RELATIONS BETWEEN FINITE HOMOLOGY AND HOMOTOPY

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1. Introduction. For a finite abelian group G let $\lambda(G)$ be the least positive integer such that $\lambda(G)G = 0$. Let $\bar{\lambda}(G)$ be the least integer such that $\lambda(G) \mid \bar{\lambda}(G)$ ($\lambda(G)$ divides $\bar{\lambda}(G)$) and if $2 \mid \lambda(G)$ then $4 \mid \bar{\lambda}(G)$. For a finitely generated abelian group G let G_T be the subgroup of G made up of all elements of G of finite order, and let $G_F = G/_{G_T}$. For a simply-connected C-W complex X, let $\mathcal{C}(\pi X; r)$ be the smallest class of abelian groups containing the groups $\pi_i(X)$, $i = 2, \ldots, r$.

We will assume throughout this paper that the topological spaces under discussion have the homotopy type of C-W complexes with only finitely many cells in each dimension. We write $H_*(X)$ ($H^*(X)$) for reduced homology (cohomology) with integer coefficients.

The point of this paper is to prove

THEOREM 1*. Suppose

i) $\pi_i(X) = 0, \quad i = 1, \ldots, k-1$ where $k-1 \geq 1$

ii) $\pi_i(X)$ is finite for $i = k, \ldots, n$,

then $\displaystyle\prod_{i=n-k+1}^{n} \lambda(\pi_i(X)) \; H_n(X) \in \mathcal{C}(\pi X; n-k)$.

COROLLARY 2.1. If $\dim X = k$ and $H_*(X)$ is finite then

$$\lambda(\{X, X\}) \mid \prod_{i=1}^{k} \bar{\lambda}(H_i(X))$$

*Hoo [1] proves a slightly different version of this theorem for the special case $k = 2$.

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and

$$\lambda(\{X, X\}) \mid \prod_{i=1}^{k} \lambda(\Sigma_i(X)) \qquad [\Sigma_i(X) = \{S^i, X\}].$$

THEOREM 3. *If* $H^i(X) = 0$ *for* $i > n+k$ *and* $H_i(Y) = 0$ *for* $i < n$ *then*

$$\{X, Y\}_F = (\sum_{i=n}^{n+k} H^i(X; H_i(Y)))_F$$

and

$$\lambda(\{X, Y\}_T) \mid \lambda((\sum_{i=n}^{n+k} H^i(X; H_i(Y)))_T) \prod_{i=1}^{k} \lambda(\Sigma_i(S^o)).$$

2. *Some algebraic lemmas*. For an element a in an abelian group A, $o(a)$ will mean the order of a.

Definition. For finite abelian groups A and B we will write $A < B$ if and only if for every $a \in A$ there exists a $b \in B$ such that $o(a) \mid o(b)$. ($o(a)$ divides $o(b)$).

The following algebraic lemmas are all trivial.

LEMMA 1. $A < B \Leftrightarrow \lambda(A) \mid \lambda(B)$.

LEMMA 2. $A < 0 \Rightarrow A = 0$.

LEMMA 3. *If* $A < B$ *and* B *is in some class of abelian groups* $₵$, *then* $A \in ₵$.

LEMMA 4. $A < B$ *and* $B < C \Rightarrow A < C$.

LEMMA 5. *If* $A < B$ *then for any integer* r, $rA < rB$.

LEMMA 6. *If* $A \to B \to C$ *is exact* $\lambda(B) \mid \lambda(A)\lambda(C)$ *and so*

i) $rA = 0 \Rightarrow rB < C$

ii) $sC = 0 \Rightarrow sB < A$.

LEMMA 7. *Suppose* $A_i \to B_i \to C_i$ *is exact* $i = 1, \ldots, t$;

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i) *if* $n_i A_i = 0$ $i = 1, \ldots, t$ *and* $C_i = B_{i+1}$ *for* $i = 1, \ldots, t-1$ *then* $n_t \ldots n_1 B_1 < C_t$;

ii) if $m_i C_i = 0$ $i = 1, \ldots, t$ *and* $A_i = B_{i+1}$ $i = 1, \ldots, t-1$ *then* $m_t m_{t-1} \ldots m_1 B_1 < A_t$.

LEMMA 8. *If* R *is a finite ring with identity* I_R *then* $\lambda(R) = 0(I_R)$.

