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TOEPLITZ OPERATORS BETWEEN FOCK SPACES

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Abstract

Given a positive Borel measure μ on the n-dimensional Euclidean space $\mathbb{C}^n$, we characterise the boundedness (and compactness) of Toeplitz operators T_μ between Fock spaces $F^\infty(\varphi)$ and $F^p(\varphi)$ with $0 < p \le \infty$ in terms of t-Berezin transforms and averaging functions of μ. Our result extends recent work of Mengestie ['On Toeplitz operators between Fock spaces', *Integral Equations Operator Theory* **78** (2014), 213–224] and others.

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1. Introduction

Let dv denote the Lebesgue measure on $\mathbb{C}^n$ and $\omega_0 = dd^c|z|^2$ the Euclidean Kähler form on $\mathbb{C}^n$, where $d^c = (\sqrt{-1}/4)(\overline{\partial} - \partial)$. Throughout the paper, we assume that $\varphi \in C^2(\mathbb{C}^n)$ and satisfies $0 < m\omega_0 \le dd^c\varphi \le M\omega_0$ for two positive constants m and M.

Given $0 < p < \infty$ and $\mu \ge 0$, the space $L^p(\varphi, \mu)$ consists of all μ-measurable functions f for which

$$\|f\|_{p,\varphi,\mu} = \left(\int_{\mathbb{C}^n} |f(z)|^p e^{-p\varphi(z)}\, d\mu(z)\right)^{1/p} < \infty.$$

When $d\mu = dv$, we write $L^p(\varphi, \mu)$ and $\|f\|_{p,\varphi,\mu}$ as $L^p(\varphi)$ and $\|f\|_{p,\varphi}$ for short. Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. The Fock space is defined to be

$$F^p(\varphi) = L^p(\varphi) \cap H(\mathbb{C}^n)$$

and

$$F^\infty(\varphi) = \left\{ f \in H(\mathbb{C}^n) : \|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}^n} |f(z)|e^{-\varphi(z)} < \infty \right\}.$$

It is well known that $F^p(\varphi)$ is a Banach space with norm $\|\cdot\|_{p,\varphi}$ for $1 \le p \le \infty$ and $F^p(\varphi)$ is a Fréchet space under $d(f, g) = \|f - g\|^p_{p,\varphi}$ for $0 < p < 1$. In the simplest case

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that $\varphi(z) = (\alpha/2)|z|^2$ with $\alpha > 0$, $F^p(\varphi)$ is the classical Fock space F^p_α, which has been studied by many authors (see [2, 7] and the references therein).

With the Bergman kernel $K_\varphi(\cdot,\cdot)$ for $F^2(\varphi)$ (see [6]), the orthogonal projection $P : L^2(\varphi) \to F^2(\varphi)$ can be represented as

$$Pf(z) = \int_{\mathbb{C}^n} K_\varphi(z,w)f(w)e^{-2\varphi(w)}\,dv(w).$$

By [6], we have $Pf = f$ for $f \in F^p(\varphi)$ and $0 < p \le \infty$. Given a positive Borel measure μ on $\mathbb{C}^n$ (written as $\mu \ge 0$ for short), we define the Toeplitz operator T_μ on $F^p(\varphi)$ as

$$T_\mu f(z) = \int_{\mathbb{C}^n} K_\varphi(z,w)f(w)e^{-2\varphi(w)}\,d\mu(w), \quad z \in \mathbb{C}^n,$$

if it can be well defined.

Positive Toeplitz operators on Fock spaces have been studied by many authors. In 2010, Isralowitz and Zhu [4] identified the boundedness, compactness and Schatten class of T_μ with $\mu \ge 0$ on the classical Fock space F^2_α with $\alpha > 0$, in terms of the average function and the t-Berezin transform. Hu and Lv discussed the same problems from one Fock space F^p_α to another F^q_α for all possible $1 < p, q < \infty$ in [2]. Mengestie [5] extended the corresponding problems from F^p_α to F^∞_α and from F^∞_α to F^p_α for $1 < p < \infty$, respectively. In 2012, Schuster and Varolin [6] gave the definition of the Fock space $F^p(\varphi)$ and obtained the characterisation on $\mu \ge 0$ such that T_μ are bounded or compact on $F^p(\varphi)$ for $1 < p < \infty$. Hu and Lv discussed the same problems from $F^p(\varphi)$ to $F^q(\varphi)$ for $0 < p, q < \infty$ in [3].

