$. For $v \in V(G)$, the degree of v, denoted by $d_G(v)$, is the number of edges incident to v. G is *regular* if all its vertices have the same degree, otherwise it is *irregular*. There are many approaches, including those in [3–6, 8–11], to characterise how irregular a given graph is. In this paper, we focus on the so-called *total irregularity* of a graph [1], defined as

$$\mathrm{irr}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|.$$

The total irregularity is related to the *irregularity* of a graph, defined as $\mathrm{irr}(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|$. The latter measurement was introduced by Albertson [5] and investigated in several works, including [2, 14, 15]. For the motivation for introducing the total irregularity as a new irregularity measure, we refer to [1]. Both the irregularity of a graph and the total irregularity of a graph depend on a single parameter, namely the pairwise difference of vertex degrees. A comparison of irr and irr_t was considered in [12]. There, it was shown that $\mathrm{irr}_t(G) \le n^2\,\mathrm{irr}(G)/4$ and, when G is a tree, $\mathrm{irr}_t(G) \le (n-2)\,\mathrm{irr}(G)$. Also, it was shown that among all trees of the same order, the star has the maximal total irregularity.

In [1], graphs with maximal total irregularity were fully characterised and the upper bound on the total irregularity of a graph was presented.

COROLLARY 1.1 [1]. *For a graph G with n vertices,*

$$\mathrm{irr}_t(G) \le \begin{cases} \frac{1}{12}(2n^3 - 3n^2 - 2n) & n \text{ even,} \\ \frac{1}{12}(2n^3 - 3n^2 - 2n + 3) & n \text{ odd.} \end{cases}$$

Moreover, the bounds are sharp.

In [17, 18], the unicyclic and bicyclic graphs, respectively, with maximal total irregularity were determined.

The lower bound on the total irregularity of general graphs is trivial, since it is obvious that the total irregularity of a graph is zero if and only if the graph is regular. Also by definition the total irregularity is nonnegative. However, it is not trivial to determine lower bounds for the total irregularity of special classes of graphs, or the total irregularity of nonregular graphs. In [19], Zhu *et al.* investigated the minimal total irregularity of the connected graphs and determined the minimal, the second minimal and the third minimal total irregularity of trees, unicyclic graphs and bicyclic graphs. They also proposed the following conjecture.

CONJECTURE 1.2 [19]. Let G be a simple connected graph with n vertices. If G is a nonregular graph, then $\mathrm{irr}_t(G) \ge 2n - 4$.

In the next section, we characterise the nonregular graphs with minimal total irregularity and thereby resolve the above conjecture. We show that Conjecture 1.2 is true only for nonregular connected graphs of even order, while the actual sharp lower bound of $n - 1$ is achieved by graphs of odd order.

By $D(G)$ we denote the set of the vertex degrees of a graph G, that is, $D(G) = \{d(v) \mid v \in V\}$. Given an undirected graph, a *degree sequence* is a monotonic nonincreasing sequence of the degrees of its vertices. A *graphical sequence* is a sequence of numbers which can be the degree sequence of some graph. In general, several graphs may have the same graphical sequence. In order to show that a given sequence of nonnegative integers is graphical, one may use the following characterisation by Erdős and Gallai.

THEOREM 1.3 [13]. *A sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ of nonnegative integers with even sum is graphical if and only if*

$$\sum_{i=1}^{r} d_i \le r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i), \tag{1.1}$$

for all $1 \le r \le n - 1$.

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Tripathi and Vijay [16] showed that (1.1) need be checked only for as many r as there are distinct terms in the sequence, not for all $1 \le r \le n-1$. Denote the indices $1 \le i \le n-1$, with $d_i > d_{i+1}$, by $\sigma_1, \sigma_2, \dots, \sigma_l$, and define $\sigma_l = n$.

