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MULTIPARAMETER WEIGHTED ERGODIC THEOREMS

ROGER L. JONES AND JAMES OLSEN

ABSTRACT. In this paper we show that multi-dimensional bounded Besicovitch weights are good weights for the pointwise ergodic theorem for Dunford-Schwartz operators and positively dominated contractions of L^p. This in particular implies new weighted results for multi-parameter measure preserving point transformations. The proofs show that Besicovitch weights are a very natural class when considered from the operator point of view. We also show that for $1 \leq r < \infty$, the r-bounded Besicovitch classes are all the same, generalizing a result of Bellow and Losert.

0. **Introduction.** Let (X, Σ, μ) denote a σ-finite measure space. Let $T_1, T_2, \ldots, T_d$ denote a family of d linear operators mapping $L^p(X)$ to itself. In some cases we will assume the operators are on all L^P spaces, $1 \leq p \leq \infty$ and in other cases we will assume $p > 1$ fixed. We will be concerned with classes of weights $\{a(\mathbf{k}) : \mathbf{k} \in Z_d^+\}$ such that the limit of averages

$$\lim_{\mathbf{N}\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f(x), \tag{1}$$

exists a.e. for all $f \in L^p$, where $\mathbf{T}^{\mathbf{k}} = T_1^{k_1} T_2^{k_2} \cdots T_d^{k_d}$. Here and throughout the paper, we will write $\mathbf{N}$ for $(N_1, N_2, \ldots, N_d)$, $\mathbf{0}$ for $(0, 0, \ldots, 0)$, *etc.* If the limit (1) exists, we will say the sequence of weights $\{a(\mathbf{k})\}_{\mathbf{k}\in Z_d^+}$ is a good sequence for $T_1, T_2, \ldots, T_d$.

The first question that must be addressed before we can discuss multiparameter theorems is what we mean when we write $\lim_{\mathbf{N}\to\infty} A(\mathbf{N})$. Unless we specify to the contrary in the statement of the theorem, we will mean unrestricted rectangular convergence, that is, if $\mathbf{N} = (N_1, N_2, \ldots, N_d)$, then $\lim_{\mathbf{N}\to\infty} A(\mathbf{N}) = L$ means that given $\epsilon > 0$ there is an integer N_0 such that if $\min\{N_1, N_2, \ldots, N_d\} > N_0$ then $|A(\mathbf{N}) - \mathbf{L}| < \epsilon$. In a similar way, if we write $\limsup_{\mathbf{N}\to\infty} A(\mathbf{N})$ we mean

$$\lim_{N_0\to\infty} \sup_{\min\{N_1, N_1, \ldots, N_d\} > N_0} A(\mathbf{N}).$$

In special situations we will also consider "square convergence" and "restricted rectangular convergence".

The class of weights we will consider are the Besicovitch sequences in Z_d^+. In the case of a single transformation, Besicovitch weights are defined to be the class of sequences

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$a(k)$ such that given $\epsilon > 0$, there is a trigonometric polynomial ψ_ϵ such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |a(k) - \psi_\epsilon(k)| < \epsilon,$$

and bounded Besicovitch weights are bounded weights in this class. In one dimension, bounded Besicovitch weights have been studied by several authors, including Baxter and Olsen [BO], Ryll-Nardzewski [R], Bellow and Losert [BL] and Olsen [O]. In particular [BO] show that for positively dominated contractions of L^p, p fixed, $1 < p < \infty$, r-bounded Besicovitch sequences (that is, sequences where the ℓ^r norm replaces the ℓ^1 norm in the definition), are good weights for L^p pointwise convergence when $r > q$ and $\frac{1}{p} + \frac{1}{q} = 1$. Later [BL] showed that all the r-bounded Besicovitch classes coincide, thus showing that bounded Besicovitch weights are good weights for L^p pointwise convergence. Multiparameter versions of their results are also known in the special case where the weights are products of bounded Besicovitch sequences. (See [C] or [O3] where techniques of Frangos and Sucheston [FS] are used to reduce the problem to the one-dimensional case. These techniques do not directly apply to our case because the weights are not products of one dimensional weights.)

In this paper we generalize the notion of Besicovitch sequences to the multiparameter case, and show that the results of [BO] and [BL] extend to this case as well. We also show that although the multi-dimensional r-bounded Besicovitch classes coincide, the restriction that the Besicovitch classes be bounded is unnecessary to obtain the results analogous to those of [BO]. It also follows from our methods that for commuting Dunford-Schwartz operators and bounded Besicovitch sequences, the convergence of (1) occurs for $f \in L^1(X)$ when convergence is "restricted rectangular convergence". The methods used to obtain the results in this paper are closely related to the methods used in [BO] to obtain the one-parameter results. The proof of Theorem 3.1, showing that the r-bounded Besicovitch classes are all the same, is different, even in one dimension, from what is given in [BL], and is self contained.

The convergence theorems obtained below are new in the case of each T_i induced by a measure preserving point transformation. That is, $T_i f(x) = f(\tau_i x)$, where τ_i is measure preserving. The proofs of the theorems in this special case are particuarily simple when given in the more general operator setting. The key point of the proof is that if T is a Dunford-Schwartz operator, and λ is a complex number of modulus 1, then the operator λT is also a Dunford-Schwartz operator. A similar remark holds for positively dominated contractions of L^p, p fixed, $1 < p < \infty$. Because of this, the proofs of Theorems 1.2 and 2.4 below show that Besicovitch weights are a very natural class of weights from an operator theory point of view. In particular, our results show that if $T_1, \ldots, T_d$ are induced by measure preserving point transformations, then bounded Besicovitch sequences are good for $T_1, \ldots, T_d$ on all L^p spaces for $1 < p < \infty$, and for the special L^1 cases given in Section 1. We also have that r-Besicovitch sequences are good for $T_1, \ldots, T_d$ considered as operators on L^p where $\frac{1}{p} + \frac{1}{q} = 1$ and $q < r$.