The purpose of this paper is to characterise those measures $\mu \ge 0$ for which the induced Toeplitz operators T_μ are bounded (or compact) from $F^p(\varphi)$ to $F^\infty(\varphi)$ (or from $F^\infty(\varphi)$ to $F^p(\varphi)$), where $0 < p \le \infty$. Our result extends those in [2–6]. Notice that $F^\infty(\varphi)$ is not self-adjoint and neither is $F^p(\varphi)$ a Banach space for $0 < p < 1$, so the approach in [2, 4–6] does not work.

Throughout the paper, the symbol C will stand for a positive constant, which may change from line to line, but does not depend on the functions being considered. Two quantities A and B are called equivalent, denoted by '$A \simeq B$', if there exists some C such that $C^{-1}A \le B \le CA$.

2. Main results

In this section, we state our main results. Before that, let us give some notation and lemmas.

For $z \in \mathbb{C}^n$ and $r > 0$, let $B(z,r) = \{w \in \mathbb{C}^n : |w - z| < r\}$. Given $\mu \ge 0$, the average of μ is $\mu(B(z,r))/v(B(z,r))$. We simply write the average function of μ as

$$\widehat{\mu}_r(\cdot) = \mu(B(\cdot,r)),$$

since $v(B(\cdot,r)) \simeq r^{2n}$. Fixing $t > 0$, the t-Berezin transform of μ is defined to be

$$\widetilde{\mu}_t(z) = \int_{\mathbb{C}^n} \left| \frac{K_\varphi(z,w)}{\sqrt{K_\varphi(z,z)K_\varphi(w,w)}} \right|^t d\mu(w).$$

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Notice that $\widetilde{\mu}_2$ is just the classical Berezin transform $\widetilde{\mu}$ defined in [6]. When $\varphi(z) = \frac{1}{2}|z|^2$, the t-Berezin transform is closely connected with the heat flow as mentioned in [1].

Given some $r > 0$, we call a sequence $\{a_k\}_{k=1}^{\infty}$ in $\mathbb{C}^n$ an r-lattice if the balls $\{B(a_k, r)\}_{k=1}^{\infty}$ cover $\mathbb{C}^n$ and $\{B(a_k, r/2)\}_{k=1}^{\infty}$ are pairwise disjoint. For $r > 0$, much more easily than in the Bergman space setting, we can find r-lattices in $\mathbb{C}^n$ and there exists an integer N such that

$$1 \le \sum_{k=1}^{\infty} \chi_{B(a_k, 2r)}(z) \le N, \quad z \in \mathbb{C}^n.$$

For $0 < p < \infty$, the Lebesgue space L^p is defined as

$$L^p = \Big\{ f \text{ is Lebesgue measurable on } \mathbb{C}^n : \|f\|_{L^p} = \Big(\int_{\mathbb{C}^n} |f|^p \, dv \Big)^{1/p} < \infty \Big\}.$$

The space l^p consists of all sequences $\{b_k\}_{k=1}^{\infty} \subset \mathbb{C}^n$ with

$$\|\{b_k\}_k\|_{l^p} = \Big(\sum_{k=1}^{\infty} |b_k|^p \Big)^{1/p} < \infty.$$

Lemma 2.1 lists some well-known results, which will be useful in the following section (see [3]).

