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INCLUSION THEOREMS FOR THE ABSOLUTE SUMMABILITY OF DIVERGENT INTEGRALS

BY

HARVEY DIAMOND, BRIAN KUTTNER AND LOUISE A. RAPHAEL*

ABSTRACT. Some inclusion theorems are obtained relating the absolute summability of divergent integrals of the form $\int_0^\infty f(x)\,dx$ under three summability methods: Abelian $A(x)$, Abelian $A(\ln x)$ and Stieltjes $S(x)$.

1. **Introduction.** If the application of a summability method to a divergent series (or integral) yields summability means of bounded variation in the summation parameter, then the series (or integral) is said to be absolutely summable. It is natural to ask which results of summability theory, in particular the inclusion theorems, hold in analogous form for absolute summability. In [6], for instance, D. Rath proved that a classical inclusion theorem for Abelian summability due to Hardy [2] remains true if summability is replaced throughout by absolute summability.

In this paper we obtain some inclusion theorems relating the absolute summability of divergent integrals of the form $\int_0^\infty f(x)\,dx$ under three summability methods: Abelian $A(x)$, Abelian $A(\ln x)$, and Stieltjes $S(x)$. Our results constitute an absolute summability analogue of two inclusion theorems appearing in [5]; these theorems are restated in Propositions 1 and 2 in the next section. We also provide examples which demonstrate proper inclusion. Such examples were lacking for the inclusion theorems of [5].

The Abelian methods $A(x)$ and $A(\ln x)$ are well known, employing the multipliers e^{-sx} and x^{-s} respectively, where s is the summation parameter tending to 0^+. The Stieltjes summability method being less well known, we will provide a brief background.†

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† The term "Stieltjes summability" has previously appeared in the physics literature [8], where it denotes a power series which can be expressed as the asymptotic expansion of a (convergent) Stieltjes integral having the form $\int_0^\infty f(x)\,dx/(1+sx)$. Our usage of the term is different.

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The name "Stieltjes" was introduced by Raphael in [5] to refer to summability methods having multipliers of the form $(1+s\lambda_n)^{-1}$ in the case of divergent sums (λ_n representing a sequence increasing to infinity with n), and $(1+sg(x))^{-1}$ in the case of divergent improper integrals ($g(x)$ increasing to infinity with x); these methods are denoted by $S(\lambda_n)$ and $S(g(x))$ respectively. In connection with sums, Stieltjes methods have been studied from time to time in the classical literature (see [5] for some references); the first systematic treatment for integrals is in [5]. Recently, Stieltjes summability has arisen naturally in the Tikhonov regularization of eigenfunction expansions associated with Sturm-Liouville equations, providing a stable method of summing such expansions if the coefficients are known only approximately. In particular, using the Tikhonov regularization method, it was proved in [7] that for regular Sturm-Liouville systems, the expansion of an L_2 function is summable $S(\lambda_n)$ at continuity points, where the λ_n are the eigenvalues; in [5] a class of singular Sturm-Liouville expansions on $[0,\infty)$ was shown to be summable $S(x)$ at continuity points to its corresponding L_2 functions (these expansions taking the form of improper integrals with respect to the spectral measure of the eigenvalues).

2. **Basic Definitions and Results.** The functions to be integrated are assumed Lebesgue measurable, locally integrable real valued functions defined on the half line $[0,\infty)$. The integral $\int_0^\infty f(x)\,dx$ means $\lim_{\alpha\to\infty}\int_0^\alpha f(x)\,dx$ provided the limit exists or is infinite. We write $f(x)\in BV(0,\infty)$ if f has bounded variation on $(0,\infty)$.

Definition 1. The integral $\int_0^\infty f$ is summable by the Abel method $A(x)$ to the sum L (written $\int_0^\infty f = L\ A(x)$) if $\phi(s)=\int_0^\infty f(x)\exp(-sx)\,dx$ converges for $s>0$ and $\lim_{s\to 0}\phi(s)=L$. The integral is said in addition to be *absolutely summable* if $\phi(s)\in BV(0,\infty)$ also holds. In this case we write $\int_0^\infty f = L\ |A(x)|$.