1. **Dunford-Schwartz operators.** In this section we define the d-dimensional analogs of Besicovitch sequences, and prove that they are good weights for the pointwise ergodic theorem and Dunford-Schwartz operators.

DEFINITION 1.1. We say that $\{a(\mathbf{k})\}$ *is r-Besicovitch* if for every $\epsilon > 0$ there is a sequence of trigonometric polynomials in d variables, ψ_ϵ, such that

$$\limsup_{\mathbf{N}\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^r < \epsilon. \tag{2}$$

We denote this class by $B(r)$.

We say that $\{a(\mathbf{k})\}$ *is r-bounded Besicovitch* if $\{a(\mathbf{k})\} \in B(r) \cap \ell^\infty$.

By Besicovitch sequences we will mean sequences in $B(1)$, and bounded Besicovitch sequences will mean sequences in $B(1) \cap \ell^\infty$.

With this notation we can state our first Theorem, which is a direct generalization of the one-parameter version in [O].

THEOREM 1.2. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d)$ *denote d (possibly not commuting) Dunford-Schwartz operators, i.e.* $\|T_i\|_1 \leq 1$ *and* $\|T_i\|_\infty \leq 1$ *for all i,* $1 \leq i \leq d$. *For* $\mathbf{T}^{\mathbf{k}} = T_1^{k_1} T_2^{k_2} \cdots T_d^{k_d}$, *we have*

$$\lim_{\mathbf{N}\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f(x)$$

exists a.e. for every $f \in L^p$, $1 < p \leq \infty$, *and all bounded Besicovitch sequences* $\{a(\mathbf{k})\}$.

PROOF. Let $(\lambda_1, \ldots, \lambda_d) \in C^d$, with $|\lambda_i| = 1$, $i = 1, 2, \ldots, d$, then the theorem holds when $a(\mathbf{k}) = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_d^{k_d}$ since $\tilde{\mathbf{T}}^{\mathbf{k}} = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_d^{k_d} \mathbf{T}^{\mathbf{k}}$ is also a d-parameter sequence of Dunford-Schwartz operators, and the theorem holds for d-parameter sequences of Dunford-Schwartz operators when $a(\mathbf{k}) = 1$. Clearly the theorem holds for finite linear combinations of such sequences, and hence holds for trigonometric polynomials in d variables.

Let $|T_i|$ be the linear modulus of T_i and $|\mathbf{T}| = |T_1| \cdots |T_d|$. If $f \in L^\infty(X)$, then for every $\epsilon > 0$ we have

$$\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f = \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \big(a(\mathbf{k}) - w_\epsilon(\mathbf{k})\big) \mathbf{T}^{\mathbf{k}} f + \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} w_\epsilon(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f,$$

showing that for any $f \in L^\infty(X)$, we have

$$\begin{aligned}
\left| \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f - \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} w_\epsilon(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f \right| &\leq \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - w_\epsilon(\mathbf{k})| \, |\mathbf{T}^{\mathbf{k}} f| \\
&\leq \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - w_\epsilon(\mathbf{k})| \, |\mathbf{T}|^{\mathbf{k}} |f| \\
&\leq \|f\|_\infty \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - w_\epsilon(\mathbf{k})| < \epsilon \|f\|_\infty.
\end{aligned}$$

We already know that we have a.e. convergence for the averages

$$\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} w_\epsilon(\mathbf{k})\mathbf{T}^{\mathbf{k}}f,$$

and hence we have a.e. convergence of

$$\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f$$

for all $f \in L^\infty(X)$. Since $L^\infty(X)$ is dense in each $L^p(X)$, $p \geq 1$, we have a.e. convergence of these averages for all f in a dense class of $L^p(X)$ for $1 < p \leq \infty$. We will have a.e. convergence for all $f \in L^p(X)$ if we have an appropriate maximal inequality. The necessary maximal inequality follows from the observation that since the sequence $\{a(\mathbf{k})\}$ is bounded Besicovitch, there is a bound b such that $|a(\mathbf{k})| \leq b$ for all $\mathbf{k}$. Consequently

$$\left|\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f\right| \leq \sup_{\mathbf{k}} |a(\mathbf{k})| \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |\mathbf{T}^{\mathbf{k}}f| \leq b\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |\mathbf{T}|^{\mathbf{k}}|f|.$$

The theorem now follows from the maximal ergodic theorem for Dunford-Schwartz operators, and by the fact that the linear modulus of a Dunford-Schwartz operator is a Dunford-Schwartz operator.

REMARK. As a special case, this result includes the case of measure preserving point transformations, however, this result is itself a special case of Theorem 2.4 below. The proof was given in this special case to show how natural the bounded Besicovitch classes are in this setting, and to provide motivation for the more general results proven later.

REMARK. The problem with the case $p = 1$ in Theorem 1.2 is the failure of a weak type $(1, 1)$ maximal inequality for multiparameter operators with the unrestricted rectangular convergence we are using. However in the case of finite measure we can extend the result to the class $L(\log L)^{d-1}L$, and in the case of infinite measure spaces, to a class of functions introduced by Fava [F]. See also [ES].