Lemma 2.1. *The Bergman kernel $K_\varphi(\cdot, \cdot)$ has the following properties.*

(1) *There exist positive constants θ and C such that for all $z, w \in \mathbb{C}^n$,*

$$|K_\varphi(z, w)| e^{-\varphi(z)} e^{-\varphi(w)} \le C e^{-\theta|z-w|} \le C.$$

(2) *There exists C such that for $z \in \mathbb{C}^n$ and $w \in B(z, r_0)$,*

$$|K_\varphi(z, w)| e^{-\varphi(z)} e^{-\varphi(w)} \ge C.$$

(3) *For $z \in \mathbb{C}^n$,*

$$K_\varphi(z, z) \simeq e^{2\varphi(z)}.$$

(4) *For $z \in \mathbb{C}^n$ and $0 < p \le \infty$,*

$$\|k_{\varphi,z}\|_{p,\varphi} \simeq 1, \quad \text{where } k_{\varphi,z}(\cdot) = \frac{K_\varphi(\cdot, z)}{\sqrt{K_\varphi(z, z)}}.$$

Lemma 2.2. *Suppose $\mu \ge 0$, $R > 0$ and $0 < p, q \le \infty$. Set $\mu_R(E) = \int_{E \cap \{z : |z| \le R\}} d\mu$ for $E \subseteq \mathbb{C}^n$ measurable. If $\widehat{\mu}_r \in L^s$ for some $r, s > 0$, then the Toeplitz operator $T_{\mu_R} : F^p(\varphi) \to F^q(\varphi)$ is compact.*

Proof. Suppose that $\{f_m\}_{m=1}^{\infty}$ is a bounded sequence in $F^p(\varphi)$ which converges uniformly to 0 on compact subsets of $\mathbb{C}^n$ as $m \to \infty$. By Montel's theorem, we only need to show that

$$\|T_{\mu_R} f_m\|_{q,\varphi} \to 0 \quad \text{as } m \to \infty. \tag{2.1}$$

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Since $\widehat{\mu}_r \in L^s$, we have $\widehat{\mu}_r \in L^\infty$. Therefore,

$$
\begin{aligned}
|(T_{\mu_R} f_m)(z)|e^{-\varphi(z)} &\le \int_{\mathbb{C}^n} |f_m(w)||K_\varphi(z,w)|e^{-2\varphi(w)}\,d\mu_R(w)e^{-\varphi(z)} \\
&\le C\int_{|w|\le R+r} |f_m(w)||K_\varphi(z,w)|e^{-2\varphi(w)}\widehat{\mu}_r(w)\,dv(w)e^{-\varphi(z)} \\
&\le C\|\widehat{\mu}_r\|_{L^\infty}\int_{|w|\le R+r} |f_m(w)||K_\varphi(z,w)|e^{-2\varphi(w)}\,dv(w)e^{-\varphi(z)}.
\end{aligned}
$$

This, together with Lemma 2.1, gives

$$
\begin{aligned}
\|T_{\mu_R} f_m\|_{\infty,\varphi} &\le C\sup_{z\in\mathbb{C}^n}\int_{|w|\le R+r} |f_m(w)||K_\varphi(z,w)|e^{-2\varphi(w)}\,dv(w)e^{-\varphi(z)} \\
&\le C\sup_{z\in\mathbb{C}^n}\int_{|w|\le R+r} |f_m(w)|e^{-\varphi(w)}e^{-\theta|w-z|}\,dv(w) \\
&\le C\sup_{|w|\le R+r}|f_m(w)|e^{-\varphi(w)}\sup_{z\in\mathbb{C}^n} e^{-\theta q|z|} \\
&\le C\sup_{|w|\le R+r}|f_m(w)| \to 0
\end{aligned}
$$

as $m\to\infty$. Hence, (2.1) is true if $q=\infty$. For $0<q<\infty$,

$$
\begin{aligned}
&\int_{\mathbb{C}^n} |(T_{\mu_R} f_m)(z)e^{-\varphi(z)}|^q\,dv(z) \\
&\quad\le C\int_{\mathbb{C}^n}\bigg|\int_{|w|\le R+r}|f_m(w)|e^{-\varphi(w)}e^{-\theta|w-z|}\,dv(w)\bigg|^q\,dv(z) \\
&\quad\le C\sup_{|w|\le R+r}|f_m(w)|e^{-\varphi(w)}\int_{\mathbb{C}^n} e^{-\theta q|z|+\theta q(R+r)}\,dv(z) \\
&\quad\le C\sup_{|w|\le R+r}|f_m(w)|.
\end{aligned}
$$

Thus, (2.1) is still true. This completes the proof. □

Theorem 2.3. *Suppose $0<p\le\infty$ and $\mu\ge 0$. Then the following statements are equivalent:*

