Canad. Math. Bull. Vol. **31** (1), 1988

A METHOD OF FORCED MONOTONICITY FOR CONJUGATE TYPE BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. Let $I = [a, b] \subseteq \mathbf{R}$ and let L be an nth order linear differential operator defined on $C^n(I)$. Let $2 \leqq k \leqq n$ and let $a \leqq x_1 < x_2 < \ldots < x_n \leqq b$. A method of forced monotonicity is used to construct monotone sequences that converge to solutions of the conjugate type boundary value problem (BVP) $Ly = f(x, y)$, $y^{(i-1)}(x_j) = r_{ij}$, where $1 \leqq i \leqq m_j$, $1 \leqq j \leqq k$, $\sum_{j=1}^{k} m_j = n$, and $f:I \times \mathbf{R} \to \mathbf{R}$ is continuous. A comparison theorem is employed and the method requires that the Green's function of an associated BVP satisfies certain sign conditions.

Let $n \geqq 2$, $2 \leqq k \leqq n$, let $I = [a, b] \subseteq \mathbf{R}$, and let $a = x_1 < x_2 < \ldots < x_k = b$. Let $p_i(x) \in C(I)$, $1 \leqq i \leqq n$, and consider the nth order linear differential operator

$$(1) \qquad Ly \equiv y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y.$$

We are concerned with the existence of solutions of the boundary value problem (BVP)

$$(2) \qquad Ly = f(x, y),$$

$$(3) \qquad y^{(i-1)}(x_j) = r_{ij}, \quad 1 \leqq i \leqq m_j, \quad 1 \leqq j \leqq k,$$

where $f:I \times \mathbf{R} \to \mathbf{R}$ is continuous, $\sum_{j=1}^{k} m_j = n$, and $r_{ij} \in \mathbf{R}$, $1 \leqq i \leqq m_j$, $1 \leqq j \leqq k$. We shall denote the conjugate type boundary conditions (3) by $By = r$.

The equation $Ly = 0$ is disconjugate on I if the only solution of $Ly = 0$ having n zeros on I, counting multiplicities, is $y \equiv 0$. Levin [4] (or see [2]) showed that if $Ly = 0$ is disconjugate on I, then the Green's function, $G(x, s)$, of the BVP $Ly = 0$, $By = 0$ exists and satisfies the inequality

$$(4) \qquad G(x, s)(x - x_1)^{m_1} \ldots (x - x_k)^{m_k} \geqq 0, \quad x_1 < s < x_k.$$

We point out that Levin [4] obtained (4) in a more general setting where the

Received by the editors June 8, 1986 and, in revised form January 19, 1987.

AMS Subject Classification (1980): Primary 34B10; Secondary 34B27.

coefficients $p_i(x)$, $1 \leqq i \leqq n$, in (1) are locally integrable on I. We shall assume that $Ly = 0$ is disconjugate on I and define subsets I_1 and I_2 of I such that $I = I_1 \cup I_2$, $I_1 \cap I_2 = \phi$, and

(5) $\quad G(x, s) \leqq 0$ for $(x, s) \in I_1 \times I$, $\quad G(x, s) \geqq 0$ for $(x, s) \in I_2 \times I$.

Šeda [6] employs [5] to obtain the existence of solutions of the BVP (2), (3). Assuming that $f(x, y)$ is monotone decreasing in y for each $x \in I_1$ and monotone increasing in y for each $x \in I_2$, he uses the method of upper and lower solutions to construct monotone iteration schemes to approximate solutions of the BVP (2), (3). Moreover, the limiting functions of these iteration schemes are solutions of the BVP (2), (3). Eloe and Grimm [3] also employ (5) to show existence of solutions of the BVP (2), (3). They assume that $f(x, y)$ satisfies a Lipschitz condition in y and construct a method of forced monotonicity in which the limiting functions of the iteration schemes provide *a priori* bounds on solutions of the BVP (2), (3) and, in general, are not solutions of the BVP (2), (3).