THEOREM 1.3. *Define the class*

$$\mathbf{R_d} = \left\{f : \int_{\{|f|>t\}} \frac{|f|}{t}\left(\log\frac{|f|}{t}\right)^d d\mu < \infty \text{ for each } t > 0\right\}.$$

Then bounded Besicovitch sequences $\{a(\mathbf{k})\}_{\mathbf{k}\in Z_+^d}$ are good for all Dunford-Schwartz operators $T_1, \ldots, T_d$, and all $f \in \mathbf{R_{d-1}}$, i.e, the averages (1) converge a.e. for all $f \in \mathbf{R_{d-1}}$.

PROOF. Let $T_1, \ldots, T_d$ be Dunford-Schwartz operators, $\{a(\mathbf{k})\}$ a bounded Besicovitch sequence. Let $\mathbf{T}^{\mathbf{k}} = T_1^{k_1}\cdots T_d^{k_d}$ where $\mathbf{k} = (k_1, \ldots, k_d)$. Define

$$A_{\mathbf{N}}f = \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f$$

and

$$Mf(x) = \sup_{\mathbf{N}} |B_{\mathbf{N}} f(x)|$$

where $B_{\mathbf{N}}$ is the operator $A_{\mathbf{N}}$ defined above, with each $a(\mathbf{k}) = 1$, and the operators $|T_1|, \ldots, |T_d|$ replacing $T_1, \ldots, T_d$. By Corollary 1.7, p. 200 of [K], we know there exists a constant C_d such that

$$\mu\big(x : Mf(x) > 4t\big) \leq C_d \int_{\{|f|>t\}} \frac{|f|}{t} \left(\log \frac{|f|}{t}\right)^{d-1} d\mu$$

for all $f \in \mathbf{R}_{d-1}$. For $\mathbf{N}_1$ and $\mathbf{N}_2$ and $f > 0$, we have

$$|A_{\mathbf{N}_1} f - A_{\mathbf{N}_2} f| = |A_{\mathbf{N}_1}(f - f_n) - A_{\mathbf{N}_2}(f - f_n) + A_{\mathbf{N}_1} f_n - A_{\mathbf{N}_2} f_n|$$

where $\{f_n\}$ is an increasing sequences of functions in L^p, p fixed, $p > 1$ converging to f a.e. We have

$$|A_{\mathbf{N}_1}(f - f_n) - A_{\mathbf{N}_2}(f - f_n)| \leq 2\|a(\mathbf{k})\|_{\ell^\infty} M(f - f_n)$$

Corollary 1.7 of [K] now implies that $M(f - f_n)$ approaches 0 a.e. Let $\epsilon > 0$. For a.e. x, choose n such that $f - f_n < \frac{\epsilon}{2}$. By the theorem for bounded Besicovitch sequences and Dunford Schwartz operators in L^P, $p > 1$, we know we can choose an $\mathbf{N}$ such that $\mathbf{N}_1, \mathbf{N}_2 > \mathbf{N}$ implies

$$|A_{\mathbf{N}_1} f_n - A_{\mathbf{N}_2} f_n| < \frac{\epsilon}{2}.$$

Thus for a.e. x, the sequence $A_{\mathbf{N}} f$ is Cauchy, and hence converges.

If we are willing to place additional restrictions on the form of convergence, and on the operators, then we can weaken the restrictions on the functions. In particular Theorem 1.4 below shows that if we consider "square convergence", that is, with $\mathbf{N}$ restricted to the form $\mathbf{N} = (n, n, n, \ldots, n)$ or "restricted rectangular convergence", that is, for a fixed constant c, each $\mathbf{N}$ is restricted to be of the form $\mathbf{N} = (n_1, n_2, \ldots, n_d)$ with $\frac{n_i}{n_j} < c$ for all $1 \leq i, j \leq d$.

THEOREM 1.4. *Let $T_1, T_2, \ldots, T_d$ be commuting Dunford-Schwartz operators. Then the limit (1) exists for all $f \in L^1(X)$, that is, bounded Besicovitch weights are good for $T_1, \ldots, T_d$ when convergence is "square convergence" or "restricted rectangular convergence".*

The proof of this result depends on the following lemma, due to Brunel. (See [K] p. 231.)

LEMMA 1.5. *For any $d > 1$ there exists a constant $c > 0$ and a family $\{\mu(\mathbf{k}) : \mathbf{k} \in Z_d^+\}$ of strictly positive numbers summing to 1 such that the following holds: If $T_1, T_2, \ldots, T_d$ are commuting contractions of $L^1(X)$ and $|T_i|$ denotes the linear modulus of T_i, then the operator $U = \sum_{\mathbf{k} \in Z_d^+} \mu(\mathbf{k}) |T_1|^{k_1} \cdots |T_d|^{k_d}$ satisfies for all $n > 0$ and all non-negative $f \in L^1(X)$*

$$\frac{1}{n^d} \sum_{k_1=0}^{n-1} \cdots \sum_{k_d=0}^{n-1} |T_1^{k_1} \cdots T_d^{k_d}| f \leq \frac{c}{m_d} \sum_{j=0}^{m_d-1} U^j f,$$

where m_d depends on n and d, and $m_d \longrightarrow \infty$ as $n \longrightarrow \infty$. (Actually $m_d = [\sqrt{n}+1]^{[1+log_2 d]}$, but the exact value is not important to us.)

