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MAXIMAL ARCS AND GROUP DIVISIBLE DESIGNS

ALAN RAHILLY

The existence of maximal arcs of a certain type in symmetric designs is shown to yield semiregular group divisible designs whose duals are also semiregular group divisible. Two infinite families of such group divisible designs are constructed. The group divisible designs in these families are, in general, not symmetric.

1. INTRODUCTION

Group divisible designs whose duals are also group divisible have received some attention (see [**1**, 4]). Under the assumption that repeated points and repeated blocks are not permitted, Mitchell [4] has shown that, if $\mathcal{G}$ and its dual $\mathcal{G}^d$ are both group divisible, then $\mathcal{G}$ and $\mathcal{G}^d$ are each semiregular or $\mathcal{G}$ is symmetric and $\mathcal{G}$ and $\mathcal{G}^d$ are each regular. A range of examples of symmetric, regular group divisible designs whose duals are also group divisible is known. As examples we cite the substructures complementary to Baer subdesigns of symmetric designs [**1**, pp.95–96] and also finite proper uniform projective Hjelmslev planes [**3**, p.294]. On the other hand, non-symmetric, semiregular, group divisible designs whose duals are also group divisible seem to be rare. In this paper we use maximal arcs of a certain type in symmetric designs to construct two infinite families of semiregular, group divisible designs whose duals are also group divisible. In general, the semiregular group divisible designs we obtain are not symmetric. We also investigate briefly working our construction method for semiregular group divisible designs in reverse to obtain symmetric designs.

Before proceeding, some remarks on the terminology and notation we shall use are in order. First, we treat designs and group divisible designs as incidence structures in the manner of [**3**]. Our usage of the term "design", without "group divisible" attached, is that of [**3**], Chapter 2. Our basic terminology and notation concerning group divisible designs is that of [**2**].

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2. Maximal arcs

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a (v, k, λ)-design, $\mathcal{P}_1 \subseteq \mathcal{P}$ and $\mathcal{P} \neq \mathcal{P}_1 \neq \emptyset$. $\mathcal{P}_1$ is said to be a *maximal u-arc* of $\mathcal{D}$ if any block of $\mathcal{B}$ is incident with precisely u points of $\mathcal{P}_1$ or with no points of $\mathcal{P}_1$. If $\mathcal{B}_1$ is the set of blocks of $\mathcal{D}$ incident with u points of $\mathcal{P}_1$ and $\mathcal{P}_2$ and $\mathcal{B}_2$ are the complements of $\mathcal{P}_1$ and $\mathcal{B}_1$ respectively, then $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{B}_1$, and $\mathcal{B}_2$ form a tactical decomposition [**3**, pp.7 and 17] with incidence matrices

$$C = \begin{bmatrix} u & 0 \\ k-u & k \end{bmatrix}$$

and

$$D = \begin{bmatrix} k & k - \frac{k-\lambda}{u} \\ 0 & \frac{k-\lambda}{u} \end{bmatrix}.$$

$$|\mathcal{P}_1| = \frac{k(u-1)}{\lambda} + 1,$$

$$|\mathcal{P}_2| = \frac{k(k-u)}{\lambda},$$

Also

$$|\mathcal{B}_1| = \frac{k}{u}|\mathcal{P}_1|,$$

and

$$|\mathcal{B}_2| = \frac{(k-\lambda)(k-u)}{\lambda u}$$

and the substructure $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, \mathcal{I}_{11})$ of $\mathcal{D}$ defined by $\mathcal{P}_1$ and $\mathcal{B}_1$ ([**3**, p.2]) is an $(|\mathcal{P}_1|, |\mathcal{B}_1|, k, u, \lambda)$-design, provided $u > 1$. Furthermore, $\mathcal{B}_2$ is a maximal $(k-\lambda)/u$-arc in the dual $\mathcal{D}^d$ of $\mathcal{D}$ which we call the *dual* maximal arc of $\mathcal{P}_1$. In this situation the substructure $\mathcal{D}_2 = (\mathcal{B}_2, \mathcal{P}_2, \mathcal{I}^d_{22})$ of $\mathcal{D}^d$ defined by $\mathcal{B}_2$ and $\mathcal{P}_2$ is an $(|\mathcal{B}_2|, |\mathcal{P}_2|, k, (k-\lambda)/u, \lambda)$-design, provided that $k - \lambda > u$. We refer to $\mathcal{D}_1$ and $\mathcal{D}_2$ as the (sub)designs "associated with" $\mathcal{P}_1$ and $\mathcal{B}_2$ respectively. These results concerning maximal u-arcs are straightforward to establish. For some of the details see [**5**, pp.8–9].