Theorem 1.4 [16]. *In Theorem 1.3 it suffices to check (1.1) for $r = \sigma_1, \sigma_2, \dots, \sigma_l$.*

Remark 1.5. For $\sigma_l = n$, (1.1) always holds since $\sum_{i=1}^{n} d_i \le n(n-1)$.

To show that a sequence of nonnegative integers is the degree sequence of a connected graph, we use the following characterisation from [7, Theorem 9, pages 117–118], slightly reformulated to fit the notation introduced above.

Theorem 1.6 [7]. *Let $d_1 \ge d_2 \ge \cdots \ge d_n$ be a sequence of nonnegative integers with $n \ge 2$. A necessary and sufficient condition for the existence of a simple connected graph G with degrees $d_G(v_i) = d_i$, is that:*

(i) $d_n \ge 1$;
(ii) $\sum_{i=1}^{n} d_i \ge 2(n-1)$; *and*
(iii) *the sequence is graphical.*

2. Results

The only nonregular connected graph of order at most three is the path with three vertices, whose total irregularity is $2 = n - 1$, agreeing with the sharp lower bound for graphs of odd order presented later in this section. Therefore, in the sequel we consider connected graphs of order at least four. First, we present two results that will be used to obtain the main results later in this section.

Lemma 2.1. *Let G be a connected graph of order $n > 3$ with $|D(G)| \ge 3$. Then there exists a connected graph H of the same order as G with $|D(H)| = 2$, such that each degree in H occurs at least twice and $\mathrm{irr}_t(H) < \mathrm{irr}_t(G)$.*

Proof. Assume that the claim of the proposition is false. Let $D(G) = \{d_1, d_2, \dots, d_k\}$, $3 \le k \le n-1$. Also, we assume that $d_1 > d_2 > \cdots > d_k$. Let $D_G = (d_G(v_1), d_G(v_2), \dots, d_G(v_n))$ be the degree sequence of G and p the smallest index such that $d_G(v_p) = d_3$. We apply a set of transformations to D_G, obtaining a sequence $D_H = (d_H(v_1), d_H(v_2), \dots, d_H(v_n))$, and consider the difference

$$\sum_{u,v \in V(G)} (|d_G(u) - d_G(v)| - |d_H(u) - d_H(v)|). \tag{2.1}$$

We distinguish two cases according to the size of p.

Case 1. $p < n$. We apply the following assignments to D_G:

$$d_H(v_i) := d_3 + 1, \quad i = 1, \dots, p-1 \quad \text{and} \quad d_H(v_i) := d_3, \quad i = p+1, \dots, n.$$

After these assignments, for all pairs of vertices v_i and v_j with $d_G(v_i) \ne d_G(v_j) \ne d_G(v_p)$,

$$|d_G(v_i) - d_G(v_j)| - |d_H(v_i) - d_H(v_j)| \ge 1.$$

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For the remaining pairs of vertices v_i and v_j,

$$|d_G(v_i) - d_G(v_j)| - |d_H(v_i) - d_H(v_j)| \geq 0.$$

Thus, it follows that the difference (2.1) is positive.

If $n - p + 1$ and d_3 are odd, the sum of the elements of D_H is also odd, and D_H cannot be a graphical sequence, since the parity condition of Theorem 1.3 is not satisfied. In this case we apply additional assignments such that d_3 occurs an even number of times in the sequence D_H. We distinguish two cases:

- If $p - 3 < n - p$, then $d_H(v_{p-1}) := d_3$. In this case, $p - 2$ summands in $\sum_{d_H(u),d_H(v)\in D_H} |d_H(u) - d_H(v)|$ increase by one, $n - p + 1$ summands decrease by one, and the rest remain unchanged. Thus, the total change $p - 2 - (n - p + 1)$ is negative.
- If $p - 3 \geq n - p$, then $d_H(v_p) := d_3 + 1$. After this assignment $p - 1$ summands in $\sum_{d_H(u),d_H(v)\in D_H} |d_H(u) - d_H(v)|$ decrease by one, $n - p$ summands increase by one, and the rest remain unchanged. Here also, the total change $-(p - 1) + n - p$ is negative.

