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EXAMPLES FOR THE THEORY OF INFINITE ITERATION OF SUMMABILITY METHODS

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1. Introduction. Garten and Knopp [**7**] introduced the notion of infinite iteration of Césaro (C_1) averages, which they called H_∞ summability. Flehinger [**6**] (apparently unaware of [**7**]) produced the first nontrivial example of an H_∞ summable sequence: the sequence $\{a_i\}_{i=1}^\infty$ where a_i is 1 or 0 as the lead digit of the integer i is one or not. Duran [**2**] has provided an elegant treatment of H_∞ summability as a special case of summability with respect to an ergodic semi-group of transformations. Duran showed that logarithmic summability contains H_∞ summability and that, for bounded sequences, the H_∞ method was equivalent to Banach-Hausdorff summability introduced by Eberlein [**3**].

In Section 2 of this paper it is shown that a bounded sequence can be assigned a limit by a finite number of iterations of C_1 density if and only if the sequence is C_1 summable to the same limit. The logarithmic method is introduced and shown to be equivalent to the more widely used zeta (or Dirichlet) density. An elementary proof of the inclusion of the H_∞ method in the logarithmic method is given. Similar results are given for iterates of the logarithmic method.

In Section 3 examples are given of subsets of the integers which differentiate between the summability methods of Section 2. Roughly stated, any set of integers with polynomial gaps has C_1 density; if the gaps are exponential then the set of integers has log density (but not C_1 density). The set of integers will have H_∞ density (but not C_1 density) if and only if the gaps are linear exponential.

2. Definitions and basic theorems. Let M be the Banach space of all bounded sequences of real numbers $(x_1, x_2, \ldots)$ with norm $||x|| = \sup_n |x_n|$. For $x \in M$ write

$$d(x, n, 1) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and inductively define

$$d(x, n, k) = \frac{1}{n} \sum_{i=1}^{n} d(x, n, k-1)$$

for $k > 1$. Clearly $\liminf_n d(x, n, k) \geqq \liminf_n d(x, n, k-1)$. Similarly the upper limits are decreasing in k.

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Definition. $x \in M$ is said to be H_k *summable to c* if $\lim_n d(x, n, k) = c$ (H_1 summability is usually called C_1 summability). x is said to be H_∞ *summable to c* if $\lim_i \liminf_n d(x, n, i) = \lim_i \limsup_n d(x, n, i)$.

LEMMA 1. *For $x \in M$, $k < \infty$, x is H_k summable to c if and only if x is C_1 summable to c.*

Proof. Lemma 1 is immediate from Theorems 49, 55 and 92 of Hardy [**9**].

H_k summability is discussed at some length in Hardy [**10**]. A consequence of Lemma 1 is that C_1 summability to c implies H_∞ summability to c.

Definition. $x \in M$ is said to be *log summable to c* if

$$\lim_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{x_i}{i} = c.$$

x is *zeta summable to c* if

$$\lim_{s \to 1^+} (s-1) \sum_{i=1}^{\infty} \frac{x_i}{i^s} = c.$$

Zeta summability is used fairly regularly in analytic number theory where it is also known as *Dirichlet density* or *analytic density* (see Hasse [**10**, pp. 223-226], Serre [**12**, pp. 125], or Golomb [**8**]). Ishiguro [**11**] and Hardy [(**10**, p. 87]) discuss other summability methods equivalent to log summability.

THEOREM 1. *Log and zeta summability are equivalent on M.*

Proof. Let $x \in M$. Since the sequence x_i is bounded, there is no loss in generality in assuming $x_i \geqq 0$. Define a measure on the positive real numbers with mass x_i/i at the points $\log i$. Let the distribution function of the measure be

$$U(x) = \sum_{\log i \leqq x} \frac{x_i}{i}.$$

The Laplace transform of U is

$$\omega(t) = \sum_{i=1}^{\infty} \frac{x_i}{i} e^{-t \log i} = \sum_{i=1}^{\infty} \frac{x_i}{i^{t+1}}.$$

Theorem 2 in Feller [**5**, p. 445], implies that $\lim_{t \to \infty} t\,\omega(t) = l$ if and only if $\lim_{x \to \infty} U(x)/x = l$. Let $s = t + 1$, $x = \log y$; this becomes

$$\lim_{s \to 1^+} (s-1) \sum_{i=1}^{\infty} \frac{x_i}{i^s} = l \quad \text{if and only if} \quad \lim_{y \to \infty} \frac{1}{\log y} \sum_{i \leqq y} \frac{x_i}{i} = l.$$

Duran [**2**] gives a useful necessary and sufficient condition for a matrix method to dominate H_∞ summability. The proof depends on a theorem announced by Eberlein [**4**]. As a special case, Duran showed that if $x \in M$ has

H_∞ limit c then x is log summable to c. The next theorem is an elementary proof of this result.