Definition 2. The integral $\int_0^\infty f$ is summable by the Abel method $A(\ln x)$ to the sum L (written $\int_0^\infty f = L\ A(\ln x)$) if $\phi(x)=\int_0^\infty f(x)x^{-s}\,dx$ converges for $s>0$ sufficiently small and $\lim_{s\to 0}\phi(s)=L$. The integral is said in addition to be *absolutely summable* if $\phi(s)\in BV(0,c)$ for some $c>0$. In this case we write $\int_0^\infty f = L\ |A(\ln x)|$.

Definition 3. The integral $\int_0^\infty f$ is summable by the Stieltjes method $S(x)$ to the sum L (written $\int_0^\infty f = L\ S(x)$) if $\phi(s)=\int_0^\infty f(x)(1+sx)^{-1}\,dx$ converges for $s>0$ and $\lim_{s\to 0}\phi(s)=L$. The integral is said in addition to be *absolutely summable* if $\phi(s)\in BV(0,\infty)$ also holds. In this case we write $\int_0^\infty f = L\ |S(x)|$.

Remark. The summability means $\phi(s)$, in Definitions 1, 2, and 3 above will be referred to as the Abel, Mellin, and Stieltjes means of f respectively.

The following inclusion theorems are from [5].

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PROPOSITION 1. *If* $\int_0^\infty f = L\ A(x)$ *and* $\int_0^\infty f(x)(1+sx)^{-1}\,dx$ *converges for* $s>0$ *then* $\int_0^\infty f = L\ S(x)$.

PROPOSITION 2. *If* $\int_0^\infty f = L\ S(x)$ *and* $\int_0^\infty f(x)x^{-s}\,dx$ *converges for* $s>0$ *sufficiently small then* $\int_0^\infty f(x)x^{-s}\,dx = L\ A(\ln x)$.

The next proposition contains identities which were used in the proofs of Propositions 1 and 2 and which will prove similarly useful later in this paper.

PROPOSITION 3. *If* $\int_0^\infty f(x)e^{-sx}\,dx$ *is bounded for* $s>0$ *and* $\int_0^\infty f(x)(1+sx)^{-1}\,dx$ *converges for* $s>0$, *then* $\int_0^\infty f(x)(1+sx)^{-1}\,dx = (1/s)\int_0^\infty \exp(-t/s)$ $\int_0^\infty f(x)\exp(-xt)\,dx\,dt$ *for* $s>0$. *If* $\int_0^\infty f(x)(1+sx)^{-1}\,dx$ *is bounded for* $s>0$ *and* $\int_0^\infty f(x)x^{-s}$ *converges for* $s>0$ *sufficiently small, then for* $s>0$ *sufficiently small we have*

$$\int_0^\infty f(x)x^{-s}\,dx = [(\sin \pi s)/\pi]\int_0^\infty t^{s-1}\int_0^\infty f(x)(1+tx)^{-1}\,dx\,dt.$$

REMARK. Proposition 3 shows that the Stieltjes means can be obtained directly from the iterated Laplace transform of f. This allows us to exploit the large body of Laplace Transform results and examples in studying the Stieltjes summability method.

The following Lemma is a slight restatement of a result due to Knopp [3].

LEMMA 1 (Knopp). *Suppose that*

1. $g(s)\in BV(0,\infty)$,
2. $\int_0^\infty h(b,s)\,ds$ *exists for* $b\in(0,c)$ *where* c *may be infinite,*
3. $\int_t^\infty h(b,s)\,ds\in BV(0,c)$ *uniformly for* $t\geq 0$.

Then $\phi(b)=\int_0^\infty h(b,s)g(s)\,ds\in BV(0,c)$.

Knopp's Lemma was used by Rath to prove the results in [6].

