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ERGODIC PROPERTIES OF BROWNIAN MOTION

by OLABISI O. UGBEBOR

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0. Introduction

Since Brownian motion is point recurrent in $\boldsymbol{R}^1$, recurrent in $\boldsymbol{R}^2$ and transient in $\boldsymbol{R}^n$, $n \geqq 3$ (see (**7**)), it follows that the total time spent in a bounded open set in $\boldsymbol{R}^1$ or $\boldsymbol{R}^2$ is unbounded. With the following ergodic theorems for Brownian motion in $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$ as motivation, we examine the rate of convergence in these theorems. Note that there is no ergodic property in $\boldsymbol{R}^n$ for $n \geqq 3$ since Brownian motion is not dense there.

Theorem 0.1. *If $\{X(t), 0 \leqq t < \infty\}$ is a separable Brownian motion process in $\boldsymbol{R}^1$ and if f and g are any two Baire functions with finite integrals $\bar{f}$ and $\bar{g} \neq 0$ respectively over $(-\infty, \infty)$, then*

$$\lim_{T\to\infty} \frac{\int_0^T f\{X(t)\}\,dt}{\int_0^T g\{X(t)\}\,dt} = \frac{\bar{f}}{\bar{g}},$$

with probability one. See (**3**).

Corollary (*Ergodic Theorem for Brownian motion in* $\boldsymbol{R}^1$). *Let $\{X(t), 0 \leqq t < \infty\}$ be a separable Brownian motion process in $\boldsymbol{R}^1$. If A and B are bounded measurable non-empty subsets of $\boldsymbol{R}^1$, then*

$$\lim_{T\to\infty} \frac{\text{total time spent in } A \text{ by } X(t) \text{ up to time } T}{\text{total time spent in } B \text{ by } X(t) \text{ up to time } T} = \frac{|A|}{|B|} \text{ a.s.}$$

Theorem 0.2 (*Ergodic Theorem for Brownian motion in* $\boldsymbol{R}^2$). *Let $X(t)$ be a Brownian motion process in $\boldsymbol{R}^2$. Let D_1 and D_2 be bounded open sets in the plane such that $D_2 \neq \phi$. Then*

$$\lim_{T\to\infty} \frac{\text{total time spent in } D_1 \text{ by } X(t) \text{ up to time } T}{\text{total time spent in } D_2 \text{ by } X(t) \text{ up to time } T} = \frac{m(D_1)}{m(D_2)} \text{ a.s.},$$

where $m(D_i)$ is Lebesgue measure of D_i in $\boldsymbol{R}^2$. See (**8**).

In $\boldsymbol{R}^2$ there are some independence problems for any sets D_i which are overcome by considering a stationary Markov chain determined by the process. We are able to prove that, almost surely,

$$\int_0^T [m(D_2)\chi_{D_1}(X(t)) - m(D_1)\chi_{D_2}(X(t))]\,dt$$

which measures the difference in time spent in the sets D_1, D_2 is unbounded as $T\to\infty$ although the ratio of this difference to

$$\int_0^T \chi_{D_i}(X(t))\,dt$$

tends to zero at a rate given by a suitable law of iterated logarithm.

Throughout this paper we shall assume that we are dealing with a separable version of Brownian motion process denoted by $X(t)=X(t,\omega)=X_t$.

c, C', C_0, $c_1,\ldots$ will denote a finite positive constant whose value is not important and not necessarily the same at different occurrences. Other notations are

χ_A	for indicator function of set A,
a.s.	for almost surely,
$O(x)$	for "large order" of x,
$[t]$	for integer part of t,
A^c	for complement of set A,
$\bar{A}$	for closure of set A,
$d(x, y)$	for distance between x and y,
∂A	for boundary of set A,
P_x, E_x	for conditional probability and expectation respectively given $X(0)=x$.

1. Rate of convergence in R^2

Theorem 1.1. *For any bounded sets A and B in R^2,*

$$g(t)=E_x\int_0^T m(B)\chi_A(X(t))\,dt-E_x\int_0^T m(A)\chi_B(X(t))\,dt$$

conver≃0es to a finite limit as $T\to\infty$; where $m(.)$ denotes Lebesgue measure in R^2.

Proof. $$g(t)=\int_0^T\int_A \frac{m(B)}{2\pi t}e^{-|x-y|^2/2t}\,dy\,dt-\int_0^T\int_B\frac{m(A)}{2\pi t}e^{-|x-z|^2/2t}\,dz\,dt$$
$$=\int_0^T f_A(t)-f_B(t)\,dt \text{ say.}$$

Since A and B are bounded sets, we have

$$\left|\int_{T_0}^T (f_A(t)-f_B(t))\,dt\right|=O\left(\frac{1}{T_o}\right),$$

for sufficiently large T_0. To complete the proof we apply the Cauchy condition for infinite integrals (see e.g. page 433 of (**1**)) to the function $f_A(t)-f_B(t)$.