PROOF OF THEOREM 1.4. Let A denote the subset of Z_d^+ such that each $\mathbf{N} = (n_1, \ldots, n_d) \in A$ satisfies $\frac{n_i}{n_j} < \alpha$ for all $1 \leq i, j \leq d$. Theorem 1.2 shows that we have convergence for $f \in L^p, p > 1$, a dense subset of $L^1(X)$. Hence we need only show the maximal function:

$$\sup_{\mathbf{N} \in A} \left| \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \mathbf{T}^{\mathbf{k}} f \right| < \infty \quad \text{a.e.}$$

We will show that we have a weak type (1, 1) maximal inequality for "restricted rectangular averages". For each $\mathbf{N} \in A$, let $n = \max(n_1, n_2, \ldots, n_d)$. Then we have

$$\begin{aligned} \left| \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \mathbf{T}^{\mathbf{k}} f \right| &\leq \frac{1}{|\mathbf{N}|} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} |T_1^{k_1} \cdots T_d^{k_d} f| \\ &\leq \frac{\alpha^d}{n^d} \sum_{k_1=0}^{n-1} \cdots \sum_{k_d=0}^{n-1} |T_1^{k_1} \cdots T_d^{k_d} f| \end{aligned}$$

where recall $\alpha \geq \max(\frac{n_i}{n_j})$, $1 \leq i, j \leq d$. Now we have

$$\left| \frac{1}{|N|} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f \right| \leq \|a(k)\|_\infty \frac{c\alpha^d}{m_d} \sum_{j=0}^{m_d-1} U^j f.$$

Since U^j is a Dunford-Schwartz operator,

$$\lim_{m_d \to \infty} \frac{1}{m_d} \sum_{j=0}^{m_d-1} U^j f$$

exists a.e. Thus we have a weak type (1, 1) maximal inequality and the theorem follows.

2. Positive contractions of L^p, $p > 1$ fixed.

To proceed to operators on $L^p(X)$, p fixed, we need some additional notation.

DEFINITION 2.1. We say that the sequence of complex numbers $\{a(\mathbf{k})\}_{\mathbf{k} \in Z_d^+}$ *has a mean* if

$$\lim_{\mathbf{N} \to \infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-1} a(\mathbf{k})$$

exists. We denote by Λ^r the sequences $\{a(\mathbf{k})\}$ such that

$$\limsup_{\mathbf{N} \to \infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-1} |a(\mathbf{k})|^r < \infty.$$

LEMMA 2.2. *The classes Λ^r and $B(r)$ satisfy the following properties. (See [BO].)*

1) *If $\{a(\mathbf{k})\} \in \Lambda^p$ and $\{b(\mathbf{k})\} \in \Lambda^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $c(\mathbf{k}) = a(\mathbf{k})b(\mathbf{k})$, then $\{c(\mathbf{k})\} \in \Lambda^1$.*

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2) *If* $\{a_\ell(\mathbf{k})\}$ *has a mean for every* ℓ, *and* $\{a_\ell(\mathbf{k})\} \longrightarrow \{a(\mathbf{k})\}$ *in the semi-norm induced on* Λ^1, *then* $\{a(\mathbf{k})\}$ *has a mean.*
3) *If* $\{a_\ell(\mathbf{k})\} \longrightarrow \{a(\mathbf{k})\}$ *in the* Λ^r *semi-norm, and* $\{a_\ell(\mathbf{k})b(\mathbf{k})\}$ *has a mean for every* ℓ *where* $\{b(\mathbf{k})\} \in \Lambda^q$, $\frac{1}{r} + \frac{1}{q} = 1$, *then* $\{a(\mathbf{k})b(\mathbf{k})\}$ *has a mean.*
4) *For any sequence* $\{a(\mathbf{k})\} \in B(r)$ *and any* $\lambda \in \mathbf{R}^d$ *the sequence* $\{a(\mathbf{k})e^{i\lambda\cdot\mathbf{k}}\} \in B(r)$ *and has a mean. In particular, sequences in* $B(r)$ *have a mean.*

PROOF. 1) is a consequence of Hölder's inequality. 2) follows from the definitions. 3) is a consequence of 1) and 2). 4) follows from the fact that the sequence $\{a(\mathbf{k})\}$ can be approximated by trigonometric polynomials ψ_ϵ in the Λ^r seminorm. Since we have a limit for each $\psi_\epsilon(\mathbf{k})e^{i\lambda\cdot\mathbf{k}}$, the result follows from 3.

We are now ready to consider operators indexed by Z_d^+ which are positive contractions of L^p, p fixed, $1 < p < \infty$.

DEFINITION 2.3. Let $T: L^p \longrightarrow L^P$, p fixed, $1 < p < \infty$. If $\|T\|_p \leq 1$ we say T is a contraction. If $f \geq 0$ implies $Tf \geq 0$ we say that T is positive. If there exists a contraction $S: L^p \longrightarrow L^p$ such that the $|Tf| \leq S|f|$ we say that T is positively dominated. In that case we necessarily have that T is a contraction and S is positive.

We have the following theorem (For a related result, see [T]).

THEOREM 2.4. *Let* $T_1, \ldots, T_d$ *be positively dominated contractions of* $L^p(X)$, $1 < p < \infty$, *and let* $\{a(\mathbf{k})\}$ *be s-Besicovitch, where* $s > q$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\lim_{\mathbf{N}\to\infty} \frac{1}{|\mathbf{N}|} \sum_{k=0}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k})\mathbf{T}^{\mathbf{k}} f$$

exists a.e. for all $f \in L^p(X)$.

The proof of this theorem will follow easily from the following lemma.

LEMMA 2.5. *Let* $T_1, \ldots, T_d$ *be positively dominated contractions of* $L^p(X)$, p *fixed,* $1 < p < \infty$. *If*

$$V_r f = \sup_{\mathbf{N}\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |\mathbf{T}^{\mathbf{k}} f|^r, \quad 1 \leq r < p$$

then

$$\|V_r f\|_{p/r} \leq c\Big(\frac{p}{r}, d\Big)\|f\|_p$$

where $c(\frac{p}{r}, d)$ *is a constant that depends only on* $\frac{p}{r}$ *and* d.