Note that for a (v, k, λ)-design to possess a maximal u-arc $\mathcal{P}_1$ we must have

$$u \mid k - \lambda \quad \text{and} \tag{1}$$

$$\lambda \mid k(u-1). \tag{2}$$

Next, let $\mathcal{P}_1$ be a maximal u-arc of a (v, k, λ)-design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$. Suppose the design $\mathcal{D}_1$ associated with $\mathcal{P}_1$ is an n-multiple of an $(|\mathcal{P}_1|, u, \lambda_1)$-design. Since $|\mathcal{B}_1| = (k/u)|\mathcal{P}_1|$ we have $k = un$. Equally clearly we have $\lambda = \lambda_1 n$. Using (1) we can infer that $u \mid \lambda$, whence $\lambda = ut$ for some positive integer t. Then, from (2), we infer that $t \mid n(u-1)$. But $v = (nu(nu-1))/(tu) + 1 = \big(n^2(u-1)\big)/t + (n(n-1))/t + 1$ and so we also have $t \mid n(n-1)$.

Clearly $t = ac$, where $a \mid n$, $c \mid n-1$, $a \geqslant 1$ and $c \geqslant 1$. Obviously a and c are relatively prime. But $t \mid n(u-1)$ and so we have $c \mid u-1$. Let $n = \alpha a = \beta c + 1$ and

$u = \gamma c + 1$. At this stage we have that $\mathcal{D}$ is an

$$(\alpha\gamma(\beta c + 1) + \alpha\beta + 1, \alpha a(\gamma c + 1), ac(\gamma c + 1))\text{-design}$$

and $\mathcal{D}_1$ is an (αa)-multiple of a $(\gamma\alpha + 1, \gamma c + 1, c(\gamma c + 1)/\alpha)$-design. Readily we also have that the subdesign $\mathcal{D}_2$ associated with the dual maximal are $\mathcal{B}_2$ of $\mathcal{P}_1$ is an

$$((\alpha - c)\beta, \alpha\beta(\gamma c + 1), \alpha a(\gamma c + 1), a(\alpha - c), ac(\gamma c + 1))\text{-design.}$$

Now suppose that $\mathcal{D}_2$ is also a multiple of a symmetric design. Immediately we have that $\mathcal{D}_2$ is an $(\alpha(\gamma c + 1)/(\alpha - c))$-multiple of an $((\alpha - c)\beta, a(\alpha - c), ac(\alpha - c)/\alpha)$-design. At this point we conclude that $\alpha \mid c(\gamma c + 1)$, $\alpha \mid ac(\alpha - c)$ and $\alpha - c \mid \alpha(\gamma c + 1)$. But $\alpha a = \beta c + 1$, whence α and c are relatively prime, and so we have $\alpha \mid \gamma c + 1$, $\alpha \mid a$ and $\alpha - c \mid \gamma c + 1 = \gamma(c - \alpha) + \gamma\alpha + 1$. It follows that $\alpha - c \mid \gamma\alpha + 1$ also. Let $a = \tau\alpha$, $\gamma\alpha + 1 = \rho(\alpha - c)$ and $\gamma c + 1 = \pi\alpha(\alpha - c)$. Clearly we have $\rho - \gamma = \pi\alpha$. Using this and $\gamma\alpha + 1 = \rho(\alpha - c)$ we can solve for γ and ρ to obtain

$$\rho = \frac{\pi\alpha^2 - 1}{c}$$

and

$$\gamma = \frac{\pi\alpha(\alpha - c) - 1}{c}.$$

Since $\beta c + 1 = \alpha a = \tau\alpha^2$ we also have

$$\beta = \frac{\tau\alpha^2 - 1}{c}.$$

Expressing the parameters of $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ in terms of π, τ, α and c we obtain that

(i) $\mathcal{D}$ is a $\mathrm{Sym}(\pi\tau\alpha^2, \alpha, c)$,
(ii) $\mathcal{D}_1$ is a $(\tau\alpha^2)$-multiple of a $\mathrm{Sym}(\pi, \alpha, c)$, and
(iii) $\mathcal{D}_2$ is a $(\pi\alpha^2)$-multiple of a $\mathrm{Sym}(\tau, \alpha, c)$,

where, for any positive integers x, y and z such that $z \mid xy^2 - 1$, we call an $(((xy^2 - 1)/z)(y - z), xy(y - z), xz(y - z))$-design a $\mathrm{Sym}(x, y, z)$.