(A) *$T_\mu : F^p(\varphi)\to F^\infty(\varphi)$ is bounded;*
(B) *$\widetilde{\mu}_t$ is bounded on $\mathbb{C}^n$ for some (or any) $t>0$;*
(C) *$\widehat{\mu}_\delta$ is bounded on $\mathbb{C}^n$ for some (or any) $\delta>0$;*
(D) *the sequence $\{\widehat{\mu}_r(a_k)\}_{k=1}^\infty$ is bounded for some (or any) r-lattice $\{a_k\}_{k=1}^\infty$.*

Furthermore,

$$
\|T_\mu\|_{F^p(\varphi)\to F^\infty(\varphi)} \simeq \sup_{z\in\mathbb{C}^n}\widetilde{\mu}_t(z) \simeq \sup_{z\in\mathbb{C}^n}\widehat{\mu}_\delta(z) \simeq \sup_k \widehat{\mu}_r(a_k). \tag{2.2}
$$

Proof. By [3, Lemma 2.3], we obtain the equivalence between statements (B), (C) and (D) with the corresponding norm estimates in (2.2). Now, we assume that

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$T_\mu : F^p(\varphi) \to F^\infty(\varphi)$ is bounded. For $z \in \mathbb{C}^n$, using Lemma 2.1 and [3, (2.6)],

$$\begin{aligned} \widetilde{\mu}_2(z) &\simeq e^{-\varphi(z)} \int_{\mathbb{C}^n} k_{\varphi,z}(w) K_\varphi(z,w) e^{-2\varphi(w)}\, d\mu(w) \\ &= |T_\mu k_{\varphi,z}(z)| e^{-\varphi(z)} \\ &\le \|T_\mu k_{\varphi,z}\|_{\infty,\varphi} \\ &\le C \|T_\mu\|_{F^p(\varphi)\to F^\infty(\varphi)} \|k_{\varphi,z}\|_{p,\varphi} \\ &\le C \|T_\mu\|_{F^p(\varphi)\to F^\infty(\varphi)}. \end{aligned} \tag{2.3}$$

This shows that $\widetilde{\mu}_2$ is bounded on $\mathbb{C}^n$. Moreover, (2.3) gives

$$\sup_{z\in\mathbb{C}^n} \widetilde{\mu}_2(z) \le C \|T_\mu\|_{F^p(\varphi)\to F^\infty(\varphi)}. \tag{2.4}$$

On the other hand, suppose that $\widehat{\mu}_\delta$ is bounded on $\mathbb{C}^n$ for $\delta > 0$. By [3, Theorem 2.6], the embedding $i : F^1(\varphi) \to L^1(\varphi, \mu)$ is bounded and

$$\|i\|_{F^1(\varphi)\to L^1(\varphi,\mu)} \simeq \sup_{z\in\mathbb{C}^n} \widehat{\mu}_\delta(z) \le C.$$

Given any $f \in F^p(\varphi)$, by the fact that $\|\cdot\|_{\infty,\varphi} \le C\|\cdot\|_{p,\varphi}$ and Lemma 2.1,

$$\begin{aligned} |T_\mu f(z) e^{-\varphi(z)}| &\le \int_{\mathbb{C}^n} |k_{\varphi,z}(w)||f(w)| e^{-2\varphi(w)}\, d\mu(w) \\ &\le \|f\|_{\infty,\varphi} \int_{\mathbb{C}^n} |k_{\varphi,z}(w)| e^{-\varphi(w)}\, d\mu(w) \\ &\lesssim \|i\|_{F^1(\varphi)\to L^1(\varphi,\mu)} \|f\|_{p,\varphi} \|k_{\varphi,z}\|_{1,\varphi} \\ &\simeq \|i\|_{F^1(\varphi)\to L^1(\varphi,\mu)} \|f\|_{p,\varphi}. \end{aligned}$$

Therefore, T_μ is bounded from $F^p(\varphi)$ to $F^\infty(\varphi)$. This, together with (2.4), gives (2.2) and completes the proof. □

Theorem 2.4. *Suppose $0 < p \le \infty$ and $\mu \ge 0$. Then the following statements are equivalent:*