PROOF. We first assume that (X, Σ, μ) is a probability space. We will use the fact that if T is a positive contraction of $L^p(X)$, $1 < p < \infty$, p fixed, then there exists a larger L^p space $L^p(\tilde{X})$ say, a dilation operator $D: L^p(X) \longrightarrow L^p(\tilde{X})$ that is positive, invertible and isometric; a positive invertiable isometry Q of $L^p(\tilde{X})$; and a conditional expectation operator E such that $DT^n f = EQ^n Df$, $n \geq 0$. (See the paper by Akcoglu and Sucheston [AS], for the details of this important and very useful representation.) We note that D and D^{-1} induce linear isometries $\tilde{D}$ and $\tilde{D}^{-1}$ between $L^{\frac{p}{r}}(X)$ and $L^{\frac{p}{r}}(\tilde{X})$ by $\tilde{D}f = (Df^{\frac{1}{r}})^r$.

Without loss of generality, we may assume each T_i is positive, and consider operators of the form $\{EQ^{k_1}D(T_{\tilde{\mathbf{k}}})\}$ where $\mathbf{k} = (k_1, \ldots, k_d)$, $\tilde{\mathbf{k}} = (k_2, \ldots, k_d)$, $\tilde{\mathbf{N}} = (N_2, \ldots, N_d)$, and E, Q and D are as above. Now let $f \geq 0$ and proceed by induction, noting the result for a Z_1^+ representation, and the fact that $Q^k g = w_k(x)g(\tau^k x)$ where τ is a non-singular point transformation. Let S be the $L^{\frac{p}{r}}(\tilde{X})$ isometry defined such that

$$S^{k_1}h(\tilde{x}) = \big(w_{k_1}(\tilde{x})\big)^r h(\tau^{k_1}\tilde{x}) = \big(Q^{k_1}(h^{1/r})(\tilde{x})\big)^r.$$

Fix $\mathbf{M} \in Z_d^+$. Then

$$\begin{aligned}
&\int\Big(\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |(\mathbf{T}^{\mathbf{k}}f)|^r\Big)^{p/r} d\mu \\
&\quad = \int\Big(\tilde{D}\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} (\mathbf{T}^{\mathbf{k}}f)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad \leq \int\Big(\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \tilde{D}\big((\mathbf{T}^{\mathbf{k}}f)^r\big)\Big)^{p/r} d\tilde{\mu} \\
&\quad \leq \int\Big(\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} (D\mathbf{T}^{\mathbf{k}}f)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad = \int\Big(\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \big(EQ^{k_1}D(\mathbf{T}^{\tilde{\mathbf{k}}}f)\big)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad \leq \int\Big(\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \big(Q^{k_1}D(\mathbf{T}^{\tilde{\mathbf{k}}}f)\big)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad \leq \int\Big(\sup_{\mathbf{N}<\mathbf{M}} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} S^{k_1}\big(D(\mathbf{T}^{\tilde{\mathbf{k}}}f)\big)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad \leq \int\Bigg(\sup_{N_1<M_1} \frac{1}{N_1}\sum_{k_1=0}^{N_1-1} S^{k_1}\Big(\sup_{\tilde{\mathbf{N}}<\tilde{\mathbf{M}}} \frac{1}{|\tilde{\mathbf{N}}|}\sum_{\tilde{\mathbf{k}}=\mathbf{0}}^{\tilde{\mathbf{N}}-\tilde{\mathbf{1}}} \big(D(\mathbf{T}^{\tilde{\mathbf{k}}}f)\big)^r\Big)\Bigg)^{p/r} d\tilde{\mu} \\
&\quad \leq c\int\Big(\sup_{\tilde{\mathbf{N}}<\tilde{\mathbf{M}}} \frac{1}{|\tilde{\mathbf{N}}|}\sum_{\tilde{k}=0}^{\tilde{N}-\tilde{1}} \big(D(\mathbf{T}^{\tilde{\mathbf{k}}}f)\big)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad = c\int\Big(\tilde{D}\sup_{\tilde{\mathbf{N}}<\tilde{\mathbf{M}}} \frac{1}{|\tilde{\mathbf{N}}|}\sum_{\tilde{\mathbf{k}}=\mathbf{0}}^{\tilde{\mathbf{N}}-\tilde{\mathbf{1}}} (\mathbf{T}^{\tilde{\mathbf{k}}}f)^r\Big)^{p/r} d\tilde{\mu} \\
&\quad \leq c(p/r,d)\int\Big(\sup_{\tilde{\mathbf{N}}<\tilde{\mathbf{M}}} \frac{1}{|\tilde{\mathbf{N}}|}\sum_{\tilde{\mathbf{k}}=\mathbf{0}}^{\tilde{\mathbf{N}}-\tilde{\mathbf{1}}} (\mathbf{T}^{\tilde{\mathbf{k}}}f)^r\Big)^{p/r} d\mu,
\end{aligned}$$

where the dominated estimate for for S was used at the crucial step. The lemma follows in this case by first letting $\mathbf{M}$ go to ∞ and then using induction. Standard arguments allow us to pass from the finite case to the sigma-finite case.