3. Construction of group divisible designs with group divisible duals

In this and the next section we shall refer to a group divisible design with the parameters

$$v = \xi y^2\left(\frac{\eta y^2 - 1}{z}\right)(y - z), \qquad b = \eta y^2\left(\frac{\xi y^2 - 1}{z}\right)(y - z), \qquad r = \eta y(\xi y^2 - 1)(y - z),$$

$$k = \xi y(\eta y^2 - 1)(y - z), \qquad \lambda_1 = \eta y(\xi yz - 1)(y - z), \qquad \lambda_2 = \eta z(\xi y^2 - 1)(y - z),$$

$$m = \frac{\eta y^2 - 1}{z}(y - z) \qquad \text{and } n = \xi y^2,$$

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where z divides each of $\xi y^2 - 1$ and $\eta y^2 - 1$, as a $\mathrm{GDD}(\xi, \eta, y, z)$. Clearly v, b et cetera as just given in terms of ξ, η, y and z, satisfy the basic group divisible design equations $vr = bk$ and $v = mn$. Straightforward verification shows that $(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1)$ is also satisfied. We note that a $\mathrm{GDD}(\xi, \eta, y, z)$ is a semiregular group divisible design.

THEOREM 1. *Suppose $\mathcal{D}$ is a $\mathrm{Sym}(\pi\tau\alpha^2, \alpha, c)$ possessing a maximal arc $\mathcal{P}_1$ whose associated design $\mathcal{D}_1$ is a $(\tau\alpha^2)$-multiple of a $\mathrm{Sym}(\pi, \alpha, c)$. Suppose further that the associated design $\mathcal{D}_2$ of the dual maximal arc $\mathcal{B}_2$ of $\mathcal{P}_1$ is a $(\pi\alpha^2)$-multiple of a $\mathrm{Sym}(\tau, \alpha, c)$. Then the substructure $\mathcal{G}$ of $\mathcal{D}$ defined by $\mathcal{P}_1^c$ and $\mathcal{B}_2^c$ is a $\mathrm{GDD}(\pi, \tau, \alpha, c)$ whose dual $\mathcal{G}^d$ is a $\mathrm{GDD}(\tau, \pi, \alpha, c)$.*

PROOF: It is sufficient to show that $\mathcal{G}^d$ is a $\mathrm{GDD}(\tau, \pi, \alpha, c)$ for then the fact that $\mathcal{G}$ is a $\mathrm{GDD}(\pi, \tau, \alpha, c)$ follows by duality.

Consider a block B_1 of $\mathcal{B}_1 = \mathcal{B}_2^c$. There are $\tau\alpha^2 - 1$ other blocks of $\mathcal{B}_1$ which meet $\mathcal{P}_1$ in precisely the same points as B_1. The group of $\mathcal{G}^d$ containing B_1 consists of these $\tau\alpha^2 - 1$ blocks along with B_1. The verification that $\mathcal{G}^d$ is a $\mathrm{GDD}(\tau, \pi, \alpha, c)$ is straightforward. We offer the following specimen of this verification and leave the rest to the reader: Every pair of points of $\mathcal{G}^d$ in the same group are on $\pi\tau\alpha^2 c(\alpha - c) - \pi\alpha(\alpha - c)$ ($=$ index of $\mathcal{D}-$ blocksize of $\mathcal{D}_1$) $= \pi\alpha(\tau\alpha c - 1)(\alpha - c)$ blocks of $\mathcal{G}^d$. ∎

For a $\mathrm{GDD}(\xi, \eta, y, z)$ we must have $y > z > 0$. Thus the smallest values for y and z we can have are $y = 2$ and $z = 1$. We note that a $\mathrm{Sym}(x, 2, 1)$ is the complement of a Hadamard design of order x.