3. **Main Theorems.** The following two inclusion theorems for absolute summability are analogous to the inclusion results of Propositions 1 and 2.

THEOREM 1. *If* $\int_0^\infty f = L\ |A(x)|$ *and* $\int_0^\infty f(x)(1+sx)^{-1}\,dx$ *converges for* $s>0$ *then* $\int_0^\infty f = L\ |S(x)|$.

Proof. From Proposition 1, $\int_0^\infty f = L\ S(x)$ so that we need only show that $\int_0^\infty f(x)(1+sx)^{-1}\in BV(0,\infty)$. We show this using Knopp's Lemma. We have $\int_0^\infty f(x)(1+bx)^{-1}\,dx = (1/b)\int_0^\infty \exp(-s/b)\ \int_0^\infty f(x)\exp(-xs)\,dx\,ds$. By assumption $\int_0^\infty f(x)\exp(-sx)\,dx\in BV(0,\infty)$ so the first hypothesis of Knopp's Lemma is satisfied. For the second hypothesis, $(1/b)\int_0^\infty \exp(-s/b)\,ds=1$ for all $b>0$. Finally, for each t, $(1/b)\int_t^\infty \exp(-s/b)\,ds=\exp(-t/b)$ is positive, monotone increasing and bounded by 1, hence its total variation on $(0,\infty)$ is bounded by 1 uniformly in t. This proves the theorem.

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THEOREM 2. *If* $\int_0^\infty f = L\ |S(x)|$ *and* $\int_0^\infty f(x)x^{-s}\,dx$ *converges for* $0<s<c$ *then* $\int_0^\infty f = L\ |A(\ln x)|$ *and* $\int_0^\infty f(x)x^{-s}\,dx \in BV(0, c']$ *for any* $c'<c$.

Proof. Define $h(s)=\int_0^\infty f(x)x^{-s}\,dx$, $0<s<c$. By Proposition 3, $\int_0^\infty f = L$ $A(\ln x)$ so we need only show that $h(s)\in BV(0, c']$. Given any a, c' such that $0<a<c'<c$ it is easy to prove that $h(s)$ is infinitely differentiable for $s\in [a, c']$. This in turn implies the boundedness of $h'(s)$ on $[a, c']$ so that $h(s)\in BV[a, c']$. It remains then to show that $h(s)\in BV(0, a]$ for some $a<c$. In what follows, we let a be any number satisfying $4a<\min(1, c)$. From Proposition 3 we have

$$\int_0^\infty f(x)x^{-s}\,dx=[(\sin \pi s)/\pi]\int_0^\infty b^{s-1}\left[\int_0^\infty f(x)(1+bx)^{-1}\,dx\right]db$$
$$=[(\sin \pi s)/\pi]\int_0^\infty b^{s-1}(1+b)^{-2a}\left[(1+b)^{2a}\int_0^\infty f(x)\times(1+bx)^{-1}\,dx\right]db, \qquad 0<s\le a.$$

To apply Knopp's Lemma we must show first that $g(b)= (1+b)^{2a}\int_0^\infty f(x)(1+bx)^{-1}\,dx \in BV(0,\infty)$. By assumption, $\int_0^\infty f(x)(1+bx)^{-1}\,dx\in BV(0,\infty)$ so $g(b)\in BV(0, 1]$ is certainly true. We will show that $g(b)\in BV[1,\infty)$ by proving that $\int_0^\infty |g'(b)|\,db<\infty$. Differentiating,

$$g'(b)=2a(1+b)^{2a-1}\int_0^\infty f(x)(1+bx)^{-1}\,dx-(1+b)^{2a}\int_0^\infty xf(x)(1+bx)^{-2}\,dx \tag{1}$$