LEMMA 9. *For a finite abelian group* G *let* I_G *be the identity homomorphism*. ($I_G \in \mathrm{Hom}(G\ G)$.) *Then* $\lambda(\mathrm{Hom}(G, G)) = 0(I_G) = \lambda(G)$.

LEMMA 10. *Suppose that* G *is an abelian group and that* R *is a finite ring with identity* I_R . *If* $m : R \times G \to G$ *satisfies*

i) $m(I_R, g) = g$ for all $g \in G$, *and*

ii) $m(r_1 + r_2, g) = m(r_1, g) + m(r_2, g)$ *for all* $r_1, r_2 \in R$, $g \in G$ *then* $\lambda(G) \mid \lambda(R)$.

LEMMA 11. *For finite abelian groups* G *and* H, $\lambda(G) \mid \lambda(G \oplus H)$, and $\bar{\lambda}(G) \mid \bar{\lambda}(G \oplus H)$.

LEMMA 12. G *is a finite abelian group*. *If* $p^k \mid \lambda(G)$ *and* $p^{k+1} \nmid \lambda(G)$ *then* $G = Z_{p^k} \oplus G'$.

LEMMA 13. G *is a finite abelian group*. $\lambda(G) \neq \bar{\lambda}(G)$ *if and only if* $G = G' \oplus Z_2 \oplus \ldots \oplus Z_2$, *where* $\lambda(G')$ *is odd*.

3. *Eilenberg-MacLane spaces and Moore spaces*. Let $K(G, n)$ be the Eilenberg-MacLane space of type (G, n) and $M(G, n)$ the Moore space of type (G, n).

PROPOSITION 1. *Let* G *be a finite abelian group and let* I_K *be the homotopy class of the identity map of* $K(G, n)$. *Then*

$$\lambda([K(G, n), K(G, n)]) = o(I_K) = \lambda(G).$$

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Proof. $[K(G,n), K(G,n)] \approx \mathrm{Hom}(G,G)$ and I_K corresponds to I_G, under the isomorphism. Since $[K(G,n), K(G,n)]$ is a ring with identity I_K, $\lambda([K(G,n), K(G,n)]) = o(I_K)$ Lemma 8

$$= o(I_G) = \lambda(G) \quad \text{Lemma 9}.$$

PROPOSITION 2. *If G is a finite abelian group and X is a C-W complex, then* $\lambda(G)H^i(X;G) = \lambda(G)H_i(K(G,n)) = 0$.

Proof. $[K(G,n), K(G,n)]$ acts on both $H^i(X;G)$ and $H_i(K(G,n))$ as a ring of operators, so the statement follows from Lemma 10 and Proposition 1.

PROPOSITION 3. *If p is a prime and $p \neq 2$ then*

$$\{M(Z_{p^k}, n), M(Z_{p^k}, n)\} = Z_{p^k}.$$

Proof. Let $f: S^n \to S^n$ be a map of degree p^k. Then the cone of f, C_f, has the homotopy type of $M(Z_{p^k}, n)$. The Hurewicz Theorem implies that $\{S^n, C_f\} = Z_{p^k}$.

The fact that $\{S^{n+1}, C_f\} = 0$ follows from the exact sequence

$$\begin{array}{ccccccccc} \{S^{n+1}, S^n\} & \xrightarrow{f_*} & \{S^{n+1}, S^n\} & \to & \{S^{n+1}, C_f\} & \to & \{S^{n+1}, S^{n+1}\} & \xrightarrow{(Sf)_*} & \{S^{n+1}, S^{n+1}\} \\ \| & & \| & & & & \| & & \| \\ Z_2 & & Z_2 & & & & Z & & Z \end{array}$$

and the observation that f_* is onto and $(Sf)_*$ is a monomorphism. The sequence

$$\begin{array}{ccccccc} \{S^n, C_f\} & \xleftarrow{f^*} & \{S^n, C_f\} & \longleftarrow & \{C_f, C_f\} & \longleftarrow & \{S^{n+1}, C_f\} \\ \| & & \| & & & & \| \\ Z_{p^k} & & Z_{p^k} & & & & 0 \end{array}$$

is exact and f^* is the zero homomorphism, since it is multiplication by

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p^k, and so $\{C_f, C_f\} = Z_{p^k}$.