Case 2. $p = n$. In this case D_G is comprised of three degrees and the degree d_3 occurs once. Assume that d_1 occurs x times, $n - 2 \geq x \geq 1$. Then d_2 occurs $n - x - 1$ times. We perform the following assignments:

$$d_H(v_{p-1}) := d_3 \quad \text{and} \quad d_H(v_i) := d_3 + 1, \quad i = 1, \dots p - 2.$$

For every $v \in V(G)$, consider the pair $(d_G(v), d_H(v))$. After the above transformation there are x pairs $(d_1, d_3 + 1)$, $n - x - 2$ pairs $(d_2, d_3 + 1)$, one pair (d_2, d_3) and one pair (d_3, d_3). Consequently.

$$\sum_{\substack{i=1,\dots,x\\ j=x+1,\dots,n-2}} (|d_G(v_i) - d_G(v_j)| - |d_H(v_i) - d_H(v_j)|) = x(n - x - 2)(d_1 - d_2),$$
$$\sum_{i=1,\dots,x} (|d_G(v_i) - d_G(v_{n-1})| - |d_H(v_i) - d_H(v_{n-1})|) = x(d_1 - d_2 - 1),$$
$$\sum_{i=1,\dots,x} (|d_G(v_i) - d_G(v_n)| - |d_H(v_i) - d_H(v_n)|) = x(d_1 - d_3 - 1),$$
$$\sum_{i=x+1,\dots,n-2} (|d_G(v_i) - d_G(v_{n-1})| - |d_H(v_i) - d_H(v_{n-1})|) = -(n - x - 2),$$
$$\sum_{i=x+1,\dots,n-2} (|d_G(v_i) - d_G(v_n)| - |d_H(v_i) - d_H(v_n)|) = (n - x - 2)(d_2 - d_3 - 1),$$

and

$$|d_G(v_{n-1}) - d_G(v_n)| - |d_H(v_{n-1}) - d_H(v_n)| = d_2 - d_3.$$

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Thus, the difference (2.1) is

$$x(n-x-2)(d_1-d_2)+x(d_1-d_2-1)+x(d_1-d_3-1)-(n-x-2)$$
$$+(n-x-2)(d_2-d_3-1)+d_2-d_3. \tag{2.2}$$

Since $d_1-d_2 \geq 1$, $d_1-d_2-1 \geq 0$, $d_1-d_3-1 \geq 1$, $d_2-d_3-1 \geq 0$, $d_2-d_3 \geq 1$ and $x \leq n-2$, the lower bound on (2.2) is

$$x(n-x-2)+x-(n-x-2)+1=(x-1)(n-x-2)+x+1. \tag{2.3}$$

Since $x \geq 1$, it follows that (2.3), and therefore (2.2) and (2.1) are positive.

In both cases 1 and 2, the sequence D_H is comprised of degrees d_3+1 and d_3, where d_3 occurs an even number of times and $d_3 \leq n-3$. Also, note that D_H does not necessarily satisfy the parity condition of Theorem 1.3. For example, this is the case precisely when n is odd and d_3 is even. In this case, we apply the assignment

$$d_H(v_i) := d_H(v_i)+1, \quad i=1,\dots n.$$

In the case where $d_3=1$ and d_3 occurs more than twice in D_H, we apply the assignment

$$d_H(v_i) := d_H(v_i)+2, \quad i=1,\dots n.$$

Now we have a degree sequence D'_H with $2 \leq y \leq n-2$ occurrences of degree $d+1$ and $n-y$ occurrences of degree d, where d occurs an even number of times in D'_H and

$$d=\begin{cases} d_3=1 & \text{and } d_3 \text{ occurs twice in } D_H,\\ d_3+2=3 & \text{that is, } d_3=1, \text{ and } d_3 \text{ occurs at least four times in } D_H,\\ d_3+1\geq 3 & \text{where } d_3 \text{ is even and } n \text{ is odd},\\ d_3 \geq 2 & \text{otherwise.}\end{cases} \tag{2.4}$$

It follows that for D'_H conditions (i) and (ii) of Theorem 1.6 are satisfied.