We rewrite the first integral as $\int_0^\infty f(x)x^{-3a}(x^{3a}/1+bx)\,dx$. Since $3a<c$, $\int_0^\infty f(x)x^{-3a}\,dx$ exists. We also have $x^{3a}/(1+bx)<1/b^{3a}$ and for each b, the function has a unique maximum. Applying Bonnet's second mean value theorem, we can obtain $\int_0^\infty f(x)(1+bx)^{-1}\,dx\le b^{-3a}\sup_{0<\alpha<\beta<\infty}|\int_\alpha^\beta f(x)x^{-3a}\,dx|< Mb^{-3a}$ for some constant M. The first term in (1) is then bounded in absolute value by $4aN/b^{1+a}$ for $b\ge 1$. The second integral in (1) may be treated in a similar fashion, so that we obtain $\int_0^\infty xf(x)(1+bx)^{-2}\,dx\le M/b^{1+3a}$ and can then bound the second term in (1) to be less than $2M/b^{1+a}$ in absolute value for $b\ge 1$. Finally, we have $|g'(b)|<3M/b^{1+a}$, proving that $\int_0^\infty |g'(b)|\,db<\infty$ and hence that $g(b)\in BV(0,\infty)$, using the arguments above.

To complete the proof using Knopp's theorem, we must show that $[(\sin \pi s)/\pi]\int_t^\infty b^{s-1}(1+b)^{-2a}\,db\in BV(0, a]$ uniformly for $t\ge 0$. Integrating by parts gives

$$\frac{\sin \pi s}{\pi}\int_t^\infty \frac{b^{s-1}}{(1+b)^{2a}}\,db=\frac{\sin \pi s}{\pi s}\frac{t^s}{(1+t)^{2a}}+2a\frac{\sin \pi s}{\pi s}\int_t^\infty \frac{b^s}{(1+b)^{1+2a}}\,db. \tag{2}$$

Clearly, $(\sin \pi s)/\pi s\in BV[0, a]$. Next, it is easily seen that $t^s/(1+t)^{2a}\le 1$ for $(s, t)\in[0, a]\times[0,\infty)$; and furthermore, for any fixed $t\in[0,\infty)$, the function is either non-increasing or non-decreasing in s. Thus the total variation of

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$t^s/(1+t)^{2a}$ is less than 1, uniformly in $t \geq 0$. We show next that the integral in (2) is of bounded variation on $[0, a]$ uniformly in t by showing that its derivative is uniformly bounded:

$$\left|\frac{d}{ds}\int_t^\infty \frac{b^s}{(1+b)^{1+2a}}\,db\right| \leq \int_t^\infty \frac{|\ln b|}{(1+b)^{1+a}}\frac{b^s}{(1+b)^a}\,db \leq \int_0^\infty \frac{|\ln b|}{(1+b)^{1+a}}\,db$$

where the last step follows from the bound $b^s/(1+b)^a \leq 1$ for $s \in [0, a]$ and all $b \geq 0$. The final integral exists and its value is of course the required bound. This completes the proof.

An immediate corollary of Theorems 1 and 2 is the following:

THEOREM 3. *If* $\int_0^\infty f = L\,|A(x)|$ *and* $\int_0^\infty f(x)x^{-s}\,dx$ *converges for* $0<s<c$ *then* $\int_0^\infty f = L\,|A(\ln x)|$ *and* $\int_0^\infty f(x)x^{-s}\,dx \in BV(0, c']$ *for any* $c'<c$.

Proof. The existence of $\int_0^\infty f(x)x^{-s}\,dx$ for $0<s<c$ implies the existence of the Stieltjes means for all $b>0$. Theorems 1 and 2 may then be applied successively to complete the proof.

Theorem 3 may be considered an integral analogue of Rath's result in [6]; or a second generation analogue of Hardy's result in [2].

4. **Examples.** The examples below are intended to help delineate the boundaries of the various summability classes. Some of our examples employ complex-valued functions to help simplify the analysis; real-valued examples may be obtained by considering real and imaginary parts of these functions.

(a) Summability $|A(x)|$ but not $S(x)$ or $A(\ln x)$: $f(x)=x\sin x$. The Stieltjes and Mellin means of this function do not exist; the Abel means may be obtained from a table of Laplace transforms and summability $|A(x)|$ verified.