(A) *$T_\mu : F^p(\varphi) \to F^\infty(\varphi)$ is compact;*
(B) *$\widetilde{\mu}_t(z) \to 0$ as $z \to \infty$ for some (or any) $t > 0$;*
(C) *$\widehat{\mu}_\delta(z) \to 0$ as $z \to \infty$ for some (or any) $\delta > 0$;*
(D) *$\widehat{\mu}_r(a_k) \to 0$ as $k \to \infty$ for some (or any) r-lattice $\{a_k\}_{k=1}^\infty$.*

Proof. The equivalence between (B), (C) and (D) comes from [3, Theorem 2.7]. Now we suppose $\lim_{z\to\infty} \widehat{\mu}_\delta(z) = 0$ and take μ_R as in Lemma 2.2. Then $\mu - \mu_R \ge 0$, $T_{\mu-\mu_R}$ is bounded from $F^p(\varphi)$ to $F^\infty(\varphi)$ and

$$\|(\widehat{\mu - \mu_R})_r\|_{L^\infty} \to 0 \quad (R \to \infty)$$

for $r > 0$. By Theorem 2.3,

$$\|T_\mu - T_{\mu_R}\|_{F^p(\varphi)\to F^\infty(\varphi)} = \|T_{\mu-\mu_R}\|_{F^p(\varphi)\to F^\infty(\varphi)} \simeq \|(\widehat{\mu - \mu_R})_r\|_{L^\infty} \to 0$$

as $R \to \infty$. Lemma 2.2 shows that $T_{\mu_R} : F^p(\varphi) \to F^\infty(\varphi)$ is compact. So, the operator T_μ is compact from $F^p(\varphi)$ to $F^\infty(\varphi)$.

On the other hand, let T_μ be compact from $F^p(\varphi)$ to $F^q(\varphi)$. By Theorem 2.3, we know that $\widehat{\mu}_\delta$ is bounded for $\delta > 0$. Since $\{k_{\varphi,z} : z \in \mathbb{C}^n\}$ is bounded in $F^p(\varphi)$, $\{T_\mu k_{\varphi,z} : z \in \mathbb{C}^n\}$ is relatively compact in $F^\infty(\varphi)$. Thus, for any sequence $\{z_k\}_{k=1}^\infty \subset \mathbb{C}^n$ satisfying $\lim_{j\to\infty} z_j = \infty$, there exists a subsequence of $\{T_\mu k_{\varphi,z_j}\}_{j=1}^\infty$ which converges to some h in $F^\infty(\varphi)$. Without loss of generality, we may assume that

$$\lim_{j\to\infty} \|T_\mu k_{\varphi,z_j} - h\|_{\infty,\varphi} = 0. \tag{2.5}$$

We claim that $h \equiv 0$. In fact, for any $w \in \mathbb{C}^n$, since $\widehat{\mu}_\delta$ is bounded,

$$T_\mu k_{\varphi,z_j}(w) = \langle T_\mu k_{\varphi,z_j}, K_\varphi(\cdot, w)\rangle = \int_{\mathbb{C}^n} k_{\varphi,z_j}(u) K_\varphi(w,u) e^{-2\varphi(u)}\, d\mu(u),$$

by Fubini's theorem. For any $\varepsilon > 0$, since $\|K_\varphi(w,\cdot)\|_{2,\varphi} \le Ce^{\varphi(w)}$, there is some $R > 0$ such that

$$\int_{|u|>R} |K_\varphi(w,u)|^2 e^{-2\varphi(u)}\, dv(u) < \varepsilon^2.$$

Now, by Lemma 2.1,

$$|k_{\varphi,z_j}(u)| \le Ce^{-\theta|z_j-u|+\varphi(u)} \le Ce^{-\theta|z_j|} e^{\varphi(u)+\theta|u|} \to 0$$

uniformly on any compact subset of $\mathbb{C}^n$ as $j \to \infty$. This, together with [3, Lemma 2.2], shows that