THEOREM 2. *If $x \in M$ is H_∞ summable to l then x is log summable to l.*

Proof. Writing $S_n = \sum_{i=1}^{n} x_i$, summation by parts shows that for $m < n$,

$$\text{(2-1)} \quad \sum_{i=m}^{n} \frac{x_i}{i} = \sum_{i=m}^{n} \frac{S_i}{i(i+1)} + O(1) = \sum_{i=m}^{n} \left(\frac{S_i}{i}\right)\frac{1}{i} + O(1).$$

Inductively from (2-1), for each fixed k,

$$\text{(2-2)} \quad \begin{aligned} \sum_{i=m}^{n} \frac{x_i}{i} &= \left\{\sum_{i=m}^{n} d(x,i,k)\frac{1}{i}\right\} + O_k(1) \\ &\geqq \inf_{i>m} d(x,i,k)\{\log n - \log m + O(1)\} + O_k(1). \end{aligned}$$

Divide both sides of (2-2) by $\log n$ and let n go to ∞ to get

$$\text{(2-3)} \quad \lim\inf_n \frac{1}{\log n}\sum_{i=1}^{n}\frac{x_i}{i} = \lim\inf_n \frac{1}{\log n}\sum_{i=m}^{n}\frac{x_i}{i} \geqq \inf_{i\geqq m} d(x,i,k).$$

As this last inequality holds for all m, we have for each k,

$$\lim\inf_n \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{i} \geqq \lim\inf_i d(x,i,k).$$

A similar inequality holds for the upper limits; thus

$$\begin{aligned} \lim_k \lim\inf_n d(x,n,k) &\leqq \lim\inf_n \frac{1}{\log n}\sum_{i=1}^{n}\frac{x_i}{i} \\ &\leqq \lim\sup_n \frac{1}{\log n}\sum_{i=1}^{n}\frac{x_i}{i} \leqq \lim_k \lim\sup_n d(x,n,k). \end{aligned}$$

This proves the theorem.

Entirely similar results can be derived for iterates of log density. As an example the analog of Lemma 1 will be given in detail. The pıoof requires the following Tauberian theorem which is an extension of a theorem given by Ishiguro [**11**].

THEOREM 3. *Let a_i be a sequence of real numbers with*

$$\sum_{i=1}^{n}\frac{a_i}{i} = o(\log n).$$

If $n \log n|a_n - a_{n-1}| < k$ *for some* $k > 0$, *then* $\lim_n a_n = 0$.

Proof. Summation by parts yields

$$\left| a_{n+1}\sum_{i=1}^{n}\frac{1}{i} + \sum_{i=1}^{n}\left\{(a_i - a_{i+1})\sum_{j=1}^{i}\frac{1}{j}\right\}\right| = \left|\sum_{i=1}^{n}\frac{a_i}{i}\right| \leqq k_1 \log n.$$

Thus

$$|a_{n+1}| \leqq \frac{k_2}{\log n}\left\{\sum_{i=1}^{n} |a_i - a_{i+1}|\log i\right\} + k_1 \leqq k_3$$

where k_i are positive constants. Thus the hypothesis imply that the a_i are bounded.

For notational simplicity let $d = d(x, \delta) = [x^\delta]$ where $0 \leqq \delta \leqq 1$ and $[y]$ is the greatest integer less than or equal to y. The hypothesis imply that