PROPOSITION 4. If $K \geq 2$ then $\{M(Z_{2^k}, n), M(Z_{2^k}, n)\} = Z_{2^k} \oplus Z_2$.

Proof. Let $f: S^n \to S^n$ be a map of degree 2^k. Let $\alpha: S^n \to C_f$ be the inclusion and let $\beta: C_f \to S^{n+1}$ be the inclusion of C_f into the cone of α. Let $h^t: S^{t+1} \to S^t$ be the appropriate suspension of the Hopf map. We have $\{S^n, C_f\} = Z_{2^k}$ (with generator α) and $\{S^{n+1}, C_f\} = Z_2$ (with generator αh^n).

The sequence $0 \longleftarrow \{S^n, C_f\} \overset{\alpha_*}{\longleftarrow} \{C_f, C_f\} \overset{\beta_*}{\longleftarrow} \{S^{n+1}, C_f\} \longleftarrow 0$ is exact. So $\{C_f, C_f\} = Z_{2^{k+1}}$ or $Z_{2^k} \oplus Z_2$.

Assume that $\{C_f, C_f\} = Z_{2^{k+1}}$. Let I be the stable homotopy class of the identity map $C_f \to C_f$. Then I is the multiplicative identity in the ring $\{C_f, C_f\}$ and consequently must be a generator of the cyclic group. Since $\alpha^*(2^k I) = 2^k \alpha = 0$, and $2^k I \neq 0$, we must have $2^k I = \beta^*(\alpha h^n) = \alpha h^n \beta$. We will arrive at a contradiction by showing that this last equation is false.

The sequence $0 \to \{S^{n+2}, S^n\} \overset{\alpha_*}{\to} \{S^{n+2}, C_f\} \overset{\beta_*}{\to} \{S^{n+2}, S^{n+1}\} \to 0$ is exact and $\{S^{n+2}, S^n\} = Z_2$ (with generator $h^n h^{n+1}$); $\{S^{n+2}, S^{n+1}\} = Z_2$ (with generator h^{n+1}). Since β^* is onto, there exists an element $\gamma \in \{S^{n+2}, C_f\}$ such that $\beta\gamma = h^{n+1}$. Since α_* is a monomorphism $\alpha h^n h^{n+1} \neq 0$. So $\alpha h^n \beta\gamma \neq 0$. But γ is an element in a group of order 4 and $k \geq 2$, so $2^k I\gamma = 2^k \gamma = 0$. Therefore $\alpha h^n \beta \neq 2^k I$ and this contradiction implies that $\{C_f, C_f\} = Z_{2^k} \oplus Z_2$. Let $n \geq 4$ and let X be the space in the Postnikov system for $M(Z_2, n)$ made up of the first two non-zero homotopy groups of $M(Z_2, n)$. Then the Postnikov system of X looks like this:

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$$\begin{array}{ccc} K(Z_2, n+1) & \xrightarrow{a} & X \\ & & \downarrow b \\ & & K(Z_2, n) \xrightarrow{k} K(Z_2, n+2). \end{array}$$

The Postnikov invariant k is either 0 or Sq^2. If $k = 0$, we have $X = K(Z_2, n) \times K(Z_2, n+1)$ and $H^{n+1}(X; Z_2) = Z_2 \oplus Z_2$. But this contradicts $H^{n+1}(X, Z_2) = H^{n+1}(M(Z_2, n); Z_2) = Z_2$. So $k = Sq^2$. (We begin now not to distinguish, in our symbols, between the following objects: a cohomology operation, a cohomology class, a map, and its homotopy class.

PROPOSITION 5. $[X, X] = Z_4$ <u>and</u> $2I_X = aSq^1b$.

<u>Proof</u>. The sequence

$$0 = H^{n-1}(X; Z_2) \to H^{n+1}(X; Z_2) \xrightarrow{a_*} [X, X] \xrightarrow{b_*} H^n(X; Z_2) \xrightarrow{Sq^2} H^{n+2}(X; Z_2)$$

is exact. $H^{n+1}(X; Z_2) = Z_2$ with generator Sq^1b, $H^n(X; Z_2) = Z_2$ with generator b, and $Sq^2b = 0$. So $0 \to Z_2 \to [X, X] \to Z_2 \to 0$ is exact and $[X, X] = Z_2 \oplus Z_2$ or Z_4 depending on whether or not $2I_X = 0$. If $2I_X \neq 0$ we have $b_*(2I_X) = 0$ and $2I_X \in \operatorname{Im} a_*$, that is, $2I_X = aSq^1b$. We will prove that $2I_X \neq 0$ by showing that for some $\gamma \in H^{n+3}(X; Z_4)$, $2\gamma \neq 0$.

