A PRODUCT OF TWO E-FUNCTIONS EXPRESSED AS A SUM OF TWO E-FUNCTIONS

by F. M. RAGAB

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§ 1. *Introductory.* The formula to be proved is

$$2\pi i E(\alpha, 1-\alpha : : z) E(\beta, 1-\beta : : z)$$

$$= \frac{\pi\sqrt{\pi} e^z}{\sin \alpha\pi \sin \beta\pi} \left[\begin{array}{l} E\left(\frac{\alpha+\beta}{2}, \frac{1+\alpha-\beta}{2}, \frac{1-\alpha+\beta}{2}, \frac{2-\alpha-\beta}{2} : \tfrac{1}{2} : e^{i\pi} z^2/4\right) \\ -E\left(\frac{\alpha+\beta}{2}, \frac{1+\alpha-\beta}{2}, \frac{1-\alpha+\beta}{2}, \frac{2-\alpha-\beta}{2} : \tfrac{1}{2} : e^{-i\pi} z^2/4\right) \end{array} \right]. \quad \ldots\ldots(1)$$

In proving (1) the following formulae are required:

$$F(\alpha;\ 2\alpha;\ z) F(\beta;\ 2\beta;\ -z) = F\left\{ \begin{array}{l} \tfrac{1}{2}(\alpha+\beta), \tfrac{1}{2}(\alpha+\beta+1);\ \frac{z^2}{4} \\ \alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \alpha+\beta \end{array} \right\}, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$E(p;\ \alpha_r :\ q;\ \rho_s :\ z) = \sum_{r=1}^{p} \prod_{s=1}^{p}{}' \Gamma(\alpha_s - \alpha_r) \left\{ \prod_{t=1}^{q} \Gamma(\rho_t - \alpha_r) \right\}^{-1} \Gamma(\alpha_r)$$

$$\times z^{\alpha_r} F\left\{ \begin{array}{l} \alpha_r, \alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1;\ (-1)^{p-q} z \\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_p + 1 \end{array} \right\}, \quad \ldots\ldots\ldots\ldots(3)$$

where $p \geqslant q+1$, (**1**).

Formula (2) is proved in § 2, formula (1) in § 3; in § 4 an expression for $K_m(z) K_n(z)$ as a sum of E-functions is deduced, and this is used to evaluate two integrals.

§ 2. *Proof of subsidiary formula.* The L.H.S. of (2) can be written

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(\alpha;\ n)}{(2\alpha;\ n)} F\left(\begin{array}{l} -n, 1-2\alpha-n, \beta;\ 1 \\ 1-\alpha-n, 2\beta \end{array} \right).$$

On applying the formula (**2**)

$$F\left(\begin{array}{l} \alpha, \quad \beta, \quad \gamma;\quad 1 \\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}, 2\gamma \end{array} \right) = \frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2})\Gamma(\gamma+\tfrac{1}{2})\Gamma(\gamma-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2})\Gamma(\tfrac{1}{2}\beta+\tfrac{1}{2})\Gamma(\gamma-\tfrac{1}{2}\alpha+\tfrac{1}{2})\Gamma(\gamma-\tfrac{1}{2}\beta+\tfrac{1}{2})}, \quad \ldots\ldots\ldots\ldots(4)$$

where, if α or β is not zero or a negative integer, $R(2\gamma-\alpha-\beta) > -1$, the expression becomes

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(\alpha;\ n)}{(2\alpha;\ n)} \frac{\Gamma(\tfrac{1}{2})\Gamma(1-\alpha-n)\Gamma(\beta+\tfrac{1}{2})\Gamma(\alpha+\beta+n)}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n)\Gamma(1-\alpha-\tfrac{1}{2}n)\Gamma(\beta+\tfrac{1}{2}n+\tfrac{1}{2})\Gamma(\alpha+\beta+\tfrac{1}{2}n)}.$$

Now, when n is an odd positive integer $1/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n) = 0$; hence, on replacing n by $2m$ the expression can be written

$$\sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!} \frac{(\alpha;\ 2m)}{(2\alpha;\ 2m)} \frac{\Gamma(\tfrac{1}{2})\Gamma(1-\alpha-2m)\Gamma(\beta+\tfrac{1}{2})\Gamma(\alpha+\beta+2m)}{\Gamma(\tfrac{1}{2}-m)\Gamma(1-\alpha-m)\Gamma(\beta+\tfrac{1}{2}+m)\Gamma(\alpha+\beta+m)},$$

and this reduces to the R.H.S. of (2).