$$\sum_{x \leqq i \leqq d} \frac{a_i}{i} = o(\log x)$$

while summation by parts shows

$$\begin{aligned}\sum_{x \leqq i \leqq d} \frac{a_i}{i} &= a_d\left(\sum_{x \leqq i \leqq d} \frac{1}{i}\right) + \sum_{1 \leqq i \leqq x^{1+\delta}-x}\left\{\left(\sum_{x \leqq j \leqq x+i} \frac{1}{j}\right)(a_{[x+i]} - a_{[x+i+1]})\right\} \\ &= a_d \log\left(\frac{x^{1+\delta}}{x}\right) + O\left(\frac{1}{x}\right) \\ &\quad + \sum_{1 \leqq i \leqq x^{1+\delta}-x}\left\{\log(x+i) - \log(x) + O\left(\frac{1}{x}\right)\right\}(a_{[x+i]} - a_{[x+i+1]}) \\ &\leqq \delta\, a_d \log x + k \sum_{1 \leqq i \leqq x^{1+\delta}-x} \frac{\log(x+i) - \log x}{(x+i)\log(x+i)} \\ &\qquad + O\left(\frac{1}{x}\sum_{1 \leqq i \leqq x^{1+\delta}-x} \frac{1}{(x+i)\log(x+i)}\right) + O\left(\frac{1}{x}\right).\end{aligned}$$

The sum in the error term is $O(1/x)$. The first sum is easily seen to be

$$\log x\{\delta - \log(1+\delta)\} + O(1/x).$$

Combining these estimates and letting x go to infinity leads to

$$0 \leqq \liminf_n \delta\, a_d + \{\delta - \log(1+\delta)\}.$$

Dividing by δ and letting δ go to zero leads to

$$0 \leqq \lim_\delta \liminf_n a_d$$

from which it follows that $0 \leqq \liminf_n a_n$. Using $a_n - a_{n+1} \geqq k/(n \log n)$ above leads to the opposite inequality for the upper limit which proves the theorem.

$x \in M$ is said to be *$\log_k$ summable to c* if the kth iterate of the log method converges to c.

LEMMA 2. *For $x \in M$, $k < \infty$, x is $\log_k$ summable to c if and only if x is log summable to c.*

Proof. It is elementary that if x is log summable to c then x is $\log_k$ summable to c for all k. For the converse, the Tauberian condition of Theorem 3 must be checked. If

$$\lim_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{b_i}{i} = c$$

where

$$b_i = \sum_{j=1}^{i} \frac{a_j}{j}$$

then

$$b_i - b_{i-1} = \left(\sum_{j=1}^{i} \frac{a_j}{j}\right)\left(\frac{1}{\log i} - \frac{1}{\log i + 1}\right) - \frac{a_{i+1}}{i \log i}$$

$$\leqq \frac{k(\log i)\left(\log 1 + \frac{1}{i}\right)}{(\log i)\ (\log i + 1)} - \frac{a_{i+1}}{i \log i} < \frac{k'}{i \log i}.$$

Similar arguments show

$$b_{i+1} - b_i \leqq \frac{k''}{i \log i}.$$

Thus Theorem 3 implies $\lim_n a_n = c$. An induction completes the proof.

The infinite iteration of log summability is dominated by the matrix method with

$$(i, j) \text{ entry} = \begin{cases} \dfrac{1}{(\log \log i) j \log j}, & 2 \leqq j \leqq i \\ 0 & \text{elsewhere} \end{cases}.$$

Details may be found in Diaconis [**1**]. Duran [**2**] discusses other related results.

3. Examples. Let A be a subset of the integers $\{1, 2, 3, \ldots\} = N$. Let a_i be the indicator function of the set A. The convergence properties associated with the vector $a = (a_1, a_2, a_3, \ldots) \in M$ allow a natural definition of various notions of density of the set A. Thus A is said to have C_1 *density* l if a is C_1 summable to l. Similar conventions will be used for log and H_∞ summability.

In this section $[x]$ denotes the greatest integer less than or equal to x, $\{x\} = x - [x]$ denotes the fractional part of x, and for real numbers s and t, $\langle s, t\rangle = \{i \in N: s \leqq i \leqq t\}$. In what follows, f and g will denote polynomials written

$$f(x) = ax^n + bx^{n-1} + d_{n-2}x^{n-2} + \ldots + d_0,$$
$$g(x) = ax^n + cx^{n-1} + e_{n-2}x^{n-2} + \ldots + e_0.$$

To rule out trivial cases, assume that

(3-1) $\deg f = \deg g$, both leading coefficients are positive and equal, and $0 < (c - b)/na < 1$.

In all cases where one of the assumptions in (3-1) is violated it is straightforward to check that the set $\bigcup_{i=1}^{\infty} \langle f(i), g(i)\rangle$ is either finite or has finite complement.

THEOREM 4. *Let f and g be polynomials satisfying the assumptions* (3-1). *Let* $A = \bigcup_{i=1}^{\infty} \langle f(i), g(i)\rangle$.
Case 1. *If $n = 1$ and a is irrational, then A has C_1 density* $(c - b)/a$.
Case 2. *If $n \geqq 2$, then A has C_1 density* $(c - b)/na$.

