INTEGRAL FORMULAS FOR SUBMANIFOLDS AND THEIR APPLICATIONS

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Introduction. Liebmann [**12**] proved that the only ovaloids with constant mean curvature in a 3-dimensional Euclidean space are spheres. This result has been generalized to the case of convex closed hypersurfaces in an m-dimensional Euclidean space by Alexandrov [**1**], Bonnesen and Fenchel [**3**], Hopf [**4**], Hsiung [**5**], and Süss [**14**].

The result has been further generalized to the case of closed hypersurfaces in an m-dimensional Riemannian manifold by Alexandrov [**2**], Hsiung [**6**], Katsurada [**7**; **8**; **9**], Ōtsuki [**13**], and by myself [**15**; **16**].

The attempt to generalize the result to the case of closed submanifolds in an m-dimensional Riemannian manifold has been recently done by Katsurada [**10**; **11**], Kôjyô [**10**], and Nagai [**11**].

Our aim in the present paper is to obtain first of all the most general integral formulas for closed submanifolds in an m-dimensional Riemannian manifold, to specialize these formulas, and to apply these formulas to obtain a generalization of the theorem of Liebmann. We also discuss submanifolds of codimension 2 in an $(n + 2)$-dimensional Euclidean space.

In § 1, we recall formulas for the submanifolds in a Riemannian manifold which will be used in the later sections. In § 2, we prove integral formulas for closed submanifolds in their most general forms. We specialize these formulas in §§ 3, 4, and 5 and prove a theorem which is a generalization of the theorem of Liebmann quoted above. In the last section we study submanifolds of codimension 2 in an $(n + 2)$-dimensional Euclidean space.

1. Preliminaries. We consider an m-dimensional orientable differentiable Riemannian manifold M of class C^∞ covered by a system of coordinate neighbourhoods $\{U; x^h\}$ and denote by g_{ji}, $\{{}_j{}^h{}_i\}$, ∇_i, $K_{kji}{}^h$, and K_{ji}, the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$, the curvature tensor, and the Ricci tensor respectively, where, throughout the paper, the indices h, i, j, k, l run over the range $\{1, 2, \ldots, m\}$.

We then consider an n-dimensional compact and orientable differentiable submanifold N of class C^∞ covered by a system of coordinate neighbourhoods $\{V; u^a\}$ and C^∞ differentiably embedded in M, and denote by

$$x^h = x^h(u^a) \tag{1.1}$$

Received January 27, 1969. This paper was written while the author was a G. A. Miller Visiting Professor at the University of Illinois.

the local expressions of N, where, throughout the paper, the indices a, b, c, d, e run over the range $\{1, 2, \ldots, n\}$ $(1 < n < m)$. The Riemannian metric of N induced from that of M is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i, \tag{1.2}$$

where

$$B_b{}^i = \partial_b x^i, \qquad \partial_b = \partial / \partial u^b. \tag{1.3}$$

We denote by $\{{}_c{}^a{}_b\}$, ∇_b, $K_{dcb}{}^a$, and K_{cb}, the Christoffel symbols formed with g_{cb}, the operator of covariant differentiation with respect to $\{{}_c{}^a{}_b\}$, the curvature tensor, and the Ricci tensor of N, respectively.

We put

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \{{}_j{}^h{}_i\} B_c{}^j B_b{}^i - \{{}_c{}^a{}_b\} B_a{}^h, \tag{1.4}$$

and call this kind of covariant differentiation van der Waerden-Bortolotti covariant differentiation along the submanifold N. From (1.2) and (1.4), we find $g_{ji}(\nabla_d B_c{}^j) B_b{}^i = 0$, which shows that $\nabla_c B_b{}^h$ are orthogonal to the submanifold N.

We assume that the mean curvature vector

$$H^h = (1/n) g^{cb} \nabla_c B_b{}^h \tag{1.5}$$

never vanishes on N and take a unit vector C^h in the direction of the mean curvature vector and then we put

$$(\nabla_c B_b{}^i) C_i = h_{cb}. \tag{1.6}$$

C^h is called the mean curvature unit normal and h_{cb} the second fundamental tensor of the submanifold N with respect to the mean curvature unit normal. The eigenvalues $k_1, \ldots, k_n$ of h_{cb} are called principal curvatures of the submanifold with respect to C^h. If $k_1 = \ldots = k_n = k$, that is, $h_{cb} = k g_{cb}$, then the submanifold is said to be umbilical with respect to C^h.