Next we show that D'_H is the graphical sequence of a graph H, by showing that D'_H satisfies condition (1.1). By Theorem 1.4 and Remark 1.5, it suffices to show that (1.1) is satisfied for $r=y$. With respect to y, we consider two cases.

- $y \leq d+1$. Then, for $r=x$, (1.1) can be written as

$$y(d+1) \leq y(y-1)+y(n-y) \quad \text{or} \quad y(d+1) \leq y(n-1),$$

  which holds for all four possibilities of d given in (2.4), since $d \leq d_3+2$ and $d_3 \leq n-3$.
- $y > d+1$. In this case, (1.1) can be written as

$$y(d+1) \leq y(y-1)+(n-y)(d+1) \quad \text{or} \quad 0 \leq y(y-d-2)+(n-y)(d+1),$$

  and it is satisfied because $d+2 \leq y$ and $y < n$.

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Thus, we have shown that D'_H is a graphical sequence of a connected graph H, and

$$\mathrm{irr}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)| > \frac{1}{2}\sum_{u,v\in V(H)} |d_H(u) - d_H(v)| = \mathrm{irr}_t(H),$$

which is a contradiction to the initial assumption that G is a nonregular connected graph with minimal total irregularity. □

Lemma 2.1 shows that a nonregular graph with minimal total irregularity must have degree set of cardinality two and both degrees must occur more than once. Obviously, the total irregularity is smallest if the two degrees differ as little as possible, that is, if they differ by one. In the next lemma we present sharper conditions on a nonregular graph of odd degree with minimal total irregularity.

Lemma 2.2. *Let G be a connected graph of odd order $n > 3$ with $|D(G)| = 2$, such that each degree occurs at least twice. Then there exists a connected graph H of the same order n, with one of the following degree sequences:*

$$\begin{gathered}(d+1, d, \ldots, d, d), \quad 3 \le d \le n-2, \quad \textit{or} \\ (d+2, d+2, \ldots, d+2, d+1), \quad 1 \le d \le n-4,\end{gathered} \tag{2.5}$$

where d is odd and $\mathrm{irr}_t(H) < \mathrm{irr}_t(G)$.

Proof. All possible sequences, where one degree occurs only once and the difference between the two degrees is one, are given in (2.5). In the sequel, we will show that they are indeed graphical sequences. Since n is odd, d must be odd as well, otherwise the parity condition of Theorem 1.3 will not be satisfied. Observe that the range of values of d in (2.5) follows from the fact that n and d are odd and $d \le n - 1$.

First, we show that, for a fixed d with $3 \le d \le n - 2$, the sequence $(d + 1, d, \ldots, d, d)$ is graphical. For that, we need to show in addition that (1.1) holds. In this case, by Theorem 1.4 and Remark 1.5, it suffices to show that (1.1) is satisfied for $r = 1$. Then, (1.1) can be written as

$$d + 1 \le \sum_{i=2}^{n} \min(r, d),$$

which obviously holds since $d \le n - 2$ and $r = 1$. Since $d \ge 3$, conditions (i) and (ii) from Theorem 1.6 are also satisfied.

Next, we show that, for a fixed d with $1 \le d \le n - 4$, the sequence $(d + 2, d + 2, \ldots, d + 2, d + 1)$ is graphical. Since $(n - 1)(d + 2) + d + 1$ is even, the parity condition of Theorem 1.3 is satisfied. By Theorem 1.4 and Remark 1.5, it suffices to show that (1.1) is satisfied for $r = n - 1$. Then, (1.1) can be written as

$$(n-1)(d+2) + d + 1 \le (n-1)(n-2) + d + 1 \quad \text{or} \quad 0 \le (n-1)(n-d-4) + d + 1.$$

The last expression holds since $d \le n - 4$. Since $d + 1 \ge 2$, conditions (i) and (ii) from Theorem 1.6 are also satisfied for the sequence $(d + 2, d + 2, \ldots, d + 2, d + 1)$.