In the following discussion the row and column consisting entirely of ones of a normalised Hadamard matrix are the first row and the first column. Suppose H_i is a normalised Hadamard matrix of order $4n_i$, for $i = 1, 2$. Let H be the Kronecker product of H_2 and H_1, obtained by replacing each 1 in H_2 by H_1 and each -1 in H_2 by $-H_1$. Deleting the first row and column of H and converting the 1 (respectively, -1) entries of the matrix so obtained to 0 (respectively, 1) we obtain the incidence matrix A of a design $\mathcal{H}^c$ which is the complement of a Hadamard design $\mathcal{H}$ of order $4n_1n_2$. The points corresponding to the first $4n_1 - 1$ rows of A are easily shown to be a maximal $(2n_1)$-arc $\mathcal{P}_1$ of $\mathcal{H}^c$ whose associated design is a $(4n_2)$-multiple of $\mathcal{H}_1^c$, where $\mathcal{H}_1$ is the Hadamard design obtainable from H_1. The exterior blocks of $\mathcal{P}_1$ are those corresponding to columns $4in_1 + 1$, for $i = 1, \ldots, 4n_2 - 1$. Careful scrutiny of A reveals that the subdesign of $\mathcal{H}^c$ associated with the dual maximal arc of $\mathcal{P}_1$ is a $(4n_1)$-multiple of the dual of $\mathcal{H}_2^c$, where $\mathcal{H}_2$ is the Hadamard design obtainable from H_2. We thus have a maximal arc (namely $\mathcal{P}_1$) in the $\mathrm{Sym}(4n_1n_2, 2, 1)$ $\mathcal{H}^c$ whose associated subdesign is a $(4n_2)$-multiple of a $\mathrm{Sym}(n_1, 2, 1)$. Further, the associated subdesign of the dual maximal arc is a $(4n_1)$-multiple of a $\mathrm{Sym}(n_2, 2, 1)$. Applying Theorem 1 we obtain

THEOREM 2. *If there are Hadamard designs of order n_1 and n_2, then there is a* $\mathrm{GDD}(n_1, n_2, 2, 1)$ *whose dual is a* $\mathrm{GDD}(n_2, n_1, 2, 1)$*. This group divisible design is symmetric if and only if $n_1 = n_2$.*

We now turn to a second construction of semiregular group divisible designs with semiregular group divisible duals.

Let d and e be integers such that $1 \leqslant e \leqslant d-2$. Consider the symmetric design $\mathcal{D} = PG_{d-1}(d,q)$ formed by the points and hyperplanes of the d-dimensional projective geometry $PG(d,q)$ over $GF(q)$. Let U be an e-dimensional subspace of $PG(d,q)$. Hyperplanes of $PG(d,q)$ meet U in an $(e-1)$-dimensional subspace or contain U. It follows that U is a maximal arc of the $\left(\sum_{i=0}^{d} q^i, q^d, q^d - q^{d-1}\right)$-design $\mathcal{D}^c$. Furthermore, the subdesign of $\mathcal{D}^c$ associated with U is a (q^{d-e})-multiple of a $\left(\sum_{i=0}^{e} q^i, q^e, q^e - q^{e-1}\right)$-design. Now the hyperplanes of $PG(d,q)$ containing U form a $(d-e-1)$-dimensional subspace in the dual space. So the design associated with the dual maximal arc of U is a (q^{e+1})-multiple of a $\left(\sum_{i=0}^{d-e-1} q^i, q^{d-e-1}, q^{d-e-1} - q^{d-e-2}\right)$-design. Since a $\left(\sum_{i=0}^{s} q^i, q^s, q^s - q^{s-1}\right)$-design is a $\mathrm{Sym}(q^{s-1}, q, q-1)$ we obtain the following result, upon applying Theorem 1.

THEOREM 3. *For all prime powers q and integers e and d such that $1 \leqslant e \leqslant d-2$, there is a* $\mathrm{GDD}(q^{e-1}, q^{d-e-2}, q, q-1)$ *whose dual is a* $\mathrm{GDD}(q^{d-e-2}, q^{e-1}, q, q-1)$*. These designs are symmetric if and only if $d = 2e+1$.*

To complete this section we note that the complement of a semiregular group divisible design whose dual is semiregular group divisible is also semiregular group divisible with semiregular group divisible dual.