(b) Summability $|A(x)|$ and $|S(x)|$ but not $A(\ln x)$: $f(x)=x^{1/2}e^{ix}$. From Laplace Transforms, $f(x)$ is $|A(x)|$ summable; the Stieltjes means of f exist so by Theorem 1 f is also $|S(x)|$ summable. However, the Mellin means of f do not exist.

(c) Summability $|A(\ln x)|$ but not summable $S(x)$ or $A(x)$: $f(x)=x^{ic}/(1+x)$ where c is a non-zero constant. This example is motivated by the example in [2] of an infinite series, $\sum_{n=1}^\infty n^{-1-ic}$, which is summable $A(\ln n)$ but not summable $A(n)$. To verify the properties, we note that the Mellin transform of f, obtained from tables in [1] is given by

$$\int_0^\infty \frac{x^{-s+ic}}{(1+x)}\,dx = \pi\csc[\pi(1-s+ic)], \qquad 0<s<1$$

and summability $|A(\ln x)|$ then follows easily. Using contour integration or Laplace transform tables, the Stieltjes means of f may be shown to be:

$$\int_0^\infty \frac{x^{ic}}{(1+x)(1+sx)}\,dx = \Gamma(1+ic)\Gamma(-ic)(1-s)^{-1}(1-s^{-ic}).$$

As $s \to 0$, the means do not approach a limit so $f(x)$ is not summable $S(x)$. That $f(x)$ is not summable $A(x)$ may be deduced either directly from the Laplace transform of f or by applying Theorem 1.

(d) Summability $|S(x)|$ but not $A(x)$. The analysis here is more difficult than the preceding examples and our function is given implicitly in terms of the inverse Laplace transform of a specific function. In what follows, we define

$$g(s) = \int_0^\infty f(x)e^{-sx}\,dx \tag{3}$$

$$h(t) = \int_0^\infty \frac{f(x)}{1+tx}\,dx. \tag{4}$$

The construction of our example proceeds as follows: We will choose (with $1<\beta<2$) $g(s)=\exp\{i[\log(s+1/s)]^\beta\}-1$ and show that there is an $f(x)$ satisfying (3) with $f \in L_2(0,\infty)$ through the use of Fourier Transform theory. With $f \in L_2$ the Stieltjes means of f exist and by Proposition 2, can be obtained from the Abel means of g, which are then analyzed to show that f is $|S(x)|$ summable. Since $g(s)$ has no limit as $s \to 0$, f is not $A(x)$ summable, finishing the example.

The following Lemma is essentially Theorem V from [4].

Lemma 2. *Write $s=\sigma+it$. Suppose that $g(s)$*

(a) *is analytic in the half-plane $\sigma>0$,*

(b) *satisfies* $\displaystyle\int_{-\infty}^{\infty} |g(\sigma+it)|^2\,dt<C$ *for* $\sigma>0$. (5)

with C a constant independent of σ. Then there is an $f(x)\in L_2(0,\infty)$ such that (3) *holds for all $\sigma>0$.*

Remark. Lemma 2 differs slightly from Theorem V in that we have replaced x by $-x$ and replaced L_2 convergence (denoted l.i.m. in [4]) by the ordinary convergence of the improper integral in (3) which holds when $\sigma>0$ because $f(x)\in L_2(0,\infty)$.

We now show that our example satisfies the conditions of Lemma 2.

Proposition 4. *Define* $g(s)=\exp\{i[\log((s+1)/s)]^\beta\}-1$, $1<\beta<2$. *Then hypotheses* (a) *and* (b) *of Lemma* 2 *are satisfied.*

Proof. It is clear that $g(s)$ is analytic for $\sigma=\mathrm{Re}(s)>0$. To verify condition (b) it suffices to study the behavior of g when s is "small" and when it is "large". More precisely, if r, R are any fixed constants with $0<r<R$ the contribution to the integral in (5) from that part of the range of integration for which $r<|s|<R$ is clearly bounded, so that we need consider only the contributions from $|s|\ge R$ and from $|s|\le r$.