From (1.6), we have

$$g^{cb} \nabla_c B_b{}^h = h_a{}^a C^h. \tag{1.7}$$

The scalar

$$H = \frac{1}{n} \sum_{a=1}^{n} k_a = \frac{1}{n} h_a{}^a \tag{1.8}$$

is called the first mean curvature of N with respect to C^h.

Now we put $C^h = C_{n+1}{}^h$ and choose $m - n$ mutually orthogonal unit normals $C_{n+1}{}^h, \ldots, C_m{}^h$ in such a way that $(B_b{}^h, C_v{}^h)$ form a positively oriented frame along the submanifold N, where, throughout the paper, the indices u, v, w take the values $n + 1, \ldots, m$. Then $\nabla_c B_b{}^h$ can be expressed as

$$\nabla_c B_b{}^h = h_{cbv} C_v{}^h, \tag{1.9}$$

which are equations of Gauss, where $h_{cb,n+1} = h_{cb}$.

On the other hand, if we put

$$\nabla_c C_v{}^h = \partial_c C_v{}^h + \{{}_j{}^h{}_i\} B_c{}^j C_v{}^i$$

equations of Weingarten can be written as

$$(1.10) \qquad \nabla_c C_v{}^h = -h_c{}^a{}_v B_a{}^h + l_{cvw} C_w{}^h,$$

where $h_c{}^a{}_v = h_{cbv} g^{ba}$ and $l_{cvw} = -l_{cwv}$ is the so-called third fundamental tensor. The l_{cvw} define the connection induced on the normal bundle. For $v = n + 1$, we have

$$(1.11) \qquad \nabla_c C^h = -h_c{}^a B_a{}^h + l_{cw} C_w{}^h,$$

where $l_{cw} = l_{c,n+1,w}$. From (1.9), (1.11), and the Ricci identity

$$\nabla_d \nabla_c C^h - \nabla_c \nabla_d C^h = K_{kji}{}^h B_d{}^k B_c{}^j C^i,$$

we find

$$\begin{aligned} &\nabla_d(-h_c{}^a B_a{}^h + l_{cv} C_v{}^h) - \nabla_c(-h_d{}^a B_a{}^h + l_{dv} C_v{}^h) = K_{kji}{}^h B_d{}^k B_c{}^j C^i, \\ &\quad -(\nabla_d h_c{}^a) B_a{}^h - h_c{}^a (h_{dav} C_v{}^h) + (\nabla_d l_{cv}) C_v{}^h + l_{cv}(-h_d{}^a{}_v B_a{}^h + l_{dvw} C_w{}^h) \\ &\quad +(\nabla_c h_d{}^a) B_a{}^h + h_d{}^a (h_{cav} C_v{}^h) - (\nabla_c l_{dv}) C_v{}^h - l_{dv}(-h_c{}^a{}_v B_a{}^h + l_{cvw} C_w{}^h) \\ &\qquad\qquad = K_{kji}{}^h B_d{}^k B_c{}^j C^i, \end{aligned}$$

from which, taking the inner product with $B_b{}^h$,

$$-\nabla_d h_{cb} - l_{cv} h_{dbv} + \nabla_c h_{db} + l_{dv} h_{cbv} = K_{kjih} B_d{}^k B_c{}^j C^i B_b{}^h,$$

or

$$(1.12) \qquad \nabla_d h_{cb} - \nabla_c h_{db} - l_{dv} h_{cbv} + l_{cv} h_{dbv} = K_{kjih} B_d{}^k B_c{}^j B_b{}^i C^h,$$

which are equations of Codazzi.