Proof. Case 1. The set $\langle ai + b, ai + c\rangle$ contains either $[c - b]$ or $[c - b] + 1$ points. It contains $[c - b] + 1$ points if and only if $1 - \{c - b\} \leqq \{ai + b\} \leqq 1$. Since a is irrational, the number, $\gamma(k)$, of sets $\langle ai + b, ai + c\rangle$, $1 \leqq i \leqq k$, which contains $[c - b] + 1$ points is $k\{c - b\} + o(k)$. Thus

$$\limsup_n \frac{1}{n} \sum_{i \leqq n} a_i = \limsup_k \frac{1}{g(k)} \sum_{i \leqq g(k)} a_i$$
$$= \limsup_k \frac{1}{ak + c} \{k[c - b] + o(k)\} = \frac{c - b}{a}.$$

A similar argument yields the same lower limit, concluding the proof of Case 1.

Case 2. The number of integers in the set $\langle f(i), g(i)\rangle$ is $g(i) - f(i) + O(1)$ as $i \to \infty$. Thus

$$\limsup_n \frac{1}{n} \sum_{i=1}^{n} a_i = \limsup_k \frac{1}{g(k)} \sum_{i \leqq g(k)} a_i$$
$$= \limsup_k \frac{1}{g(k)} \sum_{i=1}^{k} \{g(i) - f(i) + O(1)\}$$
$$= \limsup_k \frac{1}{g(k)} \left\{\frac{(c - b)k^n}{n} + O(k^{n-1}) + O(k)\right\} = \frac{c - b}{na}.$$

A parallel argument for the lower limit concludes the proof.

THEOREM 5. *Let f and g satisfy* (3-1). *Let* $A = \bigcup_{k=0}^{\infty} \langle 10^{f(k)}, 10^{g(k)}\rangle$. *Then A has log density $(c - b)/na$ but not C_1 density.*

Proof. To simplify notation, write $t(x) = 10^x$. Standard bounds for the sum $\sum_{m=p}^{q} 1/m$ yield

$$\limsup_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{a_i}{i} = \limsup_k \frac{1}{(\log 10)\, g(k)} \sum_{i \leqq t(g(k))} \frac{a_i}{i}.$$

The last sum is

$$\sum_{i=1}^{k} \sum_{m \in \langle t(f(i)), t(g(i))\rangle} \frac{1}{m} = \sum_{i=1}^{k} \left\{\log \frac{t(g(i))}{t(f(i))} + O\left(\frac{1}{t(f(i))}\right)\right\}$$
$$= \log 10 \sum_{i=1}^{k} \{g(i) - f(i)\} + O(1).$$

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Making the substitution leads to

$$\limsup_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{a_i}{i} = \frac{c-b}{na}$$

as required. Again, the lower limit follows from similar arguments.

The proof of Theorem 6 below shows that A does not have C_1 density in the case that f and g are linear. In fact, the limit points of the sequence $1/n \sum_{i=1}^{n} a_i$ form the interval

$$\left[\frac{10^{c-b}-1}{10^a-1}, \frac{10^{a+(b-c)}(10^{c-b}-1)}{10^a-1}\right]$$

in the linear case. If f and g are quadratic or higher degree polynomials, arguments similar to that of Theorem 7 show that A does not have H_∞ density. Thus A does not have C_1 density. Detailed proofs for the nonexistence of C_1 density in those cases are recorded in Diaconis [**1**].

THEOREM 6. *Let* $0 < (c-b)/a < 1$. *Then* $A = \bigcup_{k=0}^{\infty} \langle 10^{ak+b}, 10^{ak+c} \rangle$ *has* H_∞ *density* $(c-b)/a$.

Proof. Flehinger [**6**] proved this in the special case $a = 1, b = 0, c = \log_{10} 2$. Flehinger's proof generalizes in a straightforward if somewhat longwinded way to yield the results stated. Further details can be found in Diaconis [**1**].

THEOREM 7. *The set* $A = \bigcup_{k=0}^{\infty} \langle 10^{k^2}, 10^{(k+1/2)^2} \rangle$ *does not have* H_∞ *density. Rather,* $\liminf_n d(a, n, k) = 0$, $\limsup_n d(a, n, k) = 1$ *for every* k.