In this paper, we shall assume that $f(x, y)$ satisfies a Lipschitz condition in y as in [3] and under a stronger assumption we shall construct a method of forced monotonicity by adding a linear term $P(x)y$ to both sides of equation (2). The limiting functions of the iteration schemes are solutions of the BVP (2), (3). Werner [7] employs this method for systems of first order ordinary differential equations where the forcing term $P(x)$ has continuous entries. In this paper, the scalar function $P(x)$ is piecewise continuous on I and hence, locally integrable on I.

Our method is a generalization of the procedure used by Bernfeld and Lakshmikantham [1]. Bernfeld and Lakshmikantham use a maximum principle to construct a monotone iteration scheme which yields monotone sequences that converge to extremal solutions of a second order BVP. By using signs of the Green's function, we get extremal solutions for an nth order BVP.

We begin by stating a comparison theorem without proof.

LEMMA. *Let q_1 and q_2 be locally integrable functions on I such that $0 \leqq q_1(x) \leqq q_2(x)$ almost everywhere on I. If both equations $Ly = 0$ and $Ly + q_2 y = 0$ are disconjugate on I, then the equation $Ly + q_1 y = 0$ is disconjugate on I.*

Nehari [5, Theorem 4.1] states and proves this lemma for the case where q_1 and q_2 are continuous in I. We omit the proof since Nehari's proof carries over to the case where q_1 and q_2 are locally integrable on I.

Now, let f be as in (2) and assume that there exists a constant $P > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqq P|y_1 - y_2| \tag{6}$$

for all (x, y_1), (x, y_2) in $I \times \mathbf{R}$. Let I_1 and I_2 be defined by (5) and define

$$P(x) = \begin{cases} -P, & x \in I_1, \\ P, & x \in I_2. \end{cases}$$

We now consider the BVP

$$Ly + P(x)y = f(x, y) + P(x)y, \tag{7}$$

$$By = r. \tag{8}$$

Assume that the equations $Ly - Py = 0$ and $Ly + Py = 0$ are disconjugate on I. It then follows by the Lemma that the equation $Ly + P(x)y = 0$ is disconjugate on I. Hence, again using the result of Levin [4] (or see [2]), the Green's function, $H(x, s)$, of the BVP $Ly + P(x)y = 0$, $By = 0$ exists and satisfies (4). In particular, $H(x, s) \leqq 0$ for $(x, s) \in I_1 \times I$, $H(x, s) \geqq 0$ for $(x, s) \in I_2 \times I$.

Let $l_r(x)$ be the unique solution of the BVP $Ly + P(x)y = 0$, $By = r$ and define an integral operator K on $C(I)$ by

$$Ky(x) = l_r(x) + \int_I H(x, s)[f(s, y(s)) + P(s)y(s)]ds. \tag{9}$$

By properties of $H(x, s)$ (see [4]), $K: C(I) \to C^{n-1}(I)$ where

$$\|y\| = \max\{\|y\|_0, \|y'\|_0, \ldots, \|y^{(n-1)}\|_0\},$$

$$\|y^{(j)}\|_0 = \max_{x \in I} |y^{(j)}(x)|, \; 0 \leqq j \leqq n - 1,$$

for $y \in C^{n-1}(I)$. As usual, ϕ is a solution of the BVP (7), (8) if and only if ϕ is a fixed point of the operator K defined by (9).

THEOREM. *Let L be given by* (1) *and let $f: I \times \mathbf{R} \to \mathbf{R}$ be continuous and satisfy* (6). *Assume the equations $Ly - Py = 0$ and $Ly + Py = 0$ are disconjugate on I. Assume there exist functions $v_1(x)$ and $w_1(x)$ in $C(I)$ satisfying*

$$\begin{cases} v_1(x) \geqq w_1(x),\ x \in I_1, \quad v_1(x) \leqq w_1(x),\ x \in I_2, \\ Bv_1(x) = Bw_1(x) = r, \\ Lv_1(x) \leqq f(x, v_1(x)),\ x \in I, \quad Lw_1(x) \geqq f(x, w_1(x)),\ x \in I. \end{cases} \tag{10}$$