Let $\phi : K(Z_4, n+3) \to K(Z_2, n+3)$ be the fibre map corresponding to the projection $Z_4 \to Z_2$, and let $\psi : K(Z_2, n+3) \to K(Z_4, n+3)$ be the inclusion of the fibre into the total space. The following computations are straightforward: $H^{n+3}(K(Z_2, n); Z_4) = Z_2 \oplus Z_2$ with generators ψSq^2Sq^1 and r, where $\phi r = Sq^1Sq^2$; $H^{n+3}(K(Z_2, n+2); Z_4) = Z_2$ with generator t, where $\phi t = Sq^1$. $\phi(r - tSq^2) = Sq^1Sq^2 - Sq^1Sq^2 = 0$. Therefore $r - tSq^2 = 0$ or ψSq^2Sq^1. Whatever the case, we have $tSq^2 \neq \psi Sq^2Sq^1$.

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$$H^{n+3}(K(Z_2, n+2); Z_4) \xrightarrow{(Sq^2)^*} H^{n+3}(K(Z_2, n); Z_4) \xrightarrow{b^*} H^{n+3}(X; Z_4)$$

is part of the Serre exact sequence for the fibring $X \xrightarrow{b} K(Z_2, n) \xrightarrow{Sq^2} K(Z_2, n+2)$. For the generator $t \in H^{n+3}(K(Z_2, n+2); Z_4)$, $(Sq^2)^*t = tSq^2 \neq \psi Sq^2 Sq^1$. So, $b^*(\psi Sq^2 Sq^1) = \psi Sq^2 Sq^1 b \neq 0$. Now consider the element $Sq^2 Sq^1 b \in H^{n+3}(X; Z_2)$. $Sq^1(Sq^2 Sq^1 b) = (Sq^1 Sq^2 Sq^1)b = (Sq^2 Sq^2)b = Sq^2(Sq^2 b) = 0$. Therefore for some $\gamma \in H^{n+3}(X; Z_4)$, $\phi\gamma = Sq^2 Sq^1 b$. Then $\psi\phi\gamma = \psi Sq^2 Sq^1 b \neq 0$. But $\psi\phi$ is twice the identity map $K(Z_4, n+3) \to K(Z_4, n+3)$. So $\psi\phi\gamma = 2\gamma \neq 0$, and the proof is complete.

PROPOSITION 5A. Let S be the (mod 2) Steenrod Algebra. Define $h : S \to S$ by $h(\theta) = \theta Sq^2$. Let $K = \operatorname{Ker} h$. Then $KSq^1 \subset SSq^2$.

Proof. Suppose the statement is false. Then for some θ, $\theta Sq^2 = 0$ and $\theta Sq^1 \notin SSq^2$. (In particular $\theta Sq^1 \neq 0$.) Let $r = \deg \theta$. Take X as above with the qualification that $n > r+2$. Write K_i for $K(Z_2, i)$. The sequence

$$H^{n+r+1}(K_{n-1}; Z_2) \xleftarrow{(Sq^2)^*} H^{n+r+1}(K_{n+1}; Z_2) \xleftarrow{a^*} H^{n+r+1}(X; Z_2) \xleftarrow{b^*}$$

$$\longleftarrow H^{n+r+1}(K_n; Z_2) \xleftarrow{(Sq^2)^*} H^{n+r+1}(K_{n+2}; Z_2)$$

is exact.

Consider θ as an element of $H^{n+r+1}(K_{n+1}; Z_2)$. $(Sq^2)^*\theta = \theta Sq^2 = 0$. Therefore, for some $\gamma \in H^{n+r+1}(X; Z_2)$, $\theta = a^*\gamma$. Thinking of Sq^1 as a map $K_n \to K_{n+1}$, we have $(Sq^1)^* a^* \gamma = (Sq^1)^*\theta = \theta Sq^1 \in H^{n+r+1}(K_n; Z_2)$. Also $\theta Sq^1 \neq 0$ and $\theta Sq^1 \notin \operatorname{Im}(Sq^2)^*$. Therefore $b^*(Sq^1)^* a^* \gamma = b^*(\theta Sq^1) \neq 0$. But $b^*(Sq^1)^* a^* = (aSq^1 b)^* = (2I_X)^*$. (Proposition 5.) So $b^*(Sq^1)^* a^* \gamma = 2\gamma = 0$ and this contradiction proves our proposition.