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Consider the behavior of g for large s. Since $\log((s+1)/s) = \log(1+1/s) = 0(1/|s|)$, we have $g(s) = \exp\{i[0(1/|s|^\beta)]\} - 1 = 0(1/|s|^\beta)$. As $\beta > 1$, the contribution in (3) from $|s| \geq R$ is bounded.

Consider next the behavior of g when s is small. We have

$$\log\left(\frac{s+1}{s}\right) = \log\left|\frac{s+1}{s}\right| + i\arg\left(\frac{s+1}{s}\right) = \log\left|\frac{s+1}{s}\right| + 0(1) = \log(1/|s|) + 0(1)$$

so that

$$\left[\log\left(\frac{s+1}{s}\right)\right]^\beta = [\log(1|s|)]^\beta + 0[\log(1/|s|)]^{\beta-1}. \tag{6}$$

Now

$$\exp\left\{i\left[\log\left(\frac{s+1}{s}\right)\right]^\beta\right\}\Bigg| = \exp\left\{\operatorname{Re} i\left[\log\left(\frac{s+1}{s}\right)\right]^\beta\right\} = \exp\{0[\log(1/|s|)]^{\beta-1}\} \tag{7}$$

since the first term in (6) is real. As $\beta < 2$ we see that given any positive constants K and λ we have for sufficiently large x,

$$\exp K(\log x)^{\beta-1} \leq x^\lambda.$$

Thus, the last expression in (7) is seen to be $0(|s|^{-\lambda})$ and if we choose $\lambda < \frac{1}{2}$, the contribution to the integral in (5) from $|s| < r$ is bounded. This completes the proof that the hypotheses of Lemma 2 are satisfied by $g(s)$.

THEOREM 4. *Let* $g(s) = \exp\{i[\log((s+1)/s)]^\beta\} - 1$. *Then there exists an* $f(x)$ *in* $L_2(0,\infty)$ *satisfying* (3) *which is not* $A(x)$ *summable but is summable* $|S(x)|$.

Proof. The existence of an $f \in L_2(0,\infty)$ satisfying (3) follows because g satisfies the hypotheses of Lemma 2. Further, f is not $A(x)$ summable because $g(s)$ has no limit as $s \to 0$. Since $f \in L_2(0,\infty)$, the Stieltjes means of f exist, and as $g(s)$ is bounded for s real and positive, Proposition 3 furnishes the Stieltjes means in terms of g:

$$h(t) = (1/t)\int_0^\infty \exp(-s/t)g(s)\,ds. \tag{8}$$

We must show $h(t) \in BV(0,\infty)$.

We write g as a sum of three functions, $g_i(s)$, $i = 1, 2, 3$ which are defined by:

$$g_1(s) = \begin{cases} 0, & 0 < s \leq \frac{1}{2} \\ g(s), & s > \frac{1}{2} \end{cases}$$

$$g_2(s) = \begin{cases} \exp i[\log(1/s)]^\beta, & 0 < s \leq \frac{1}{2} \\ 0, & s > \frac{1}{2} \end{cases}$$

$$g_3(s) = \begin{cases} g(s) - g_2(s), & 0 < s \leq \frac{1}{2} \\ 0, & s > \frac{1}{2} \end{cases}$$

Let $h_i(t)$, $i=1,2,3$ denote the function obtained by replacing $g(s)$ by $g_i(s)$ in the integral (8). We will prove that, for $i=1,2,3$ we have $h_i(t)\in BV(0,\infty)$. In the case of $i=1,3$ we do this by proving that $g_i(s)\in BV(0,\infty)$, and the conclusion that $h_i(t)\in BV(0,\infty)$ then follows from the application of Knopp's Lemma as in the proof of Theorem 1.