Then there exists a solution, ϕ, of the BVP *satisfying*

$$v_1(x) \geqq \phi(x) \geqq w_1(x),\ x \in I_1 \text{ and } v_1(x) \leqq \phi(x) \leqq w_1(x),\ x \in I_2.$$

Moreover, for $m = 1, 2, 3, \ldots$ define sequences $v_{m+1}(x) = Kv_m(x)$, $w_{m+1}(x) = Kw_m(x)$. Then the sequences converge to solutions $v(x)$ and $w(x)$ respectively of the BVP (2), (3) *such that*

$$v_m(x) \geqq v_{m+1}(x) \geqq v(x) \geqq w(x) \geqq w_{m+1}(x) \geqq w_m(x),\ x \in I_1, \tag{11}$$

$$v_m(x) \leqq v_{m+1}(x) \leqq v(x) \leqq w(x) \leqq w_{m+1}(x) \leqq w_m(x),\ x \in I_2,$$

for each m. Furthermore, any solution u of the BVP (2), (3) *satisfying* $v_1(x) \geqq u(x) \geqq w_1(x)$, $x \in I_1$, $v_1(x) \leqq u(x) \leqq w_1(x)$, $x \in I_2$ *satisfies* $v(x) \geqq u(x) \geqq w(x)$, $x \in I_1$, $v(x) \leqq u(x) \leqq w(x)$, $x \in I_2$.

REMARK. Note that for the BVP (7), (8), $F(x, y) = f(x, y) + P(x)y$ satisfies that F is monotone decreasing in y for each $x \in I_1$ and monotone increasing in y for each $x \in I_2$. Thus, the above theorem is essentially proved by Šeda [6, Theorem 2].

PROOF. Let $y, z \in C(I)$ such that $y(x) \geqq z(x)$, $x \in I_1$, $y(x) \leqq z(x)$, $x \in I_2$. Then

$$Ky(x) \geqq Kz(x),\ x \in I_1, \quad Ky(x) \leqq Kz(x),\ x \in I_2. \tag{12}$$

To see this, note that for $s \in I$, $f(s, y(s)) + P(s)y(s) \leqq f(s, z(s)) + P(s)z(s)$. Since $H(x, s) \leqq 0$ for $(x, s) \in I_1 \times I$, and $H(x, s) \geqq 0$ for $(x, s) \in I_2 \times I$, (12) follows from (9). Also,

$$\begin{aligned} v_1(x) \geqq v_2(x) \geqq w_2(x) \geqq w_1(x), &\quad x \in I_1, \\ v_1(x) \leqq v_2(x) \leqq w_2(x) \leqq w_1(x), &\quad x \in I_2. \end{aligned} \tag{13}$$

To see this, note that $v_1(x) = l_r(x) + \int_I H(x, s)[Lv_1(s) + P(s)v_1(s)]ds$ and by (10), $Lv_1(s) + P(s)v_1(s) \leqq f(s, v_1(s)) + P(s)v_1(s)$, $s \in I$. Thus, $v_1(x) \geqq v_2(x)$, $x \in I_1$, $v_1(x) \leqq v_2(x)$, $x \in I_2$. The inequalities for $w_1(x)$ and $w_2(x)$ follow similarly. From (12) and (13) it now follows inductively that for each $m = 1, 2, 3, \ldots,$

$$\begin{aligned} v_m(x) \geqq v_{m+1}(x) \geqq w_{m+1}(x) \geqq w_m(x),\ x \in I_1, \\ v_m(x) \leqq v_{m+1}(x) \leqq w_{m+1}(x) \leqq w_m(x),\ x \in I_2. \end{aligned} \tag{14}$$

By properties of $H(x, s)$ (see [4] or [2]) the sequences $\{v_m^{(i)}(x)\}$ and $\{w_m^{(i)}(x)\}$, $0 \leqq i \leqq n - 1$, are equicontinuous and uniformly bounded. By (14) and (9), the sequences $\{v_m^{(i)}(x)\}$ and $\{w_m^{(i)}(x)\}$ converge pointwise in I and hence, the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ converge in $C^{n-1}(I)$ to functions $v(x)$ and $w(x)$, respectively, such that (11) holds. It now follows from (9) that $v(x) = Kv(x)$ and $w(x) = Kw(x)$. Thus, $v(x)$ and $w(x)$ satisfy the BVP (7), (8); in particular, $v(x)$ and $w(x)$ are in $C^n(I)$ and satisfy the BVP (2), (3).