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PROPOSITION 6. $\{M(Z_2, n), M(Z_2, n)\} = Z_4$.

Proof. Take $n \geq 4$. Then $\{M(Z_2, n), M(Z_2, n)\} = [M(Z_2, n), M(Z_2, n)] = [M(Z_2, n), X]$. (Where X is as above.) Using the homotopy sequence for the fibring $K(Z_2, n+1) \xrightarrow{a} X \xrightarrow{b} K(Z_2, n)$, we have that $0 = H^{n-1}(M(Z_2, n); Z_2) \rightarrow H^{n+1}(M(Z_2, n); Z_2) \xrightarrow{a_*} [M(Z_2, n), X] \xrightarrow{b_*} H^n(M(Z_2, n); Z_2) \rightarrow H^{n+2}(M(Z_2, n); Z_2 = 0$ is exact; $H^n(M(Z_2, n); Z_2) = Z_2$. Call the generator g. $H^{n+1}(M(Z_2, n); Z_2) = Z_2$ with generator $Sq^1 g$. Since b_* is onto there exists a $\xi \in [M(Z_2, n), X]$ such that $b\xi = g$. Then $Sq^1 b\xi = Sq^1 g \neq 0$, and since a_* is a monomorphism $aSq^1 b\xi = aSq^1 g \neq 0$, But $aSq^1 b = 2I_X$. Therefore $0 \neq aSq^1 b\xi = 2\xi$, and $[M(Z_2, n), X] = Z_4$.

Putting together Propositions 3, 4, and 6 we have that if $G = Z_{p^k}$ (any prime p and exponent k) then $\lambda(\{M(G, n), M(G, n)\}) = \bar{\lambda}(G)$. Now we prove:

PROPOSITION 7. <u>For any finite abelian group</u> G, $\lambda(\{M(G, n), M(G, n)\}) = \bar{\lambda}(G)$.

Proof. $G = \sum_i Z_{p_i^{r_i}}$. Let $M = M(G, n)$, $M_i = M(Z_{p_i^{r_i}}, n)$. Then $M = \vee_i M_i$. Let $I_M (I_{M_i})$ be the stable homotopy class of the identity map of M (M_i). Then $I_M = \vee_i I_{M_i}$. For any positive integer t, $tI_M = \vee_i tI_{M_i}$ and $tI_M = 0$ if and only if $tI_{M_i} = 0$ for each i.

Since $o(I_{M_i}) = \bar{\lambda}(Z_{p_i^{r_i}}) \mid \bar{\lambda}(G)$ (remark above and Lemma 11) we have $\bar{\lambda}(G) I_{M_i} = 0$ for all i and $o(I_M) \mid \bar{\lambda}(G)$. Now we must show that $\bar{\lambda}(G) \mid o(I_M)$. Using Lemma 13, $\bar{\lambda}(G) = \lambda(G)$ or $G = G^1 \oplus Z_2 \oplus \ldots \oplus Z_2$ where $\lambda(G')$ is odd.

Case 1. Assume $\bar{\lambda}(G) = \lambda(G)$. For any prime p suppose $p^k \mid \lambda(G)$ and $p^{k+1} \nmid \lambda(G)$. Then $G = Z_{p^k} \oplus G''$ (Lemma 12).

($p^k \neq 2$, otherwise we would not be in Case 1.) Let $M_{p^k} = M(Z_{p^k}, n)$

and $M'' = M(G'', n)$. Then $M = M_{p^k} \vee M''$, and $M'' = M(G'', n)$.

Therefore $o(I_{M_{p^k}}) \mid o(I_M)$. That is $p^k \mid o(I_M)$ and this holds for all maximal prime power factors p^k of $\lambda(G)$. So $\lambda(G) \mid o(I_M)$. That is, if $\bar{\lambda}(G) = \lambda(G)$ then $\bar{\lambda}(G) = o(I_M)$.