§ 3. *Proof of the formula.* In (1) expand the L.H.S. by means of (3), making use of the formula (**3**).

$$F(\alpha;\ \rho;\ z) = e^z F(\rho-\alpha;\ \rho;\ -z), \quad (5)$$

and it becomes

$$2\pi i\,[\Gamma(1-2\alpha)\Gamma(\alpha)z^{\alpha}F(\alpha;\ 2\alpha;z)+\Gamma(2\alpha-1)\Gamma(1-\alpha)z^{1-\alpha}F(1-\alpha;\ 2-2\alpha;\ z)]$$
$$\times[\Gamma(1-2\beta)\Gamma(\beta)z^{\beta}e^{z}F(\beta;\ 2\beta;\ -z)+\Gamma(2\beta-1)\Gamma(1-\beta)z^{1-\beta}e^{z}F(1-\beta;\ 2-2\beta;\ -z)].$$

On multiplying and applying (2) this becomes

$$2\pi i e^{z}\left[\begin{aligned}&\Gamma(1-2\alpha)\Gamma(\alpha)\Gamma(1-2\beta)\Gamma(\beta)z^{\alpha+\beta}F\begin{Bmatrix}\tfrac12(\alpha+\beta),\tfrac12(\alpha+\beta+1);& z^2/4\\ \alpha+\tfrac12,\beta+\tfrac12,\alpha+\beta\end{Bmatrix}\\ &+\sum_{\alpha,\beta}\Gamma(1-2\alpha)\Gamma(\alpha)\Gamma(2\beta-1)\Gamma(1-\beta)z^{\alpha-\beta+1}F\begin{Bmatrix}\tfrac12(\alpha-\beta+1),\tfrac12(\alpha-\beta+2);& z^2/4\\ \alpha+\tfrac12,\tfrac32-\beta,\alpha-\beta+1\end{Bmatrix}\\ &+\Gamma(2\alpha-1)\Gamma(1-\alpha)\Gamma(2\beta-1)\Gamma(1-\beta)z^{2-\alpha-\beta}F\begin{Bmatrix}\tfrac12(2-\alpha-\beta),\tfrac12(3-\alpha-\beta);& z^2/4\\ \tfrac32-\alpha,\tfrac32-\beta,2-\alpha-\beta\end{Bmatrix}\end{aligned}\right].$$

Again, on applying (3) to the R.H.S. of (1), it becomes

$$\frac{\pi\sqrt{\pi}e^{z}}{\sin\alpha\pi\sin\beta\pi}\left[\begin{aligned}&\frac{\Gamma(\tfrac12-\beta)\Gamma(\tfrac12-\alpha)\Gamma(1-\alpha-\beta)}{\Gamma\left(\frac{1-\alpha-\beta}{2}\right)}\Gamma\left(\frac{\alpha+\beta}{2}\right)2i\sin\left(\frac{\alpha+\beta}{2}\pi\right)\left(\frac{z}{2}\right)^{\alpha+\beta}\\ &\qquad\times F\{\tfrac12(\alpha+\beta),\tfrac12(\alpha+\beta+1);\ \alpha+\tfrac12,\beta+\tfrac12,\alpha+\beta;\ \tfrac14z^2\}\\ &+\sum_{\alpha,\beta}\frac{\Gamma(\beta-\tfrac12)\Gamma(\beta-\alpha)\Gamma(\tfrac12-\alpha)}{\Gamma\left(\frac{\beta-\alpha}{2}\right)}\Gamma\left(\frac{1+\alpha-\beta}{2}\right)2i\cos\left(\frac{\alpha-\beta}{2}\pi\right)\left(\frac{z}{2}\right)^{\alpha-\beta+1}\\ &\qquad\times F\{\tfrac12(\alpha-\beta+1),\tfrac12(\alpha-\beta+2);\ \tfrac32-\beta,\alpha-\beta+1,\alpha+\tfrac12;\ \tfrac14z^2\}\\ &+\frac{\Gamma(\alpha+\beta-1)\Gamma(\alpha-\tfrac12)\Gamma(\beta-\tfrac12)}{\Gamma\left(\frac{\alpha+\beta-1}{2}\right)}\Gamma\left(\frac{2-\alpha-\beta}{2}\right)2i\sin\left(\frac{\alpha+\beta}{2}\pi\right)\left(\frac{z}{2}\right)^{2-\alpha-\beta}\\ &\qquad\times F\{\tfrac12(2-\alpha-\beta),\tfrac12(3-\alpha-\beta);\ 2-\alpha-\beta,\tfrac32-\alpha,\tfrac32-\beta;\ \tfrac14z^2\}\end{aligned}\right].$$

On comparing these two expressions it is seen that they are equal.