Consider then $g_1(s)$. For $s>\frac{1}{2}$, $g(s)$ has a continuous derivative given by

$$g'(s)=-\frac{i\beta}{s(s+1)}\left(\log\frac{s+1}{s}\right)^{\beta-1}\exp i\left[\log\left(\frac{s+1}{s}\right)^{\beta}\right]$$

so that

$$|g'(s)|=\frac{\beta}{s(s+1)}\left(\log\frac{s+1}{s}\right)^{\beta-1}.$$

The variation of $g_1(s)$ in $(0,\infty)$ is equal to

$$g(\tfrac{1}{2})+\int_{1/2}^{\infty}|g'(s)|\,ds<\infty$$

as required.

Next, we consider $g_3(s)$. The function represents the difference between g and its asymptotic behavior, $g_2(s)$, when s is small. As in the case of g_1, we show g_3 has bounded variation by showing that the integral of the absolute value of g_3' is finite. We calculate

$$g_3'(s)=\frac{i\beta}{s}\left[-\frac{1}{s+1}\left(\log\frac{s+1}{s}\right)^{\beta-1}\exp i\left(\log\frac{s+1}{s}\right)^{\beta}+\left(\log\frac{1}{s}\right)^{\beta-1}\exp i\left(\log\frac{1}{s}\right)^{\beta}\right] \tag{9}$$

Now, uniformly in $0<s\leq\frac{1}{2}$, we have

$$\log\frac{s+1}{s}=\log\frac{1}{s}+\log(1+s)=\log\frac{1}{s}+0(s)$$

whence,

$$\left[\log\left(\frac{s+1}{s}\right)\right]^{\beta-1}=\left(\log\frac{1}{s}\right)^{\beta-1}+0\left[s\left(\log\frac{1}{s}\right)^{\beta-2}\right].$$

Also,

$$\left(\log\frac{s+1}{s}\right)^{\beta}=\left(\log\frac{1}{s}\right)^{\beta}+0\left[s\left(\log\frac{1}{s}\right)^{\beta-1}\right]$$

whence

$$\exp i\left(\log\frac{s+1}{s}\right)^{\beta}=\exp i\left(\log\frac{1}{s}\right)^{\beta}\exp\left[0\left(s\left(\log\frac{1}{s}\right)^{\beta-1}\right)\right]$$

$$=\exp i\left(\log\frac{1}{s}\right)^{\beta}\left[1+0\left(s\left(\log\frac{1}{s}\right)^{\beta-1}\right)\right].$$

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Substituting these results into (9), we obtain

$$g_3'(s)=\frac{i\beta}{s}\left(\log\frac{1}{s}\right)^{\beta-1}\left\{\exp i\left(\log\frac{1}{s}\right)^{\beta}\right\}\left[-(1+0(s))\left(1+0\left[\frac{s}{\log(1/s)}\right]\right)\right.$$
$$\left.\times\left(1+0\left(s\left(\log\frac{1}{s}\right)^{\beta-1}\right)\right)+1\right]$$
$$=0\left[\left(\log\frac{1}{s}\right)^{2\beta-2}\right]$$

and we now deduce that $g_3(s)\in BV(0,\infty)$.

It remains to show that $h_2(t)$ has bounded variation. The integral which defines h_2 is:

$$h_2(t)=(1/t)\int_0^{1/2}\exp(-s/t)\exp i\left(\log\frac{1}{s}\right)^{\beta}ds.$$

The analysis requires integration by parts a number of times which results in integrals of the form

$$I(n,\alpha;t)\equiv\frac{1}{t^{n+1}}\int_0^{1/2}s^n\left(\log\frac{1}{s}\right)^{\alpha}\exp i\left(\log\frac{1}{s}\right)^{\beta}\exp\left(-\frac{s}{t}\right)ds,$$

where here, and below n denotes a non-negative integer. In the proof of Theorem 1, Knopp's Lemma was used to show that if $f(s)\in BV(0,\infty)$ then

$$\frac{1}{t}\int_0^{\infty}\exp\left(-\frac{s}{t}\right)f(s)\,ds\in BV(0,\infty).$$