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Next, we show that $\mathrm{irr}_t(H) < \mathrm{irr}_t(G)$. Assume that G has y vertices of degree $d+1$ and $n-y$ vertices of degree d, where $2 \le y < n$. Then

$$\mathrm{irr}_t(G) = y(n-y). \tag{2.6}$$

With the given constraints, $\mathrm{irr}_t(G)$ reaches its minimum of $2n-4$ for $y=2$ and $y=n-2$, which is larger than $\mathrm{irr}_t(H) = n-1$, for $n>3$. □

For an illustration of the above lemma consider the degree sequence $(n-1, n-2, \ldots, n-2, n-2)$. A graph with this degree sequence can be constructed by deleting $\lfloor n/2 \rfloor$ edges from K_n, such that no two deleted edges have a common end vertex.

For the degree sequence $(n-2, n-2, \ldots, n-2, n-3)$, a corresponding graph can be constructed by deleting $\lfloor n/2 \rfloor - 1$ edges from K_n, such that no two deleted edges have a common end vertex. There remain three vertices with degrees $n-1$. Finally, delete the two edges that connect one of those vertices with the remaining two.

We are now ready to present the sharp lower bound on the total irregularity of the connected nonregular graphs, as well as their second and the third minimal value with respect to the total irregularity.

Theorem 2.3. *Let G be a connected nonregular graph with n vertices. Then* $\mathrm{irr}_t(G) \ge n-1$. *Moreover, this bound is attained by the graphs of odd order characterised in Lemma 2.2.*

Proof. Let H be a graph with the minimal total irregularity. By Lemmas 2.1 and 2.2, the degree sequence of H consists of two different degrees that differ by one. Denote them by d and $d+1$. Assume that H has y vertices of degree d and $n-y$ vertices of degree $d+1$, where $1 \le y < n$. Its total irregularity is $y(n-y)$ (as in expression (2.6)). With the given constraints, $\mathrm{irr}_t(H)$ reaches its minimum of $n-1$ for $y=1$ and $y=n-1$. If n is even, we cannot obtain a graphical sequence for $y=1$ and $y=n-1$, since then the sum of all degrees is odd. Thus, $y=1$ or $y=n-1$ are feasible solutions only when n is odd. In this case also the degree that occurs only once must be even. In Lemma 2.2 the degree sequences of these graphs are fully characterised and the total irregularity of each one is $n-1$. □

Theorem 2.4. *The second and third smallest value of the total irregularity of connected nonregular graphs of order n are $2n-4$ and $2n-2$, respectively, and can be attained by graphs of order n with an arbitrary parity.*

Proof. As in Lemma 2.2, if a graph G has y vertices of degree $d+1$ and $n-y$ vertices of degree d, where $2 \le y < n$ and $d+1$ and d occur at least twice, then (2.6) is minimised for $y=2$ and $y=n-2$ and its value is $2n-4$. Such a sequence, with $d \ge 2$, satisfies conditions (i) and (ii) of Theorem 1.6 and the parity condition of Theorem 1.3 for both odd and even n. To show that the sequence is graphical, by Theorem 1.4 and Remark 1.5, it suffices to show that (1.1) is fulfilled for $r=y$. With respect to y, we consider two cases.