REMARK. For simplicity, we assume that the bound (6) holds on $I \times \mathbf{R}$. It is necessary only that the bound (6) holds on the compact domain bounded by the curves $v_1(x)$, $w_1(x)$ and the lines $x = a$, $x = b$.

EXAMPLE. Consider the BVP

$$y''' = y^2 + 1, \tag{15}$$

$$y(0) = y(1/2) = y(1) = 0. \tag{16}$$

Let $I_1 = [1/2, 1]$ and $I_2 = [0, 1/2)$ and let $v_1(x) = -x(x - 1/2)(x - 1)$ and $w_1(x) = -v_1(x)$. Since

$$\max_{x \in [0,1]} |v_1(x)| = \sqrt{3}/36 \quad \text{and}$$

$$|f(x, y_1) - f(x, y_2)| = |y_1 + y_2| \, |y_1 - y_2|$$

we choose $P = \sqrt{3}/18$; (6) holds on the compact domain bounded by the curves $v_1(x)$, $w_1(x)$ and the lines $x = 0$, $x = 1$.

The equations $y''' - Py = 0$ and $y''' + Py = 0$ are disconjugate on [0, 1]. To see this, we employ the Pólya criterion for disconjugacy [2]. Let $Q^3 = P$. Then $y_1(x) = e^{Qx}$, $y_2(x) = e^{-(Q/2)x} \sin(\sqrt{3}/2)\, Qx$, and $y_3(x) = -e^{-(Q/2)x} \cos(\sqrt{3}/2)\, Qx$ forms a Markov system of solutions of $y''' - Py = 0$ on [0, 1]; that is, $W(y_1)(x) > 0$, $W(y_1, y_2)(x) > 0$, and $W(y_1, y_2, y_3)(x) > 0$ on [0, 1] where W denotes the usual Wronskian determinant. Similarly, $y_1(x) = e^{-Qx}$, $y_2(x) = e^{(Q/2)x} \sin(\sqrt{3}/2)\, Qx$, and $y_3(x) = -e^{(Q/2)x} \cos(\sqrt{3}/2)\, Qx$ forms a Markov system of solutions of $y''' + Py = 0$ on [0, 1].

Since $v_1(x)$ and $w_1(x)$ satisfy (10), the theorem applies and there exists a solution of BVP (15), (16) satisfying $v_1(x) \geqq \phi(x) \geqq w_1(x)$, $x \in I_1$, and $v_1(x) \leqq \phi(x) \leqq w_1(x)$, $x \in I_2$.

Moreover, define

$$P(x) = \begin{cases} -P, & 1/2 \leqq x \leqq 1, \\ P, & 0 \leqq x < 1/2, \end{cases}$$

and construct sequences $\{v_{m+1}(x)\}$ and $\{w_{m+1}(x)\}$ such that $v_{m+1}(x)$ is the solution of the BVP

$$y''' + P(x)y = (v_m(x))^2 + 1 + P(x)v_m(x),$$
$$y(0) = y(1/2) = y(1) = 0,$$

and $w_{m+1}(x)$ is the solution of the BVP

$$y''' + P(x)y = (w_m(x))^2 + 1 + P(x)w_m(x),$$
$$y(0) = y(1/2) = y(1) = 0.$$

Then the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ converge to solutions $v(x)$ and $w(x)$, respectively, of the BVP (15), (16), such that (11) is satisfied.

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https://doi.org/10.4153/CMB-1988-012-0 Published online by Cambridge University Press