Multiplying (1.12) by g^{cb} and contracting, we find

$$(1.13) \qquad \nabla_d h_a{}^a - \nabla_a h_d{}^a - l_{dv} h_a{}^a{}_v + l_{av} h_d{}^a{}_v = K_{kjih} B_d{}^k B^{ji} C^h,$$

where

$$B^{ji} = g^{cb} B_c{}^j B_b{}^i.$$

An arbitrary vector field w^h normal to the submanifold N is expressed as

$$w^h = C_u{}^h w_u,$$

and consequently

$$\begin{aligned} \nabla_c w^h &= (-h_c{}^a{}_u B_a{}^h + l_{cuv} C_v{}^h) w_u + C_u{}^h \partial_c w_u \\ &= -h_c{}^a{}_u w_u B_a{}^h + (\partial_c w_v + l_{cuv} w_u) C_v{}^h. \end{aligned}$$

We put

$$'\nabla_c w^h = ('\nabla_c w_v) C_v{}^h = (\partial_c w_v + l_{cuv} w_u) C_v{}^h,$$

https://doi.org/10.4153/CJM-1970-046-9 Published online by Cambridge University Press

and say that the vector w^h normal to the submanifold N is parallel with respect to the connection $'\nabla$ induced on the normal bundle when

$$'\nabla_c w^h = 0,$$

that is, when $\nabla_c w^h$ is tangent to the submanifold.

In the latter sections, we assume that the mean curvature vector H^h is parallel with respect to the induced connection $'\nabla$. This assumption is equivalent to the fact that

$$\nabla_c H^h = \frac{1}{n}\nabla_c(h_a{}^a C^h) = \frac{1}{n}(\nabla_c h_a{}^a)C^h + \frac{1}{n}h_a{}^a(-h_c{}^b B_b{}^h + l_{cw}C_w{}^h)$$
$$= -\frac{1}{n}h_a{}^a h_c{}^b B_b{}^h + \frac{1}{n}(\nabla_c h_a{}^a)C^h + \frac{1}{n}h_a{}^a l_{cw} C_w{}^h$$

is tangent to the submanifold, that is,

(1.14) $$h_a{}^a = \text{const} \neq 0, \qquad l_{cw} = 0.$$

2. Integral formulas. We now assume the existence of a vector field v^h in M and put

(2.1) $$v_b = B_b{}^i v_i.$$

From this equation we have

(2.2) $$\nabla_c v_b = (\nabla_c B_b{}^i)v_i + B_c{}^j B_b{}^i(\nabla_j v_i),$$

from which

$$g^{cb}\nabla_c v_b = (g^{cb}\nabla_c B_b{}^i)v_i + B^{ji}(\nabla_j v_i)$$
$$= h_a{}^a C^i v_i + \tfrac{1}{2}B^{ji}(\nabla_j v_i + \nabla_i v_j),$$

or

(2.3) $$g^{cb}\nabla_c v_b = \alpha h_a{}^a + \tfrac{1}{2}B^{ji}(\mathcal{L}_v g_{ji}),$$

where

(2.4) $$\alpha = C^i v_i$$

and $\mathcal{L}_v$ denotes the Lie derivative with respect to v^h.

Integrating (2.3) over N, we find

(2.5) $$\int_N \alpha h_a{}^a\, dS + \tfrac{1}{2}\int_N B^{ji}(\mathcal{L}_v g_{ji})\, dS = 0,$$

where dS is the surface element of N.

We next put

(2.6) $$w_b = h_b{}^a v_a,$$

from which

$$\nabla_c w_b = (\nabla_c h_b{}^a)v_a + h_b{}^a \nabla_c v_a,$$

and consequently,

$$g^{cb}\nabla_c w_b = (\nabla_c h_b{}^c)v^b + \tfrac{1}{2}h^{cb}(\nabla_c v_b + \nabla_b v_c). \tag{2.7}$$

On the other hand, we have, from (2.2),

$$\tfrac{1}{2}(\nabla_c v_b + \nabla_b v_c) = (\nabla_c B_b{}^i)v_i + \tfrac{1}{2}B_c{}^j B_b{}^j(\nabla_j v_i + \nabla_i v_j),$$

and consequently (2.7) becomes

$$g^{cb}\nabla_c w_b = (\nabla_c h_b{}^c)v^b + (h^{cb}\nabla_c B_b{}^i)v_i + \tfrac{1}{2}h^{cb}B_c{}^j B_b{}^i(\mathscr{L}_v g_{ji}). \tag{2.8}$$

Substituting

$$\nabla_c h_b{}^c = \nabla_b h_a{}^a - l_{bv}h_a{}^a{}_v + l_{av}h_b{}^a{}_v - K_{kjih}B_b{}^k B^{ji}C^h$$

obtained from (1.13) into (2.8), we obtain

$$g^{cb}\nabla_c w_b = (\nabla_b h_a{}^a - l_{bv}h_a{}^a{}_v + l_{av}h_b{}^a{}_v - K_{kjih}B_b{}^k B^{ji}C^h)v^b + (h^{cb}\nabla_c B_b{}^i)v_i + \tfrac{1}{2}h^{cb}B_c{}^j B_b{}^i(\mathscr{L}_v g_{ji}). \tag{2.9}$$