PROOF OF THEOREM 2.4. Put $b(\mathbf{k})(x) = \mathbf{T}^{\mathbf{k}}f(x)$. Then for a.e. x, $b(\mathbf{k})(x) \in \Lambda^{s/r}$ where $\frac{1}{r}+\frac{1}{s} = 1$, by Lemma 2.2. We can find a sequence $\{a_\ell(\mathbf{k})\}$ of d-dimensional trigonometric polynomials which converges to $\{a(\mathbf{k})\}$ in the Λ^s semi- norm. If $\lambda_1, \ldots, \lambda_d$ are complex numbers of modulus 1, each operator $\lambda_i T_i$ is a positively dominated contraction of L^p, so $\{a_\ell(\mathbf{k})b(\mathbf{k})(x)\}$ has a mean for $a_\ell(\mathbf{k})b(\mathbf{k})$ for $a_\ell(\mathbf{k}) = \lambda_1^{k_1} \cdots \lambda_d^{k_d}$ by the d-dimensional version of the ergodic theorem for positively dominated contractions of L^p (see [O2] or [M]). It follows that $\{a_\ell(\mathbf{k})b(\mathbf{k})\}$ has a mean a.e. when $a_\ell(\mathbf{k})$ is a polynomial, so $\{a(\mathbf{k})b(\mathbf{k})(x)\}$ has a mean a.e.

3. **Bounded Besicovitch sequences.** In the previous section we showed that for $T_1, \ldots, T_d$ positively dominated contractions of L^p, $p > 1$ the r-Besicovitch sequences are good for $r > q$, $\frac{1}{p} + \frac{1}{q} = 1$. For $d = 1$, Bellow and Losert [BL] have shown that $B(r) \cap \ell^\infty = B(1) \cap \ell^\infty$ for all $r > 1$, making all classes of bounded Besicovitch sequences the same, and hence good for all positively dominated contractions of any L^p, $p > 1$. In this section we show that this result holds for $d > 1$ as well, making all bounded Besicovitch sequences good for all $T_1, \ldots, T_d$ where each T_i is positively dominated contraction of a fixed L^p space, $1 < p < \infty$.

The following theorem is a generalization of a one-variable result proved by Bellow and Losert [BL]. It is in a sense an analog to the fact that for L^p spaces on a finite measure space we have $L^p \cap L^\infty = L^1 \cap L^\infty$.

THEOREM 3.1. *For all $p \geq 1$ we have $B(p) \cap \ell^\infty = B(1) \cap \ell^\infty$.*

PROOF. We will first show that $B(p) \subset B(1)$. This is the easy direction. Assume that $\{a(\mathbf{k})\} \in B(p)$. Given $\epsilon > 0$ we know that there is a trigonometric polynomial $\psi_\epsilon(\mathbf{k})$ such that

$$\limsup_{\mathbf{N}} \left(\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^p \right)^{1/p} < \epsilon^p.$$

By Hölder's inequality we have

$$\begin{aligned} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})| &\leq \left(\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^p \right)^{1/p} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} (1^q)^{1/q} \\ &= \left(\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^p \right)^{1/p}. \end{aligned}$$

Hence

$$\limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})| \leq \limsup_{\mathbf{N}} \left(\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^p \right)^{1/p} < \epsilon.$$

and thus if $\{a(\mathbf{k})\} \in B(p)$ then $\{a(\mathbf{k})\} \in B(1)$, and we have $\|a - \psi_\epsilon\|_1 \leq \|a - \psi_\epsilon\|_p$.

We now need to show that $\{a(\mathbf{k})\} \in B(1)$ implies $\{a(\mathbf{k})\} \in B(p)$. Assume that $\{a(\mathbf{k})\} \in B(1)$. We then know that given $\epsilon > 0$ there is a trigonometric polynomial ψ_ϵ such that $\|a - \psi_\epsilon\|_1 < \epsilon$, *i.e.*

$$\limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})| < \epsilon.$$

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Our goal is to show that ψ_ϵ can be selected so that

$$\limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^p < C\epsilon$$

where C depends only on $\|a\|_{\ell^\infty}$ and does not depend on ψ_ϵ. If we knew that $\|\psi_\epsilon\|_{\ell^\infty} \leq \|a\|_{\ell^\infty}$ for each choice of $\epsilon > 0$ then we could achieve the goal by the following argument:

$$\begin{aligned}\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^p &\leq \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|\, |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})|^{p-1} \\ &\leq \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |a(\mathbf{k}) - \psi_\epsilon(\mathbf{k})| (\|a\|_{\ell^\infty} + \|a\|_{\ell^\infty})^{p-1} \\ &\leq (2\|a\|_{\ell^\infty})^{p-1} \|a - \psi_\epsilon\|_1 < (2\|a\|_{\ell^\infty})^{p-1}\epsilon.\end{aligned}$$

To prove that we can select the polynomials that are used in the definition of $B(1)$ in such a way that there is a uniform bound on the ℓ^∞ norm, we first introduce certain special trigonometric polynomials, and prove some lemmas. The proof will be completed by Lemma 3.5 below.

Given a finite sequence of integers $\{n_1, n_2, \ldots, n_r\}$ and a corresponding set of real numbers $\{\beta_1, \beta_2, \ldots, \beta_r\}$ which are linearly independent over the rationals, let

$$B = B(n_1, n_2, \ldots, n_r; \beta_1, \beta_2, \ldots, \beta_r)$$

and denote by K_B the kernel

$$K_B(x) = \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_r=0}^{n_r} \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_r|}{n_r}\right) e^{2\pi i(\nu_1\beta_1 + \cdots + \nu_r\beta_r)x}.$$

To simplify notation later, let

$$d_B(\nu_1, \ldots, \nu_d) = \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_r|}{n_r}\right).$$

To extend this kernel to d dimensions, we define $K_{\mathbf{B}}(\mathbf{x}) = K_{B_1}(x_1)K_{B_2}(x_2)\cdots K_{B_d}(x_d)$ where each B_j is as above and $\mathbf{x} = (x_1, \ldots, x_d)$.

LEMMA 3.2. *The kernel $K_{\mathbf{B}}(\mathbf{x})$ satisfies:*

1) $K_{\mathbf{B}}$ is non-negative.