4. A CONSTRUCTION FOR SYMMETRIC DESIGNS

In Theorem 1 we showed that, if there is a $\mathrm{Sym}(\pi\tau\alpha^2, \alpha, c)$ with a maximal arc $\mathcal{P}_1$ whose associated design is a $(\tau\alpha^2)$-multiple of a $\mathrm{Sym}(\pi, \alpha, c)$ and for which the associated design of the dual maximal arc of $\mathcal{P}_1$ is a $(\pi\alpha^2)$-multiple of a $\mathrm{Sym}(\tau, \alpha, c)$, then there exists a $\mathrm{GDD}(\pi, \tau, \alpha, c)$ whose dual is a $\mathrm{GDD}(\tau, \pi, \alpha, c)$. We also have the following converse result.

THEOREM 4. *If there is a* $\mathrm{Sym}(\pi, \alpha, c)$*, a* $\mathrm{Sym}(\tau, \alpha, c)$ *and a* $\mathrm{GDD}(\pi, \tau, \alpha, c)$ *whose dual is a* $\mathrm{GDD}(\tau, \pi, \alpha, c)$*, then there is a* $\mathrm{Sym}(\pi\tau\alpha^2, \alpha, c)$*, $\mathcal{D}$, such that*

(a) *$\mathcal{D}$ possesses a maximal arc $\mathcal{P}_1$ whose associated subdesign is a* $(\tau\alpha^2)$-*multiple of a* $\mathrm{Sym}(\pi, \alpha, c)$ *and*

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(b) *the dual maximal arc of* $\mathcal{P}_1$ *has associated subdesign which is a* $(\pi\alpha^2)$-*multiple of a* $\mathrm{Sym}(\tau,\alpha,c)$.

PROOF: Let $\mathcal{G} = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ be a $\mathrm{GDD}(\pi,\tau,\alpha,c)$ such that $\mathcal{G}^d$ is a $\mathrm{GDD}(\tau,\pi,\alpha,c)$. Let the groups of $\mathcal{G}$ be $\{P_{ij} \mid j = 1,\ldots,\pi\alpha^2\}$, $i = 1,\ldots,\left((\tau\alpha^2-1)/c\right)(\alpha-c)$, and the groups of $\mathcal{G}^d$ be $\{B_{ij} \mid j = 1,\ldots,\tau\alpha^2\}$, for $i = 1,\ldots,\left((\pi\alpha^2-1)/c\right)(\alpha-c)$. Choose disjoint sets $\mathcal{P}_\pi$ and $\mathcal{B}_\tau$ such that $|\mathcal{P}_\pi| = \left((\pi\alpha^2-1)/c\right)(\alpha-c)$ and $|\mathcal{B}_\tau| = \left((\tau\alpha^2-1)/c\right)(\alpha-c)$ and which are each disjoint from $\mathcal{P}'$ and $\mathcal{B}'$. Construct a $(\tau\alpha^2)$-multiple $\mathcal{D}_\pi = (\mathcal{P}_\pi, \mathcal{B}', \mathcal{I}_\pi)$ of a $\mathrm{Sym}(\pi,\alpha,c)$ such that, for each fixed i, $(B_{ij})_\pi = (B_{ih})_\pi$ for all j, h, where $(X)_\pi$ denotes the set of points incident in $\mathcal{D}_\pi$ with block X. Similarly, construct a $(\pi\alpha^2)$-multiple $\mathcal{D}_\tau = (\mathcal{B}_\tau, \mathcal{P}', \mathcal{I}_\tau)$ of a $\mathrm{Sym}(\tau,\alpha,c)$ such that, for each fixed i, $(P_{ij})_\tau = (P_{ih})_\tau$ for all j, h. Then let $\mathcal{D} = \left(\mathcal{P}_\pi \cup \mathcal{P}', \mathcal{B}_\tau \cup \mathcal{B}', \mathcal{I}_\pi \cup \mathcal{I}_\tau^d \cup \mathcal{I}'\right)$, where $\mathcal{I}_\tau^d$ is defined by $(P,B) \in \mathcal{I}_\tau$ if and only if $(B,P) \in \mathcal{I}_\tau^d$.

$\mathcal{D}$ is easily shown to be symmetric. We show that $\mathcal{D}$ is a $\mathrm{Sym}(\pi\tau\alpha^2,\alpha,c)$ by showing that $\mathcal{D}$ has blocksize $\pi\tau\alpha^3 c(\alpha-c)$ and each pair of blocks of $\mathcal{D}$ meet in $\pi\tau\alpha^2 c(\alpha-c)$ points.