$$\begin{aligned}
|T_\mu k_{\varphi,z_j}(w)| &\le \int_{\mathbf{C}^n} |k_{\varphi,z_j}(u) K_\varphi(w,u) e^{-2\varphi(u)}| \widehat{\mu}_\delta(u)\, dv(u) \\
&\le C\Big(\int_{|u|\le R} + \int_{|u|>R}\Big) |k_{\varphi,z_j}(u) K_\varphi(w,u) e^{-2\varphi(u)}|\, dv(u) \\
&< C\varepsilon + \|k_{\varphi,z_j}\|_{2,\varphi} \Big(\int_{|u|>R} |K_\varphi(w,u)|^2 e^{-2\varphi(u)}\, dv(u)\Big)^{1/2} \\
&< C\varepsilon
\end{aligned}$$

if j is sufficiently large, where C is independent of ε. Thus,

$$\lim_{j\to\infty} T_\mu k_{\varphi,z_j}(w) = 0, \quad w \in \mathbb{C}^n.$$

On the other hand, by (2.5),

$$\lim_{j\to\infty} T_\mu k_{\varphi,z_j}(w) = h(w)$$

for $w \in \mathbb{C}^n$. Therefore, $h \equiv 0$. That is,

$$\lim_{j\to\infty} \|T_\mu k_{\varphi,z_j}\|_{\infty,\varphi} = 0.$$

This, combined with (2.3), implies that

$$\widetilde{\mu}_2(z_j) \simeq |T_\mu k_{\varphi,z_j}(z_j)| e^{-\varphi(z_j)} \le \|T_\mu k_{\varphi,z_j}\|_{\infty,\varphi} \to 0 \quad \text{as } j \to \infty.$$

Hence, $\lim_{z\to\infty} \widetilde{\mu}_2(z) = 0$. The proof is complete. □

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Theorem 2.5. *Suppose* $0 < p < \infty$ *and* $\mu \geq 0$. *Then the following statements are equivalent:*

(A) $T_\mu : F^\infty(\varphi) \to F^p(\varphi)$ *is bounded;*
(B) $T_\mu : F^\infty(\varphi) \to F^p(\varphi)$ *is compact;*
(C) $\widetilde{\mu}_t \in L^p$ *for some (or any)* $t > 0$*;*
(D) $\widehat{\mu}_\delta \in L^p$ *for some (or any)* $\delta > 0$*;*
(E) $\{\widehat{\mu}_r(a_k)\}_{k=1}^\infty \in l^p$ *for some (or any) r-lattice* $\{a_k\}_{k=1}^\infty$.

Furthermore,

$$\|T_\mu\|_{F^\infty(\varphi)\to F^p(\varphi)} \simeq \|\widetilde{\mu}_t\|_{L^p} \simeq \|\widehat{\mu}_\delta\|_{L^p} \simeq \|\{\widehat{\mu}_r(a_k)\}_k\|_{l^p}. \tag{2.6}$$

Proof. By [3, Lemma 2.3], we obtain the equivalence of (C), (D) and (E) with the corresponding norm estimates in (2.6). The implication (B) $\Rightarrow$ (A) is trivial, so we complete the proof by showing that (A) $\Rightarrow$ (E) and (E) $\Rightarrow$ (B).

(A) $\Rightarrow$ (E). Given any bounded sequence $\{\lambda_k\}$ and r_0-lattice $\{a_k\}$, with r_0 as in Lemma 2.1(2), set