Integrating this over N, we find

$$\int_N [v^b\nabla_b h_a{}^a + (h^{cb}\nabla_c B_b{}^i)v_i + \tfrac{1}{2}h^{cb}B_c{}^j B_b{}^i(\mathscr{L}_v g_{ji}) - K_{kjih}B_b{}^k v^b B^{ji}C^h - l_{bv}v^b h_a{}^a{}_v + l_{cv}h_b{}^c{}_v v^b]\, dS = 0. \tag{2.10}$$

On the other hand, we have, from (2.4),

$$\begin{aligned}\nabla_b\alpha &= (-h_b{}^a B_a{}^i + l_{bv}C_v{}^i)v_i + B_b{}^j C^i(\nabla_j v_i)\\ &= (-h_b{}^a v_a + l_{bv}v_v) + B_b{}^j C^i(\nabla_j v_i),\end{aligned}$$

where $v_v = C_v{}^i v_i$ and

$$\nabla_c\nabla_b\alpha = \nabla_c(-h_b{}^a v_a + l_{bv}v_v) + (\nabla_c B_b{}^j)C^i(\nabla_j v_i) + B_b{}^j(-h_c{}^a B_a{}^i + l_{cw}C_w{}^i)(\nabla_j v_i) + B_c{}^k B_b{}^j C^i(\nabla_k\nabla_j v_i),$$

from which

$$\begin{aligned}g^{cb}\nabla_c\nabla_b\alpha &= g^{cb}\nabla_c(-h_b{}^a v_a + l_{bw}v_w) + \tfrac{1}{2}h_a{}^a C^j C^i(\nabla_j v_i + \nabla_i v_j)\\ &\quad - \tfrac{1}{2}h^{cb}B_c{}^j B_b{}^i(\nabla_j v_i + \nabla_i v_j) + g^{cb}l_{cw}B_b{}^j C_w{}^i(\nabla_j v_i) + B^{kj}C^i(\nabla_k\nabla_j v_i),\\ &= g^{cb}\nabla_c(-h_b{}^a v_a + l_{bw}v_w) + \tfrac{1}{2}(h_a{}^a C^j C^i - h^{cb}B_c{}^j B_b{}^i)(\nabla_j v_i + \nabla_i v_j)\\ &\quad + g^{cb}l_{cw}B_b{}^j C_w{}^i(\nabla_j v_i) + B^{kj}C^i(\nabla_k\nabla_j v_i).\end{aligned}$$

Integrating over N, we find

$$\int_N [\tfrac{1}{2}(h_a{}^a C^j C^i - h^{cb}B_c{}^j B_b{}^i)(\mathscr{L}_v g_{ji}) + g^{cb}l_{cw}B_b{}^j C_w{}^i(\nabla_j v_i) + B^{kj}C^i(\nabla_k\nabla_j v_i)]\, dS = 0. \tag{2.11}$$

3. The case in which v^h is a conformal Killing vector field. We assume that v^h is a conformal Killing vector field, that is,

$$\mathscr{L}_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}, \tag{3.1}$$

where $\rho = (1/m)\nabla_i v^i$, and consequently

$$\mathscr{L}_v\{{}_j{}^h{}_i\} = \nabla_j\nabla_i v^h + K_{kji}{}^h v^k = \delta_j^h \rho_i + \delta_i^h \rho_j - \rho^h g_{ji}, \tag{3.2}$$

where $\rho_i = \nabla_i \rho$, $\rho^h = \rho_i g^{ih}$. In this case, (2.5) and (2.10) become

$$\int_N \alpha h_a{}^a \, dS + n \int_N \rho \, dS = 0, \tag{3.3}$$

and

$$\int_N [v^b\nabla_b h_a{}^a + (h^{cb}\nabla_c B_b{}^i)v_i + \rho h_a{}^a - K_{kjih}B_b{}^k v^b B^{ji}C^h - l_{bv}v^b h_a{}^a{}_v + l_{cv}h_b{}^c{}_v v^b] \, dS = 0, \tag{3.4}$$

respectively. From (3.2), we have

$$\begin{aligned} B^{kj}C^i(\nabla_k\nabla_j v_i) &= B^{kj}C^i(-K_{lkji}v^l + g_{ki}\rho_j + g_{ji}\rho_k - g_{kj}\rho_i) \\ &= -K_{lkji}v^l B^{kj}C^i - n\rho_i C^i. \end{aligned}$$