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- $y \leq d$. Then, for $r = y$, (1.1) can be written as
$$y(d+1) \leq y(y-1) + y(n-y) \quad \text{or} \quad 0 \leq y(n-d-2),$$
which is satisfied since $d \leq n-2$.
- $y > d$. In this case, (1.1) can be written as
$$y(d+1) \leq y(y-1) + d(n-y) \quad \text{or} \quad 0 \leq y(y-d-2) + d(n-y). \tag{2.7}$$
If $y = d+1$, it follows that $y(y-d-2) + d(n-y) = -d-1+d(n-y) = d(n-y-1)-1$. Since $y \leq n-2$ and $d \geq 2$, $d(n-y-1)-1 > 0$, and thus (2.7) holds. If $y \geq d+2$, from $y > d$ and $d \leq n-2$, it follows that $y(y-d-2)+d(n-y) \geq d(y-d-2)+d(n-y) = d(n-d-2) \geq 0$.

Thus, we have shown that a sequence comprised of y vertices of degree $d+1$ and $n-y$ vertices of degree d is graphical and the corresponding connected graphs have total irregularity $2n-4$.

A candidate that may have smaller total irregularity than $2n-4$ is a graph with two degrees which differ by more than one and where one of the degrees occurs only once. Obviously, if the difference between the two degrees is larger (than two) the total irregularity will be larger as well. Thus, let us consider first the case where the degrees differ by two. If this results in a total irregularity larger than $2n-4$, then we do not need to check the cases where the difference between the two degrees is larger than two. So, consider the degree sequences

$$\begin{aligned}
(d, d-2, \ldots, d-2, d-2), \quad & d = 3 \text{ and } n = 4;\ \text{or } 4 \leq d, d \text{ is even and } d+1 < n;\\
& \text{or } 5 \leq d, d \text{ is odd}, n \text{ is even and } d+1 < n;\\
(d, d, \ldots, d, d-2), \quad & 4 \leq d, d \text{ is even and } d+1 < n;\\
& \text{or } 3 \leq d, d \text{ is odd}, n \text{ is even and } d+1 < n.
\end{aligned} \tag{2.8}$$

The constraints on d ensure that conditions (i) and (ii) of Theorem 1.6 and the parity condition of the Erdős–Gallai theorem are satisfied, so the degree sequences in (2.8) may belong to connected graphs. Next, we show that the degree sequences in (2.8) also satisfy relation (1.1).

First consider the sequences $(d, d-2, \ldots, d-2, d-2)$. By Theorem 1.4 and Remark 1.5, it suffices to show that (1.1) is satisfied for $r = 1$. Now (1.1) can be written as
$$d \leq \sum_{i=2}^{n} \min(r, d-2).$$
Obviously, the inequality holds, since $r = 1$ and $d < n-1$.

Next, consider the sequences $(d, d, \ldots, d, d-2)$. In this case, by Theorem 1.4 and Remark 1.5, it suffices to show that (1.1) is satisfied for $r = n-1$. Then (1.1) can be written as
$$(n-1)d \leq (n-2)(n-1) + d - 2 \quad \text{or} \quad 0 \leq (n-d-2)(n-1) + d - 2.$$

Since $d < n - 1$ and $n \geq 3$, it follows that the last inequalities hold. Thus, we have shown that the degree sequences in (2.8) are graphical and the corresponding graphs have total irregularity $2n - 2$. It follows that $2n - 4$ is the second smallest value of the total irregularity of connected nonregular graphs.

Note that for every graph the total irregularity is an even number, since the number of vertices of odd degree is even. Thus, the value $2n - 3$ is excluded as a value of the total irregularity and the third smallest value is $2n - 2$. The degree sequences of graphs with such total irregularity were characterised in (2.8). □

We note that for disconnected nonregular graphs, just as for connected graphs, one may obtain the above presented bounds simply by including zero as a possible degree. For example, the sharp lower bound in the case of disconnected nonregular graphs of $n - 1$ can be obtained by a graph of odd order n with degree sequence $(1, 1, \ldots, 1, 0)$, comprising $(n - 1)/2$ pairs of adjacent vertices and one isolated vertex.

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