2) For every $\epsilon > 0$, there exists an integer $\mathbf{N_0}$ such that if $\mathbf{N} > \mathbf{N_0}$ then for all $\mathbf{m}$,

$$\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m} - \mathbf{k}) \leq 1 + \epsilon, \text{ and } \lim_{\mathbf{N}\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m} - \mathbf{k}) = 1.$$

3) $\|K_{\mathbf{B}}\|_1 = 1$.

PROOF. To see that $K_{\mathbf{B}}$ is non-negative, note that it can be written as a finite product of kernels of the form

$$\sum_{|\nu| \leq n} \left(1 - \frac{|\nu|}{n}\right) e^{-i\nu x} = \frac{1}{n}\left(\frac{\sin \frac{n}{2}x}{\sin \frac{x}{2}}\right)^2.$$

To see that $||K_{\mathbf{B}}|| = 1$ we note that since $K_{\mathbf{B}}$ is non-negative, we have

$$||K_{\mathbf{B}}||_1 = \limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} |K_{\mathbf{B}}(\mathbf{k})| = \limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{k})$$
$$= \limsup_{\mathbf{N}} \prod_{j=1}^{d} \frac{1}{N_j} \sum_{k_j=0}^{N_j-1} K_{B_j}(k_j).$$

Consequently it is enough to show that

$$\lim_{N_j \to \infty} \frac{1}{N_j} \sum_{k_j=0}^{N_j-1} K_{B_j}(k_j) = 1.$$

To see this, we note that we have a finite sum, each term of which is of the form

$$\lim_{N_j} \frac{1}{N_j} \sum_{k_j=0}^{N_j-1} e^{i\lambda k_j}.$$

where $\lambda = \nu_1\beta_1 + \cdots + \nu_r\beta_r$ for some choice of $\nu_1, \ldots, \nu_r$. Using the linear independence of the $\{\beta_1, \ldots, \beta_r\}$, we see that $\lambda \neq 0$ unless all $\nu_i = 0$. If $\lambda \neq 0$ the average converges to zero, and if all $\nu_i = 0$ the average is clearly 1.

To see that $\frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m} - \mathbf{k}) \leq 1 + \epsilon$, for $\mathbf{N}$ large enough, we use the above arguments, and note that we have a finite sum of terms of the form

$$\frac{1}{N_j} \sum_{k_j=0}^{N_j-1} e^{i\lambda(m_j - k_j)} = e^{i\lambda m_j} \frac{1}{N_j} \sum_{k_j=0}^{N_j-1} e^{-i\lambda k_j}.$$

Since these terms converge to zero (if $\lambda_j \neq 0$), independent of the choice of m_j, the result follows by selecting $\mathbf{N_0}$ large enough.

LEMMA 3.3. *If we define* $K_{\mathbf{B}} \star a(\mathbf{j}) = \lim_{\mathbf{N}} \frac{1}{\mathbf{N}} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{j}-\mathbf{k}) a(\mathbf{k})$ *then for* $\{a\} \in B(1)$ *we have*

1) $K_{\mathbf{B}} \star a$ *is a trigonometric polynomial,*
2) $||K_{\mathbf{B}} \star a||_\infty \leq ||a||_\infty$,
3) $||K_B \star a||_1 \leq ||a||_1$.

PROOF. To see 1) first note that by Lemma 2.2, for each $\lambda = (\lambda_1, \ldots, \lambda_d) \neq (0, \ldots, 0)$, $\lim_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) e^{-\lambda \cdot \mathbf{k}}$ has a mean. Denote this mean by $c(\lambda)$. Hence

$$K_{\mathbf{B}} \star a(\mathbf{m}) = \lim_{\mathbf{N}} \frac{1}{\mathbf{N}} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) K_B(\mathbf{m} - \mathbf{k}) = \lim_{\mathbf{N}} \frac{1}{\mathbf{N}} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) \prod_{j=1}^{d} K_{B_j}(\mathbf{m} - \mathbf{k}).$$

Each of the d terms in the product has the form

$$\sum_{|\nu_1| < n_{1_j}} \cdots \sum_{|\nu_{r_j}| < n_{r_j}} d_{B_j}(\nu_1, \ldots, \nu_{r_j}) \lim_{N_j} \frac{1}{N_j} \sum_{n=0}^{N_j-1} a(\mathbf{k}) e^{i(\nu_1\beta_1 + \cdots + \nu_r\beta_{r_j})(m_j - k_j)}$$
$$= \sum_{|\nu_1| < n_1} \cdots \sum_{|\nu_r| < n_{r_j}} d_{B_j}(\nu_1, \ldots, \nu_{r_j}) e^{i(\nu_1\beta_1 + \cdots + \nu_{r_j}\beta_{r_j}) m_j} \lim_{N_j} \frac{1}{N_j} \sum_{n=0}^{N_j-1} a(\mathbf{k}) e^{-i(\nu_1\beta_1 + \cdots + \nu_{r_j}\beta_{r_j}) k_j}$$
$$= \sum_{|\nu_1| < n_1} \cdots \sum_{|\nu_{r_j}| < n_{r_j}} d(\nu_1, \ldots, \nu_r) c(\nu_1\beta_1 + \cdots + \nu_{r_j}\beta_{r_j}) e^{i(\nu_1\beta_1 + \cdots + \nu_{r_j}\beta_{r_j}) m_j}.$$

The product of these d terms is a trigonometric polynomial, as required.