Case 2. Assume $G = G' \oplus Z_2 \oplus \dots \oplus Z_2$ where $\lambda(G')$ is odd. Let $M' = M(G', n)$ and $M_2 = M(Z_2, n)$. $M = M' \vee M_2 \vee \dots \vee M_2$ and $I_M = I_{M'} \vee I_{M_2} \vee \dots \vee I_{M_2}$. Since $\lambda(G') = \bar{\lambda}(G')$ we have $o(I_{M'}) = \bar{\lambda}(G')$ (Case 1) and this number is odd. $o(I_{M_2} \vee \dots \vee I_{M_2}) = o(I_{M_2}) = 4$ is prime to $o(I_{M'})$ and so $o(I_M) = o(I_{M'}) \cdot o(I_{M_2} \vee \dots \vee I_{M_2}) = 4\bar{\lambda}(G') = \bar{\lambda}(G)$.

PROPOSITION 8. If G is a finite abelian group and Y is a C-W complex, then for any $n \geq 3$,

i) $\lambda[M(G, n), M(G, n)] = \bar{\lambda}(G)$

ii) $\lambda[M(G, n), Y] \mid \bar{\lambda}(G)$.

Proof. i) is a consequence of stability and ii) is a consequence of i) and Lemma 10.

4. The Theorems.

THEOREM 1. *Suppose*

i) $\pi_i(X) = 0$, $i = 1, \dots, k-1$ *where* $k - 1 \geq 1$

ii) $\pi_i(X)$ *is finite for* $i = k, \dots, n$,

then

$$\prod_{i=n-k+1}^{n} \lambda(\pi_i(X))\, H_n(X) \in \mathcal{C}(\pi X; n-k) .$$

Proof. Let X^j be the space in the Postnikov system for X made up of the first j homotopy groups of X. We have fibrations $K(\pi_{j+1}(X), j+1) \to X^{j+1} \to X^j$ and the sequence

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$H_n(K(\pi_{j+1}(X), j+1) \to H_n(X^{j+1}) \to H_n(X^j)$ is exact when $j \geq n-k$. $\lambda(\pi_{j+1}(X)) H_n(K(\pi_{j+1}(X); j+1) = 0$ (Proposition 2) and so, using Lemma 7,

$$\prod_{i=n-k+1}^{n} \lambda(\pi_i(X)) H_n(X^n) < H_n(X^{n-k}).$$ Also $H_n(X^n) = H_n(X)$ and

$H_n(X^{n-k}) \in \mathcal{C}(\pi X; n-k)$ by the (mod $\mathcal{C}$) Hurewicz Theorem. Now use Lemma 3 and the proof is complete.

COROLLARY 1.1. _If_ $\Sigma_i(X)$ $(= \{S^i, X\})$ _is finite for_ $i = 1, \ldots, n$ _then_ $\lambda(H_n(X)) \mid \prod_{i=1}^{n} \lambda(\Sigma_i(X))$.

Proof. Choose $m > n+1$ and let $Y = S^m X$. Then $\pi_j(Y) = 0$ for $i < m$ and $\pi_j(Y) = \Sigma_{j-m}(X)$ is finite for $m \leq j \leq m+n$. Applying Theorem 1 we have $\prod_{j=m+1}^{m+n} \lambda(\pi_j(Y)) H_{m+n}(Y) \in \mathcal{C}(\pi Y; n)$. But $\mathcal{C}(\pi Y; n)$ is the trivial class and $H_{m+n}(Y) = H_n(X)$, so we have $\prod_{i=1}^{n} \lambda(\Sigma_i(X)) H_n(X) = 0$.

THEOREM 2A. _If_ $\dim X = k$ _and_ $H_*(X)$ _is finite, then for any_ Y $\prod_i \bar{\lambda}(H_i(X))[S^2 X, Y] = 0$.

Proof. Let $P = S^2 X$ and let P^i be the space in the Eckmann-Hilton decomposition of P made up of the first i homology groups of P. Note that all spaces and maps in this decomposition are double suspensions. The inclusion $P^{i-1} \to P^i$ has cone $M(H_i(P), i)$ and so for any Y the sequence $[M(H_i(P), i), Y] \to [P^i, Y] \to [P^{i-1}, Y]$ is exact. $\bar{\lambda}(H_i(P))[M(H_i(P), i) Y] = 0$ for all i (Proposition 8), and using Lemma 7 we have $\prod_{i=1}^{k+2} \bar{\lambda}(H_i(P))[P^{k+2}, Y] < [P^o, Y]$. But $P^{k+2} = P$, $P^o = pt.$, $[P^o, Y] = 0$ and $H_i(P) = H_{i-2}(X)$ and the proof is complete.