A similar argument shows that, more generally,

$$\frac{1}{t^{n+1}}\int_0^{\infty}s^n\exp\left(-\frac{s}{t}\right)f(s)\,ds\in BV(0,\infty).$$

Applying this fact with $f(s)$ replaced by $(\log(1/s))^{\alpha}\exp i(\log(1/s))^{\beta}$ for $s\le\frac{1}{2}$ and 0 for $s>\frac{1}{2}$ we obtain the result that for $n\ge 0$,

$$I(n,\alpha;t)\in BV(0,\infty)\quad\text{if}\quad\alpha<-\beta \tag{10}$$

since $|f'(s)|$ is integrable in that case.

In the integration by parts which follows, it is easy to verify that the boundary terms are of bounded variation in $(0,\infty)$. We then obtain the following recursion relation for $I(n,\alpha;t)$ (C denotes a constant and $A(t)$ the boundary terms):

$$I(n,\alpha;t)=\frac{C}{t^{n+1}}\int_0^{1/2}s^{n+1}\left(\log\frac{1}{s}\right)^{\alpha-\beta+1}\exp\left(-\frac{s}{t}\right)d_s\left[\exp i\left(\log\frac{1}{s}\right)^{\beta}\right]$$
$$=A(t)+\frac{C}{t^{n+1}}\int_0^{1/2}d_s\left[s^{n+1}\left(\log\frac{1}{s}\right)^{\alpha-\beta+1}\exp\left(-\frac{s}{t}\right)\right]\exp i\left(\log\frac{1}{s}\right)^{\beta}$$
$$=A(t)+CI(n,\alpha-\beta+1;t)+CI(n,\alpha-\beta;t)$$
$$+CI(n+1,\alpha-\beta+1;t). \tag{11}$$

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Now, since $\beta > 1$, we note that in each of the last three terms of (11), α is replaced by a quantity which is less than α by an amount at least equal to $\beta - 1$. Thus, starting with $h_2(t) = I(0, 0; t)$ and repeatedly applying (11), we eventually express $h_2(t)$ as a sum of boundary terms $A(t)$ plus terms of the form $CI(n, \alpha; t)$ where the values of α are less than $-\beta$. Using (10), it follows that $h_2(t) \in BV(0, \infty)$ and the proof is complete.

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Added in proof. The Stieltjes summability method has also been applied to the study of elliptic operators under the name resolvent summability by D. Gurarie and M. Kon, *Radial Bounds for Perturbations of Elliptic Operators*, to appear in Journal of Functional Analysis.

BIBLIOGRAPHY

1. A. Erdelyi, W. Magnus, F. Oberkettinger, F. G. Tricomi, Tables of Integral Transforms, Vol. **2** (McGraw Hill, 1954).

2. G. H. Hardy, Divergent Series (Oxford 1949) pp. 73 ff.

3. K. Knopp, *Norlund Method for Functions*, Math Z, **63** (1955), pp. 39–52.

4. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (AMS 1934) pp. 8 ff.

5. L. A. Raphael, *The Stieltjes Summability Method and Summing Sturm-Liouville Expansions*, SIAM Journal on Mathematical Analysis, **13** (1982), pp. 676–689.

6. D. Rath, *An Inclusion Theorem on Summability*, J. London Math Soc. (2), **16** (1977), pp. 493–489.

7. A. N. Tikhonov, *Stable Methods for the Summation of Fourier Series*, Soviet Math. Dokl. **5** (1964), pp. 641–644.

8. S. Graffi, *Stieltjes Summability and Convergence of the Pade Approximants for the Vacuum Polarization by an External Field*, J. Math. Phys., **14** (1973), pp. 1184–1186.

DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA 26506

DEPARTMENT OF PURE MATHEMATICS
THE UNIVERSITY OF BIRMINGHAM
BIRMINGHAM B15 2TT ENGLAND

DEPARTMENT OF MATHEMATICS
HOWARD UNIVERSITY
WASHINGTON, D.C. 20059