To see 2) note that

$$|K_{\mathbf{B}} \star a(\mathbf{m})| = \lim_{\mathbf{N}} \left| \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_B(\mathbf{m}-\mathbf{k})a(\mathbf{k}) \right| \leq \lim_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m}-\mathbf{k}) \sup_{\mathbf{k}} |a(\mathbf{k})|$$
$$\leq \|a\|_\infty \lim_{\mathbf{N}} \frac{1}{\mathbf{N}} \sum_{\mathbf{k}=0}^{\mathbf{N}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m}-\mathbf{k}) \leq \|a\|_\infty.$$

To see 3) note that

$$\begin{aligned}
\|K_B \star a\|_1 &= \limsup_{\mathbf{M}} \frac{1}{|\mathbf{M}|} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{M}-\mathbf{1}} |K_{\mathbf{B}} \star a(\mathbf{m})| \\
&= \limsup_{\mathbf{M}} \frac{1}{|\mathbf{M}|} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{M}-\mathbf{1}} \lim_{\mathbf{N}} \left| \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-1} K_{\mathbf{B}}(\mathbf{m}-\mathbf{k})a(\mathbf{k}) \right| \\
&\leq \limsup_{\mathbf{M}} \limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \left| \frac{1}{|\mathbf{M}|} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{M}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m}-\mathbf{k})a(\mathbf{k}) \right| \\
&\leq \limsup_{\mathbf{M}} \limsup_{\mathbf{N}} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} a(\mathbf{k}) \left| \frac{1}{|\mathbf{M}|} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{M}-\mathbf{1}} K_{\mathbf{B}}(\mathbf{m}-\mathbf{k}) \right| \leq \|a\|_1 \|K_{\mathbf{B}}\|_1 \leq \|a\|_1.
\end{aligned}$$

LEMMA 3.4. *If φ is a trigonometric polynomial and let $\epsilon > 0$ be given. Then we can select a Bochner-Fejer polynomial $K_{\mathbf{B}} \star \varphi$ such that $\|K_{\mathbf{B}}(\varphi) - \varphi\|_1 \leq \epsilon$.*

PROOF. To keep the notation from obscuring the simple idea, we first present the proof for one dimension. To estimate the term $\|K_B(\varphi) - \varphi\|_1$, assume that

$$\varphi(x) = \sum_{\nu_1=0}^{m_1} \cdots \sum_{\nu_r=0}^{m_r} A(\nu_1, \nu_2, \ldots, \nu_r) e^{2\pi i(\nu_1\beta_1+\cdots+\nu_r\beta_r)x},$$

with the β_j linearly independent over the rationals. We now select the sequence

$$B = B(n_1, n_2, \ldots, n_r; \beta_1, \beta_2, \ldots, \beta_r)$$

in such a way that each of the terms in the exponential is represented, and for each j we have $n_j > m_j$. Then we have

$$\begin{aligned}
&K_B(\varphi) - \varphi \\
&\quad \leq \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_r=0}^{n_r} A(\nu_1, \nu_2, \ldots, \nu_r) \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_r|}{n_r}\right) e^{2\pi i(\nu_1\beta_1+\cdots+\nu_r\beta_r)x} \\
&\qquad - \sum_{\nu_1=0}^{m_1} \cdots \sum_{\nu_r=0}^{m_r} A(\nu_1, \nu_2, \ldots, \nu_r) e^{2\pi i(\nu_1\beta_1+\cdots+\nu_r\beta_r)x} \\
&\quad = \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_r=0}^{n_r} A(\nu_1, \nu_2, \ldots, \nu_r) \left(\left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_r|}{n_r}\right) - 1 \right) e^{2\pi i(\nu_1\beta_1+\cdots+\nu_r\beta_r)x}.
\end{aligned}$$

Expand the product, and note the fact that we have only $m_1 m_2 \cdots m_r$ non-zero terms. If any $\nu_j > m_j$ then $A(\nu_1, \nu_2, \ldots, \nu_r) = 0$. Consequently, for $(n_1, n_2, \ldots, n_r)$ taken large enough, we can make

$$\left(\left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_r|}{n_r}\right) - 1\right)$$

as small as we like. Hence we can make $|K_B(\varphi) - \varphi| < \epsilon$ for all x.

The higher dimensional analog follows by the same argument, with much additional notation.

LEMMA 3.5. *Given a sequence $\{a\} \in B(1)$ and given $\epsilon > 0$ there is a Bochner-Fejer polynomial $K_{\mathbf{B}} = K_{\mathbf{B}} \star a$ such that $\|\{a\} - K_{\mathbf{B}} \star a\|_\infty < \epsilon$.*

PROOF. From the definition of B(1) we know that given $\epsilon > 0$, we know that there is a trigonometric polynomial φ such that $\|a - \varphi\|_1 < \epsilon/3$.

By Lemma 3.4 we know that B can be selected so that $\|K_{\mathbf{B}} \star \varphi - \varphi\|_1 < \epsilon/3$.

From Lemma 3.3 we have $\|K_{\mathbf{B}}(a) - K_{\mathbf{B}}(\varphi)\|_1 < \|a - \varphi\|_1 < \epsilon/3$.

Combining these estimates, we see that for $\{a\} \in B(1)$ we have

$$\|a - K_{\mathbf{B}}(a)\|_1 \leq \|a - \varphi\|_1 + \|\varphi - K_{\mathbf{B}}(\varphi)\|_1 + \|K_{\mathbf{B}}(\varphi) - K_{\mathbf{B}}(a)\|_1 < \epsilon.$$

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Department of Mathematics
DePaul University
2219 N. Kenmore
Chicago, Illinois 60614
U.S.A.
e-mail: MATRLJ@DePaul.Bitnet

Department of Mathematics
North Dakota State University
Fargo, North Dakota 58105
U.S.A.
e-mail: JOlsen@Plains.NoDak.edu