Substituting this into (2.11), we find

$$\int_N [\rho h_a{}^a - \rho h_a{}^a + g^{cb}l_{cw}B_b{}^j C_w{}^i(\nabla_j v_i) - K_{lkji}v^l B^{kj}C^i - n\rho_i C^i] \, dS = 0,$$

or

$$\int_N [n\rho_i C^i + K_{kjih}v^k B^{ji}C^h - g^{cb}l_{cw}B_b{}^j C_w{}^i(\nabla_j v_i)] \, dS = 0. \tag{3.5}$$

4. The case in which v^h is a conformal Killing vector field and $(\nabla_c B_b{}^i)v_i = \alpha h_{cb}$. The conformal Killing vector field v^h can be expressed as

$$v^h = B_a{}^h v^a + C_u{}^h \alpha_u \tag{4.1}$$

along the submanifold N, where $\alpha_{n+1} = \alpha$. Thus, from equations (1.9) of Gauss and (4.1), we have

$$\begin{aligned} (\nabla_c B_b{}^i)v_i &= h_{cbu} \cdot \alpha_u \\ &= h_{cb} \cdot \alpha + h_{cb\,n+2} \cdot \alpha_{n+2} + \ldots + h_{cb\,m} \cdot \alpha_m. \end{aligned}$$

We assume in the following that

$$h_{cb\,n+2} \cdot \alpha_{n+2} + \ldots + h_{cb\,m} \cdot \alpha_m = 0, \tag{4.2}$$

that is,

$$(\nabla_c B_b{}^i)v_i = \alpha h_{cb}. \tag{4.3}$$

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The condition (4.2) or (4.3) is satisfied if

$$h_{cb\,n+2} = 0, \ldots, \quad h_{cb\,m} = 0, \tag{4.4}$$

or

$$\alpha_{n+2} = 0, \ldots, \quad \alpha_m = 0, \tag{4.5}$$

or

$$h_{cb\,n+2} = 0, \ldots, h_{cb\,n+s} = 0, \quad \alpha_{n+s+1} = 0, \ldots, \alpha_m = 0. \tag{4.6}$$

If (4.4) is satisfied, then equations (1.9) of Gauss take the form

$$\nabla_c B_b{}^h = h_{cb} C^h, \tag{4.7}$$

which means that the van der Waerden-Bortolotti covariant derivative $\nabla_c B_b{}^h$ of $B_b{}^h$ is in the direction of mean curvature vector. If (4.5) is satisfied, then (4.1) takes the form

$$v^h = B_a{}^h v^a + \alpha C^h, \tag{4.8}$$

which means that the conformal Killing vector field v^h is contained in the linear space spanned by vectors tangent to the submanifold N and the mean curvature vector. This case has been considered by Katsurada and Nagai [**11**]. We notice that the condition (4.2) or (4.3) is automatically satisfied for the case of hypersurface.

Now, if we assume (4.3), then we have, from (3.4),

$$\int_N [v^b \nabla_b h_a{}^a + \alpha h^{cb} h_{cb} + \rho h_a{}^a - K_{kjih} v'^k B^{ji} C^h - l_{bv} v^b h_a{}^a{}_v + l_{cv} h_b{}^c{}_v v^b]\, dS = 0, \tag{4.9}$$

where v'^k is the tangent part of v^h, that is,

$$v'^k = B_a{}^h v^a = v^h - C_u{}^h v_u. \tag{4.10}$$

5. The case in which v^h is a conformal Killing vector field, $(\nabla_c B_b{}^i) v_i = \alpha h_{cb}$, and the mean curvature vector is parallel with respect to the connection induced in the normal bundle. We now assume that v^h is a conformal Killing vector field, $(\nabla_c B_b{}^i) v_i = \alpha h_{cb}$ and, moreover, the mean curvature vector $H^h = (1/n) g^{cb} \nabla_c B_b{}^h$ is parallel with respect to the connection induced in the normal bundle.