Theorem 1.2. *Let D_1 and D_2 be bounded non-empty open sets in R^2 such that $\bar{D}_1\cap\bar{D}_2=\varnothing$. Then with probability one,*

$$f(T,\omega)=\int_0^T[m(D_2)\chi_{D_1}(X_x(t,\omega))-m(D_1)\chi_{D_2}(X_x(t,\omega))]\,dt$$

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is unbounded as $T\to\infty$. However with probability one,

$$\frac{f(T,\omega)}{\int_0^T \chi_{D_1}(X_x(t,\omega))\,dt}=O\left(\left(\frac{\text{loglog } N}{N}\right)^{1/2}\right)\to 0 \quad \text{as} \quad T\to\infty,$$

where $X_x(t,\omega)$ is Brownian motion in $\mathbf{R}^2$ starting from x and $(N-1)$ is the number of new entries to D_1 after hitting D_2, up to time T.

Proof. First we obtain uniform upper and lower bounds for P_x {total time spent in D_1 before hitting $D_2>t$} for all $x\in\bar{D}_1$. Let A, B be open sets such that $\bar{A}\subset D_1$ and $\bar{D}_1\subset B$ such that $D_2\subset(\bar{B})^c=D$ say. Consider an open subset A_1 of A such that $d(A_1,\partial A)>0$. Then

$$P_x(\sigma_A<\sigma_D)\geqq P_x(\sigma_{A_1}<\sigma_D) \quad \text{for} \quad x\in\bar{D}_1\cap A^c;$$

where for any Borel set B,

$$\sigma_B(\omega)=\sigma_B=\begin{cases}\inf(t>0: X_t\in B)\\ +\infty \text{ otherwise.}\end{cases}$$

Since σ_D is the limit of a monotone increasing sequence of non-negative simple functions f_n say, and by Proposition 2.1 of (**9**)

$P_x(\sigma_{A_1}\leqq f_n)$ is lower semi-continuous in x for each n, it follows that $P_x(\sigma_{A_1}<\sigma_D)$ is bounded below and assumes its minimum for $x\in\bar{D}_1\cap A^c$. But $P_x(\sigma_{A_1}<\sigma_D)>0$, so that

$\min_x P_x(\sigma_{A_1}<\sigma_D)=C'>0$ for $x\in\bar{D}_1\cap A^c$. Observe that $P_x(\sigma_A<\sigma_D)=1$ for $x\in A$ and we have proved

$$\text{there exists a constant } C'>0 \text{ such that } P_x(\sigma_A<\sigma_D)>C' \quad \text{for all} \quad x\in\bar{D}_1. \tag{1.1}$$

Starting from $y\in A$, define

$$\tau_{\partial D_1}=\text{first passage time out of } D_1,$$
$$\beta=d(y,D_1^c)$$
$$S(y,\beta)=\text{circle centre } y \text{ and radius } \beta.$$

Then $P_y\{\tau_{\partial D1}>t\}\geqq P_y\{\tau_{\partial S(y,\beta)}>t\}$. Moreover $P_y\{\tau_{\partial S(y,\beta)}>t\}>C_2e^{-c_1t}$ by Theorem 2 of (**2**). Therefore

$$P_y\{\text{total time spent in } D_1 \text{ before leaving } B>t\}\geqq C_2e^{-c_1t};\ y\in\bar{A}. \tag{1.2}$$

Next define

$$Q_x=\text{total time spent in } D_1 \text{ before hitting } D_2, \text{ starting from } x\in\bar{D}_1$$

$$R_x=\begin{cases}0 \text{ if } X_x(t) \text{ does not hit } A \text{ before } \partial B,\\ 1 \text{ if } X_x(t) \text{ hits } A \text{ at time } \mu \text{ say before hitting } \partial B.\end{cases}$$

Then $Q_x\geqq R_x.\,Q_{X(\mu)}$, where $X(\mu)\in\bar{A}$. By the strong Markov property, (1.1) and (1.2) we obtain the lower bound of

Lemma 1.1. *There exist positive constants c, C', C_0, c_1 such that $C_0 e^{-c_1 t} \leqq P_x$ {total time spent in D_1 before hitting $D_2 > t\} \leqq C'e^{-ct}$, for all $x \in \bar{D}_1$.*

For the upper bound let $C_1 = \inf\{P_x(X(1) \in D_2) : x \in \bar{D}_1\}$. Then $C_1 > 0$. Define recursively

$$\tau_0 = 0,\ \tau_n = \text{first hitting time of } \bar{D}_1 \text{ after } \tau_{n-1} + 1.$$

Then for any positive integer n and $x \in \bar{D}_1$, P_x {total time spent in D_1 before hitting $D_2 > n\} \leqq P_x\{X(\tau_j + 1) \notin D_2,\ j = 0, \ldots, n-1\} \leqq (1 - C_1)^n < 1$.

$$\begin{aligned} \therefore\ & P_x\{\text{total time spent in } D_1 \text{ before hitting } D_2 > t\} \\ & \leqq P_x\{\text{total time spent in } D_1 \text{ before hitting } D_2 > [t]\} \\ & \leqq (1 - C_1)^{[t]} < 1. \end{aligned}$$

Therefore for suitable constants C', c,

$$P_x\{\text{total time spent in } D_1 \text{ before hitting } D_2 > t\} \leqq C'e^{-ct}, \text{for all } x \in \bar{D}_1.$$

This completes the proof of Lemma 1.1.