§ 4. *Product of two modified Bessel Functions of the Second Kind.* In (1) replace z by $2z$, put $\alpha=\tfrac12+m$, $\beta=\tfrac12+n$, and apply the formula (**4**)

$$\cos n\pi E(\tfrac12+n,\tfrac12-n::2z)=\sqrt{(2\pi z)}\,e^{z}K_n(z),\ \ldots\ldots(6)$$

and it reduces to

$$K_m(z)K_n(z)=\frac{1}{4z\sqrt{\pi}}\sum_{i,-i}\frac{1}{i}E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{1-m+n}{2},\frac{1-m-n}{2}:\tfrac12:e^{i\pi}z^2\right).\ldots(7)$$

From this it can be deduced that

$$\int_0^\infty\lambda^{l-1}K_m(\lambda)K_n(\lambda)E(p;\ \alpha_r:q;\ \rho_s:z/\lambda^2)\,d\lambda$$
$$=\tfrac14\sqrt{\pi}\,E\left(\alpha_1,\ldots,\alpha_p,\frac{l+m+n}{2},\frac{l+m-n}{2},\frac{l-m+n}{2},\frac{l-m-n}{2}:\rho_1,\ldots,\rho_q,\frac{l}{2},\frac{l+1}{2}:z\right),\ \ldots(8)$$

provided that $p\geqq q+1$, $R(l\pm m\pm n)>0$, $|\operatorname{amp} z|<\pi$. For other values of p and q the result holds if the integral converges.

In proving this use is made of the formula (**5**)

$$\int_0^\infty\lambda^{k-1}E(p;\ a_r:q;\ \rho_s:\lambda)E(l;\ \beta_t:m;\ \sigma_u:z/\lambda)\,d\lambda$$

$$=\frac{\pi}{\sin k\pi}\left\{\begin{array}{l} z^k E\begin{pmatrix}\alpha_1, \ldots, \alpha_p, \beta_1 - k, \ldots, \beta_l - k : e^{\pm i\pi} z \\ 1-k, \rho_1, \ldots, \rho_q, \sigma_1 - k, \ldots, \sigma_m - k\end{pmatrix} \\ -E\begin{pmatrix}\alpha_1+k, \ldots, \alpha_p+k, \beta_1, \ldots, \beta_l : e^{\pm i\pi} z \\ 1+k, \rho_1+k, \ldots, \rho_q+k, \sigma_1, \ldots, \sigma_m\end{pmatrix}\end{array}\right\}, \quad \ldots\ldots\ldots\ldots(9)$$

where $p \geqq q+1$, $l \geqq m+1$, $R(\alpha_r+k)>0$, $r=1, 2, \ldots, p$, $R(\beta_t-k)>0$, $t=1, 2, \ldots, l$, and $|\operatorname{amp} z|<\pi$. For other values of p, q, l, m the result holds if the integral converges.

On substituting from (7) in the L.H.S. of (8), replacing λ by $\sqrt{(z/\lambda)}$ and applying (9) formula (8) is obtained. In the first of the two integrals the lower sign in $e^{\pm i\pi}$ in (9) should be employed, in the second integral the upper sign.

In particular, if $p=q=0$,

$$\int_0^\infty e^{-\lambda^2/z} \lambda^{l-1} K_m(\lambda) K_n(\lambda)\, d\lambda$$

$$=\tfrac{1}{4}\sqrt{\pi}\, E\left(\frac{l+m+n}{2}, \frac{l+m-n}{2}, \frac{l-m+n}{2}, \frac{l-m-n}{2} : \frac{l}{2}, \frac{l+1}{2} : z\right), \quad \ldots\ldots\ldots(10)$$

provided that $R(z)>0$, $R(l \pm m \pm n)>0$.

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UNIVERSITY OF GLASGOW
and
FACULTY OF SCIENCE
EIN SHAMS UNIVERSITY
CAIRO