In this case, we have (1.14) and consequently, from (3.3), (4.9), (3.5), we obtain

$$h_a{}^a \int_N \alpha\, dS + n \int_N \rho\, dS = 0, \tag{5.1}$$

$$\int_N [\alpha h^{cb} h_{cb} + \rho h_a{}^a - K_{kjih} v'^k B^{ji} C^h]\, dS = 0, \tag{5.2}$$

$$\int_N [n \rho_i C^i + K_{kjih} v^k B^{ji} C^h]\, dS = 0, \tag{5.3}$$

respectively.

Forming the difference (5.2) – (5.1) multiplied by $(1/n)h_e^{\ e}$, we find

$$(5.4)\qquad \int_N \alpha\left(h^{cb} - \frac{1}{n}h_e^{\ e}g^{cb}\right)\left(h_{cb} - \frac{1}{n}h_d^{\ d}g_{cb}\right)dS - \int_N K_{kjih}v'^k B^{ji}C^h\,dS = 0.$$

Thus if $\alpha \neq 0$ has definite sign and $K_{kjih}v'^k B^{ji}C^h = 0$, then $h_{cb} = (1/n)h_a^{\ a}g_{cb}$, which shows that the submanifold N is umbilical with respect to the mean curvature normal. Thus we have the following result.

THEOREM 5.1. *Suppose that an orientable Riemannian manifold M admits a conformal Killing vector field v^h. If a closed and orientable submanifold N of M satisfies* (4.2) *or* (4.3), *the mean curvature vector is parallel with respect to the connection induced in the normal bundle, $\alpha \neq 0$ does not change the sign, and*

$$(5.5)\qquad K_{kjih}v'^k B^{ji}C^h = 0,$$

then the submanifold is umbilical with respect to the mean curvature normal.

We notice here that condition (5.5) is automatically satisfied when M is a space of constant curvature (see Katsurada and Nagai [**11**]).

We now assume that M admits a homothetic Killing vector field v^h, that is, ρ = const. Then we have from (5.3)

$$\int_N K_{kjih}v^k B^{ji}C^h\,dS = 0$$

or

$$\int_N K_{kjih}v'^k B^{ji}C^h\,dS + \int_N K_{kjih}v''^k B^{ji}C^h\,dS = 0,$$

where v''^k is the normal part of v^h. Thus the condition (5.5) in Theorem 5.1 can be replaced by

$$(5.6)\qquad K_{kjih}v''^k B^{ji}C^h = 0.$$

If, moreover, (4.5) is satisfied, that is, if v^h has the form

$$v^h = B_a^{\ h}v^a + \alpha C^h,$$

then (5.4) becomes

$$\int_N \alpha\left(h^{cb} - \frac{1}{n}h_e^{\ e}g^{cb}\right)\left(h_{cb} - \frac{1}{n}h_d^{\ d}g_{cb}\right)dS + \int_N K_{kjih}v''^k B^{ji}C^h\,dS = 0,$$

or

$$\int_N \alpha\left[\left(h^{cb} - \frac{1}{n}h_e^{\ e}g^{cb}\right)\left(h_{cb} - \frac{1}{n}h_d^{\ d}g_{cb}\right) + K_{kjih}C^k B^{ji}C^h\right]dS = 0.$$

Thus condition (5.5) in Theorem 5.1 can be replaced by

$$K_{kjih}C^k B^{ji}C^h = 0,$$

https://doi.org/10.4153/CJM-1970-046-9 Published online by Cambridge University Press

or

$$-K_{kjih}C^kB_c{}^jC^iB_b{}^hg^{cb} = 0. \tag{5.7}$$

This condition has the following geometrical interpretation. We choose n mutually orthogonal unit vectors $X_1, X_2, \ldots, X_n$ tangent to the submanifold and consider the sectional curvatures $\gamma(C, X_1)$, $\gamma(C, X_2), \ldots, \gamma(C, X_n)$. Then (5.7) means that the sum of these sectional curvatures is zero.

If N is a hypersurface, then (5.7) can be written as

$$K_{ji}C^jC^i = 0,$$

(see [**15**]).

6. Submanifold of codimension 2 in an $(n + 2)$-dimensional Euclidean space. We consider a submanifold N of codimension 2 in an $(n + 2)$-dimensional Euclidean space E and let the local expression of N be

$$X = X(u^a), \tag{6.1}$$

where X is the so-called position vector field.