THEOREM 2B. _If_ $\pi_i(X)$ _is finite for_ $i \leq k$ _and_ $\dim Y \leq k-2$, _then_ $\prod_{i=3}^{k} \lambda(\pi_i(X))[Y, \Omega^2 X] = 0$.

Proof. Let $L = \Omega^2 X$ and let L^i be the space in the Postnikov system of L made up of the first i homotopy groups of L. All spaces and maps in this Postnikov system are double loops. For each $i \leq k-2$ we have the fibring $K(\pi_i(L), i) \to L^i \to L^{i-1}$, and the sequence $H^i(Y; \pi_i(L)) \to [Y, L^i] \to [Y, L^{i-1}]$ is exact. $\lambda(\pi_i(L))H^i(Y; \pi_i(L)) = 0$ (Proposition 2) and using Lemma 7 $\prod_{i=1}^{k-2} \lambda(\pi_i(L)[Y, L^{k-2}] < [Y, L^o]$. However, $L^o = pt.$, $[Y, L^o] = 0$, $[Y, L^{k-2}] = [Y, L] = [Y, \Omega^2 X]$ (since $\dim Y \leq k-2$) and $\pi_i(L) = \pi_{i+2}(X)$, and so the proof is complete.

Corollary 2.1A follows immediately from Theorem 2A and Corollary 2.1B is easily proved by applying Theorem 2B to the case $[S^k X, \Omega^2 S^{k+2} X]$ ($= \{X, X\}$).

COROLLARY 2.2. *If* $\dim X = k$ *and* $H_*(X)$ *is finite then for any* C-W *complex* Y

$$\prod_{i=1}^{k} \bar{\lambda}(H_i(X)) \{X, Y\} = \prod_{i=1}^{k} \lambda(\Sigma_i(X)) \{X, Y\} = \prod_{i=1}^{k} \bar{\lambda}(H_i(X)) \{Y, X\}$$

$$= \prod_{i=1}^{k} \lambda(\Sigma_i(X)) \{Y, X\} = 0 .$$

Proof. Use Lemma 10 and Corollary 2.1.

THEOREM 3. *If* $H^i(X) = 0$ *for* $i > n+k$ *and* $H_i(Y) = 0$ *for* $i < n$ *then* $\{X, Y\}_F = (\sum_i H^i(X; H_i(Y)))_F$ *and*

$$\lambda(\{X, Y\}_T) \mid \lambda((\sum_n^{n+k} H^i(X; H_i(Y)))_T) \prod_{i=1}^{k} \lambda(\Sigma_i(S^o)) .$$

Proof. We may assume that we are in the stable range (i.e. $n > k+1$) and that Y is a finite complex. Then for some r, Y has an r-dual Y', and $\{X, Y\} = \{X \# Y', S^r\}$. $\dim X \# Y' = (n+k) + (r-n) = r+k < r+n < 2r-1$. So $\{X \# Y', S^r\} = [X \# Y', S^r]$. Let $a : S^r \to K(Z, r)$ represent the fundamental class of S^r, make it into a fibre map with fibre F. Then $[X \# Y', F] \to [X \# Y', S^r] \to [X \# Y', K(Z, r)] \to [S(X \# Y'), F]$ is exact. $\pi_i(F) = 0$ for $i \leq r$ and $\pi_{r+i}(F) = \Sigma_i(S^o)$ for $0 \leq i \leq k$.

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Using Theorem 2B we have $\lambda([X \# Y', F]) \mid \prod_{i=1}^{k} \lambda(\Sigma_i(S^o))$.

$$[X \# Y', K(Z, r)] = H^r(X \# Y') = \sum_{i=n}^{n+k} H^i(X; H_i(Y)).$$

Also $[S(X \# Y'), F]$ is finite for the same reasons that $[X \# Y', F]$ is finite. It is now easy to see that

$$[X \# Y', F] \to [X \# Y', S^r]_T \to \sum_{i=n}^{n+k} H^i(X; H_i(Y))_T$$

is exact and that $[X \# Y', S^r]_F = \sum_{i=n}^{n+k} H^i(X; H_i(Y))_F$ (although a_* need not induce this isomorphism). The theorem now follows from Lemma 6.

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