From the construction of $\mathcal{D}$ a block of $\mathcal{B}'$ has $\pi\alpha(\tau\alpha^2-1)(\alpha-c) + \pi\alpha(\alpha-c)$ points on it in $\mathcal{D}$. Also the set of points on a block of $\mathcal{B}_\tau$ is the union of $\tau\alpha(\alpha-c)$ groups of $\mathcal{G}$. Thus such a block of $\mathcal{D}$ has blocksize $\pi\alpha^2 \times \tau\alpha(\alpha-c)$.

Two blocks in the same group of $\mathcal{G}^d$ have $\pi\alpha(\tau\alpha c-1)(\alpha-c) + \pi\alpha(\alpha-c)$ points of $\mathcal{D}$ in common and blocks from different groups of $\mathcal{G}^d$ have $\pi c(\tau\alpha^2-1)(\alpha-c) + \pi c(\alpha-c)$ points of $\mathcal{D}$ in common. The set of points common to two blocks of $\mathcal{D}$ in $\mathcal{B}_\tau$ is the union of $\tau c(\alpha-c)$ groups of $\mathcal{G}$. Since $\mathcal{G}$ has $\pi\alpha^2$ points in each group the set of points common to two such blocks of $\mathcal{D}$ contains $\pi\alpha^2 \times \tau c(\alpha-c)$ points of $\mathcal{D}$. Finally, consider two blocks of $\mathcal{D}$, one from $\mathcal{B}'$ and one from $\mathcal{B}_\tau$. Since $\mathcal{G}$ is semiregular each of the $\left((\tau\alpha^2-1)/c\right)(\alpha-c)$ groups of $\mathcal{G}$ meets each block of $\mathcal{G}$ in the same number of points ([**2**, p.373]), namely $\pi\alpha c$, since the blocksize of $\mathcal{G}$ is $\pi\alpha(\tau\alpha^2-1)(\alpha-c)$. But the set of points incident with a block of $\mathcal{D}$ in $\mathcal{B}_\tau$ is the union of $\tau\alpha(\alpha-c)$ groups of $\mathcal{G}$. So the block intersection number for a block of $\mathcal{B}'$ and a block of $\mathcal{B}_\tau$ is $\tau\alpha(\alpha-c)\times\pi\alpha c$.

From the construction of $\mathcal{D}$ it follows that $\mathcal{P}_1 = \mathcal{P}_\pi$ is a maximal arc of $\mathcal{D}$ with $\mathcal{B}_\tau$ as its dual maximal arc, and that the associated subdesigns of these maximal arcs are $\mathcal{D}_\pi$ and $\mathcal{D}_\tau$, respectively. ∎

Effectively, in the construction of the $\mathrm{Sym}(\pi\tau\alpha^2,\alpha,c)\mathcal{D}$ in the proof of Theorem 4 we utilised a one-to-one correspondence between the set of blocks of a $\mathrm{Sym}(\pi,\alpha,c)$ and the set of groups of a $\mathrm{GDD}(\tau,\pi,\alpha,c)$. A one-to-one correspondence between the sets of blocks of a $\mathrm{Sym}(\tau,\alpha,c)$ and the set of groups of a $\mathrm{GDD}(\pi,\tau,\alpha,c)$ was also utilised. Since these one-to-one correspondences can be arbitrarily selected there are

$$\left(\left(\frac{\pi\alpha^2-1}{c}\right)(\alpha-c)\right)!\left(\left(\frac{\tau\alpha^2-1}{c}\right)(\alpha-c)\right)!$$

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$\mathrm{Sym}(\pi\tau\alpha^2, \alpha, c)$'s obtainable using the construction, upon choosing a $\mathrm{Sym}(\pi, \alpha, c)$, a $\mathrm{Sym}(\tau, \alpha, c)$ and a $\mathrm{GDD}(\pi, \tau, \alpha, c)$ whose dual is a $\mathrm{GDD}(\tau, \pi, \alpha, c)$. We can obtain further $\mathrm{Sym}(\pi\tau\alpha^2, \alpha, c)$'s by choosing a different (possibly isomorphic) $\mathrm{Sym}(\pi, \alpha, c)$ and also by choosing a different $\mathrm{Sym}(\tau, \alpha, c)$ or a different $\mathrm{GDD}(\pi, \tau, \alpha, c)$. Clearly there are large numbers of symmetric designs we can construct using the group divisible designs whose existence can be inferred from Theorems 2 and 3.

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Department of Mathematics
University of Queensland
St Lucia Qld 4067
Australia

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