We put

$$X_a = \partial_a X; \tag{6.2}$$

then the metric tensor g_{cb} of N is given by

$$g_{cb} = X_c \cdot X_b, \tag{6.3}$$

where $X_c \cdot X_b$ denotes the inner product of X_c and X_b.

If we put

$$\nabla_c X_b = \partial_c X_b - \left\{{}^{\,a}_{c\,b}\right\} X_a,$$

then the mean curvature vector field is given by

$$H = \frac{1}{n} g^{cb}\nabla_c X_b. \tag{6.4}$$

We assume that $H \neq 0$ and choose the first unit normal C to the submanifold N in this direction and denote by D the second unit normal.

Then the equations of Gauss can be written as

$$\nabla_c X_b = h_{cb}C + k_{cb}D, \tag{6.5}$$

where $(1/n)h_a{}^a$ is the first mean curvature of N and

$$g^{cb}k_{cb} = 0. \tag{6.6}$$

The equations of Weingarten take the form

$$\nabla_c C = -h_c{}^aX_a + l_cD, \tag{6.7}$$

$$\nabla_c D = -k_c{}^aX_a - l_cC. \tag{6.8}$$

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From the Ricci identity,

$$\nabla_d \nabla_c X_b - \nabla_c \nabla_d X_b = -K_{dcb}{}^a X_a,$$

we have, using (6.5), (6.7), and (6.8),

$$(\nabla_d h_{cb})C + h_{cb}(-h_d{}^a X_a + l_d D) + (\nabla_d k_{cb})D + k_{cb}(-k_d{}^a X_a - l_d C) - (\nabla_c h_{db})C - h_{db}(-h_c{}^a X_a + l_c D) - (\nabla_c k_{db})D - k_{db}(-k_c{}^a X_a - l_c C) = -K_{dcb}{}^a X_a,$$

from which

$$K_{dcb}{}^a = h_d{}^a h_{cb} - h_c{}^a h_{db} + k_d{}^a k_{cb} - k_c{}^a k_{db}, \tag{6.9}$$

$$\nabla_d h_{cb} - \nabla_c h_{db} - l_d k_{cb} + l_c k_{db} = 0, \tag{6.10}$$

$$\nabla_d k_{cb} - \nabla_c k_{db} + l_d h_{cb} - l_c h_{db} = 0. \tag{6.11}$$

Equations (6.9) are those of Gauss and (6.10) and (6.11) those of Codazzi.

In a similar way, from the Ricci identity

$$\nabla_d \nabla_c C - \nabla_c \nabla_d C = 0,$$

we find

$$\nabla_d l_c - \nabla_c l_d + h_d{}^a h_{ca} - h_c{}^a k_{da} = 0, \tag{6.12}$$

which are equations of Ricci.

Now the position vector X is expressed as

$$X = X_a v^a + \alpha C + \beta D, \tag{6.13}$$

and consequently we have

$$X_c = (h_{cb} C + k_{cb} D)v^b + X_a \nabla_c v^a + (\nabla_c \alpha)C + \alpha(-h_c{}^a X_a + l_c D) + (\nabla_c \beta)D + \beta(-k_c{}^a X_a - l_c C),$$

from which

$$\nabla_c v_b = g_{cb} + \alpha h_{cb} + \beta k_{cb}, \tag{6.14}$$

$$\nabla_c \alpha + h_{cb} v^b - l_c \beta = 0, \tag{6.15}$$

$$\nabla_c \beta + k_{cb} v^b + l_c \alpha = 0. \tag{6.16}$$

From (6.14), we have

$$g^{cb} \nabla_c v_b = n + \alpha h_a{}^a,$$

from which, integrating over N,

$$n \int_N dS + \int_N \alpha h_a{}^a \, dS = 0. \tag{6.17}$$

We next put

$$w_b = h_b{}^a v_a, \tag{6.18}$$

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from which

$$\nabla_c w_b = (\nabla_c h_b{}^a) v_a + h_b{}^a (\nabla_c v_a),$$
$$g^{cb}\nabla_c w_b = (\nabla_c h_a{}^c) v^a + h^{ba}(\nabla_b v_a),$$
$$= v^a \nabla_a h_c{}^c + l_c k_a{}^c v^a + h_a{}^a + \alpha h^{ba} h_{ba} + \beta h^{ba} k_{ba},$$

by virtue of (6.10) and (1.14). Thus, integrating over N, we find

$$\int_N [v^a \nabla_a h_c{}^c + l_c k_a{}^c v^a + h_a{}^a + \alpha h^{ba} h_{ba} + \beta h^{ba} k_{ba}]\, dS = 0. \tag{6.19}$$