Proof. Writing $10^y = t(y)$ and $d(x, k)$ for $d(a, x, k)$, consider x of the form $x = t((n + s/2)^2)$ where s is a real variable, $0 < \gamma_1 \leqq s \leqq 1$, for γ_1 to be chosen later.

$$d(x, 1) = \frac{1}{x}\left\{\sum_{k=1}^{n-1} \left\{t\left(\left(k+\frac{1}{2}\right)^2\right) - t(k^2) + O(1)\right\} + t\left(\left(n+\frac{s}{2}\right)^2\right) - t((n^2))\right\}.$$

The largest term in the sum, when divided by x, is

$$t\left\{\left(n-\frac{1}{2}\right)^2 - \left(n+\frac{s}{2}\right)^2\right\} = O(t(-n)),$$

where the implied constant is independent of s and n. Thus, for $\gamma_1 \leqq s \leqq 1$,

$$d(x, 1) = 1 + O(nt(-n)) + O(t(-n)) = 1 + o(1)$$

where the implied constant may depend on γ_1, but is independent of s and n. This proves $\limsup_x d(x, 1) = 1$. Assume inductively γ_i, $0 \leqq \gamma_1 \leqq \gamma_2 \ldots \leqq \gamma_j < 1$, have been found such that for $\gamma_i < s \leqq 1$, $d(t((n + s/2)^2), i) =$

$1 + o(1)$ as $n \to \infty$. We now show for any $\epsilon > 0$, $\gamma_j + \epsilon < s \leqq 1$ implies $d(t(n + s/2)^2, j + 1) = 1 + o(1)$, as $n \to \infty$.

$$d\left(t\left(\left(n + \frac{s}{2}\right)^2\right), j + 1\right)$$

$$= \frac{1}{t\left(\left(n + \frac{s}{2}\right)^2\right)} \left[\sum_{k=1}^{t((n+(\gamma_j+\epsilon)/2)^2)} d(k, j) + \sum_{t((n+(\gamma_j+\epsilon)/2)^2)}^{t((n+s/2)^2)} d(k, j) \right].$$

In the first sum, since $0 \leqq d(k, j) \leqq 1$, dividing by $t((n + s/2)^2)$ shows the first sum is $O(t((\gamma_j + \epsilon - s)n)) = o(1)$, where the implied constant may depend on ϵ but not on n, s. All terms in the second sum are $1 + o(1)$. Making this substitution, the second sum is $Y + o(Y)$ where

$$Y = \frac{t\left(\left(n + \frac{s}{2}\right)^2\right) - t\left(\left(n + \frac{\gamma_j + \epsilon}{2}\right)^2\right)}{t\left(\left(n + \frac{s}{2}\right)^2\right)} = 1 + o(1).$$

Combining these estimates gives

$$d(t((n + s/2)^2), j + 1) = 1 + o(1) + o(1 + o(1)) + o(1) = 1 + o(1),$$

as was to be shown. The result for upper limits now follows by taking $\gamma_1 = \epsilon$, $\gamma_2 = \gamma_1 + \epsilon/2$, $\gamma_3 = \gamma_2 + \epsilon/4, \ldots$. Essentially, the same estimates give the same results for the lower limits. Again x is chosen of the form $x = t((n + s/2)^2)$, but now $1 < \gamma_1 \leqq s \leqq 2$. Then $d(x, 1) = O(t((1 - s)n)) = o(1)$, the rest of the proof following similarly.

Remarks. 1) In the linear case of Theorem 4 the set A always has C_1 density even if the leading coefficient a is rational. Simple examples show that the density need not be equal to $(c - b)/a$.

2) Theorem 5 shows that the set of integers with an even number of digits has log density 1/2.

3) It is possible, but notationally awkward, to extend Theorem 7 to any polynomials $f(x)$, $g(x)$ of degree greater than 1. The set A of Theorem 7 is $\bigcup_{k=0}^{\infty} \langle 10^{k^2}, 10^{k^2+k+1/4} \rangle$. Theorem 5 shows that A has log density 1/2.

3) A computation similar to Theorem 6 shows that sets of the form $\bigcup_{k=0}^{\infty} \langle 10^{10^{f(k)}}, 10^{10^{g(k)}} \rangle$, where f and g are linear, have L_∞ density but not log density.

5) Theorem 7 together with the results of Section 6 of Duran [**2**] show that the logarithmic method is not a Hausdorff method.

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