$$f_{a_k}(z) = \sum_{k=1}^\infty \lambda_k k_{\varphi,a_k}(z).$$

By [3, Lemma 2.4], $f_{a_k} \in F^\infty(\varphi)$ and $\|f_{a_k}\|_{\infty,\varphi} \leq C \sup_k |\lambda_k|$. Since $T_\mu : F^\infty(\varphi) \to F^p(\varphi)$ is bounded, $T_\mu f_{a_k} \in F^p(\varphi)$. By Khinchine's inequality and Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbb{C}^n} \Big(\sum_{k=1}^\infty |\lambda_k T_\mu(k_{\varphi,a_k})(z)|^2\Big)^{p/2} e^{-p\varphi(z)}\, dv(z)\\
&\quad \leq C \int_{\mathbb{C}^n} \int_0^1 \Big|\sum_{k=1}^\infty \psi_k(t)\lambda_k T_\mu(k_{\varphi,a_k})(z)\Big|^p dt e^{-p\varphi(z)}\, dv(z)\\
&\quad = C \int_0^1 \Big\|T_\mu\Big(\sum_{k=1}^\infty \psi_k(t)\lambda_k k_{\varphi,a_k}\Big)\Big\|_{p,\varphi}^p dt\\
&\quad \leq C \int_0^1 \|T_\mu\|_{F^\infty(\varphi)\to F^p(\varphi)}^p \Big\|\sum_{k=1}^\infty \psi(t)\lambda_k k_{\varphi,a_k}\Big\|_{\infty,\varphi}^p dt\\
&\quad \leq C\|T_\mu\|_{F^\infty(\varphi)\to F^p(\varphi)}^p \sup_k |\lambda_k|^p,
\end{aligned}$$

where ψ_k is the kth Rademacher function on $[0, 1]$. Since the balls $\{B(a_j, r_0)\}_j$ cover $\mathbb{C}^n$ and the cover has a bounded number of sheets,

$$\begin{aligned}
&\int_{\mathbb{C}^n} \Big(\sum_{k=1}^\infty |\lambda_k T_\mu(k_{\varphi,a_k})(z)|^2\Big)^{p/2} e^{-p\varphi(z)}\, dv(z)\\
&\quad \geq C \sum_{j=1}^\infty \int_{B(a_j,r_0)} \Big(\sum_{k=1}^\infty |\lambda_k T_\mu(k_{\varphi,a_k})(z)|^2\Big)^{p/2} e^{-p\varphi(z)}\, dv(z)\\
&\quad \geq C \sum_{j=1}^\infty \int_{B(a_j,r_0)} |\lambda_j T_\mu(k_{\varphi,a_j})(z)|^p e^{-p\varphi(z)}\, dv(z)
\end{aligned}$$

$$\geq C \sum_{j=1}^{\infty} |\lambda_j|^p |T_\mu(k_{\varphi,a_j})(a_j)|^p e^{-p\varphi(a_j)}$$
$$\geq C \sum_{j=1}^{\infty} |\lambda_j|^p \widehat{\mu}_{r_0}(a_j)^p.$$

Take $\beta_j = |\lambda_j|^p$ for each j. Then $\{\beta_j\}_{j=1}^{\infty}$ is bounded and

$$\sum_{j=1}^{\infty} \beta_j \widehat{\mu}_{r_0}(a_j)^p \leq C \|T_\mu\|^p_{F^\infty(\varphi)\to F^p(\varphi)} \sup_j |\beta_j|.$$

Therefore, $\{\widehat{\mu}_{r_0}(a_j)^p\}_{j=1}^{\infty} \in l^1$ and

$$\|\{\widehat{\mu}_{r_0}(a_j)\}_j\|_{l^p} \leq C \|T_\mu\|_{F^\infty(\varphi)\to F^p(\varphi)}. \tag{2.7}$$

(E) $\Rightarrow$ (B). Define μ_R as in Lemma 2.2. Then $\mu - \mu_R \geq 0$. Similarly to the proof of Theorem 3.6 in [3],

$$\|T_\mu\|_{F^\infty(\varphi)\to F^p(\varphi)} \leq C \|\{\widehat{\mu}_r(a_k)\}_k\|_{l^p}. \tag{2.8}$$

So,

$$\|T_\mu - T_{\mu_R}\|_{F^\infty(\varphi)\to F^p(\varphi)} = \|T_{\mu-\mu_R}\|_{F^\infty(\varphi)\to F^p(\varphi)} \simeq \|\{(\widehat{\mu - \mu_R})_r(a_k)\}_k\|_{l^p} \to 0$$

as $R \to \infty$, since $\{\widehat{\mu}_r(a_k)\}_k \in l^p$. Lemma 2.2 shows that $T_{\mu_R} : F^\infty(\varphi) \to F^p(\varphi)$ is compact and so $T_\mu : F^\infty(\varphi) \to F^p(\varphi)$ is compact.

The estimate (2.6) comes from (2.7) and (2.8). The proof is complete. □

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