From (6.15), we have

$$\nabla_c \nabla_b \alpha + \nabla_c (h_{ba} v^a) - (\nabla_c l_b)\beta - l_b \nabla_c \beta = 0,$$
$$\nabla_c \nabla_b \alpha + \nabla_c (h_{ba} v^a) - (\nabla_c l_b)\beta + l_b (k_{ca} v^a + l_c \alpha) = 0,$$

from which

$$g^{cb}\nabla_c \nabla_b \alpha + \nabla_c (h_a{}^c v^a) - (\nabla_c l^c)\beta + k_{cb} l^c v^b + l_c l^c \alpha = 0.$$

Integrating over N, we obtain

$$\int_N [\alpha l_c l^c - \beta(\nabla_c l^c) + k_{cb} l^c v^b]\, dS = 0. \tag{6.20}$$

We now assume that

$$(\nabla_c X_b) \cdot X = \alpha h_{cb}, \tag{6.21}$$

which means that

$$(h_{cb} C + k_{cb} D)(X_a v^a + \alpha C + \beta D) = \alpha h_{cb},$$

or

$$\beta k_{cb} = 0. \tag{6.22}$$

We also assume that

$$\nabla_a \left(\frac{1}{n} g^{cb} \nabla_c X_b \right)$$

is tangent to the submanifold, which means that

$$\nabla_c (h_a{}^a C) = (\nabla_c h_a{}^a) C + h_a{}^a (-h_c{}^b X_b + l_c D)$$

is tangent to the submanifold, that is to say,

$$h_a{}^a = \text{const} \neq 0, \qquad l_c = 0. \tag{6.23}$$

Thus, taking account of (6.3) and (6.22), we have, from (6.17) and (6.19),

$$n \int_N dS + h_a{}^a \int_N \alpha\, dS = 0, \tag{6.24}$$

$$\int [h_a{}^a + \alpha h^{ba} h_{ba}]\, dS = 0. \tag{6.25}$$

Forming the difference (6.25) – (6.26) multiplied by $(1/n)h_d^{\ d}$, we find

$$\text{6.26)} \qquad \int_N \alpha\left(h^{ba} - \frac{1}{n} h_e^{\ e} g^{ba}\right)\left(h_{ba} - \frac{1}{n} h_d^{\ d} g_{ba}\right) dS = 0.$$

Thus, if $\alpha \neq 0$ does not change the sign, we have

$$h_{cb} = (1/n)h_a^{\ a}g_{cb},$$

from which we have the following result.

THEOREM 6.1. *Assume that a closed and orientable submanifold N of co-dimension 2 in an $(n + 2)$-dimensional Euclidean space satisfies:*

$$(\nabla_c X_b) \cdot X = \alpha h_{cb},$$

$$\nabla_c\left(\frac{1}{n} g^{ba} \nabla_b X_a\right) \textit{ is tangent to } N,$$

and that $\alpha \neq 0$ does not change the sign; then the submanifold is umbilical with respect to the mean curvature normal.

Since N is umbilical with respect to the mean curvature normal, we can put

$$\text{(6.27)} \qquad h_{cb} = \lambda g_{cb},$$

where λ is a constant different from zero. Since $h_c^a = \lambda \delta_c^{\ a}$ and $l_c = 0$, we have from (6.7)

$$\nabla_c(C + \lambda X) = 0,$$

from which

$$\text{(6.28)} \qquad X + \frac{1}{\lambda} C = C_0,$$

where C_0 is a constant vector, from which we can conclude that the sub-manifold N is on a sphere with centre at C_0 and with the radius $1/|\lambda|$. From (6.5), we see that the equations of Gauss for N as a hypersurface of a sphere are

$$''\nabla_c X_b = k_{cb} D,$$

which shows that N is minimal in the sphere. Thus we have the following result.

THEOREM 6.2. *Under the same assumptions as in Theorem* 6.1, *the submanifold N is a minimal hypersurface of a sphere.*

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