Let $\alpha_1(\omega) = \sigma_{D_1}(\omega)$, $\beta_1(\omega) = \alpha_1(\omega) + \sigma_{D_2}(\omega^+_{\alpha_1})$ and for $n \geqq 2$,

$$\alpha_n(\omega) = \beta_{n-1}(\omega) + \alpha_1(\omega^+_{\beta_{n-1}}),\ \beta_n(\omega) = \alpha_n(\omega) + \alpha_{D_2}(\omega^+_{\alpha_n})$$

where $\omega_t = X(t, \omega)$ is standard separable Brownian motion, and ω^+_s is the shifted Brownian motion $\omega^+_s : t \to X(t + s, \omega)$. Using our notation we may rewrite, for $x \in D_1$,

$$\int_0^T \chi_{D1}(X_x(t, \omega))\, dt \ \text{ as } \ \sum_{i=1}^n \int_{\alpha_i}^{\beta_i} \chi_{D_1}(X_x(t, \omega))\, dt + R^n_{D_1}$$

and (1.3)

$$\int_0^T \chi_{D_2}(X_x(t, \omega))\, dt \ \text{ as } \ \sum_{i=1}^n \int_{\beta_i}^{\alpha_{i+1}} \chi_{D_2}(X_x(t, \omega)\, dt + R^n_{D_2}$$

where, for

$$\beta_n \leqq T < \alpha_{n+1},\ R^n_{D_1} = 0 \ \text{ and } \ 0 \leqq R^n_{D_2} < \int_{\beta_n}^{\alpha_{n+1}} \chi_{D_2}(X_x(t, \omega))\, dt;$$

for

$$\alpha_{n+1} \leqq T < \beta_n,\ R^n_{D_2} = 0 \ \text{ and } \ 0 \leqq R^n_{D_1} < \int_{\alpha_{n+1}}^{\beta_{n+1}} \chi_{D_1}(X_x(t, \omega))\, dt.$$

If $x \notin D_1$, there will be an extra initial term in (1.3) which will make no difference to our results. Let

$$U_i = \int_{\alpha_i}^{\beta_i} \chi_{D_1}(X_x(t, \omega))\, dt,\ \ V_i = \int_{\beta_i}^{\alpha_{i+1}} \chi_{D_2}(X_x(t, \omega))\, dt.$$

Then $P_x\{U_i > t\} \geqq C_0 e^{-c_1 t}$ for all $x \in \bar{D}_1$ by Lemma 1.1. Therefore for every fixed λ no matter how large,

$$P_x\{U_i < \lambda\} < \delta < 1 \ \text{ for all } \ x \in \bar{D}_1.$$

By the strong Markov property and the fact that the bounds in Lemma 1.1 are uniform in $x \in \bar{D}_1$, we have

$$P\{U_1 < \lambda,\ U_2 < \lambda, \ldots, U_n < \lambda\} < \delta^n \to 0 \ \text{ as } \ n \to \infty.$$

Hence for any fixed λ, there is zero probability that all the U_i are less than λ. Since

$$f(T, \omega) = m(D_2)\left(\sum_{i=1}^{n} U_i + R_{D_1}^{n}\right) - m(D_1)\left(\sum_{i=1}^{n} V_i + R_{D_2}^{n}\right),$$

it is easy to see that, with probability one, $f(T, \omega) > \lambda$ infinitely-often as $n \to \infty$. Now consider a sequence $\lambda_n(=n)\uparrow\infty$ as $n \to \infty$ and define the event $E_n = \{\omega : f(T, \omega) > \lambda_n(=n)$ infinitely-often$\}$. Then $P\{E_n\} = 1$ and since E_n is monotone in n, $P\{\bigcap_n E_n\} = 1$. Therefore for $\omega \in \bigcap_n E_n$, $f(T, \omega)$ is unbounded as $n \to \infty$. Hence with probability one, $f(T, \omega)$ is unbounded as $T \to \infty$. For proof of the second part of Theorem 1.2 we require the machinery of stationary Markov chains. First we state a useful result.

Lemma 1.2. *Given $X(0) = x$, let $X(\alpha_i) = a_i$. Then $\{a_i\}$ is a strictly stationary Markov chain on $\bar{D}_1$. Moreover there exists, for $S \subset \bar{D}_1$, the unique stationary distribution $p(S)$, for the Markov chain $\{a_i\}$, such that*

$$|p^i(x, S) - p(S)| < c\rho^i,\ x \in \bar{D}_1,$$

for constants $c > 0$, $0 < \rho < 1$; where $p^i(x, S)$ is the i-step transition probability for the Markov chain.

For the proof of Lemma 1.2 see Proposition 4.1 in (**8**) and §5 in Chapter V of (**4**).

Precise determination of the stationary distribution for generalised sets D_1, D_2 is difficult. We did obtain $p(S)$ in two special cases (stated below as Remarks 1 and 2) using standard potential theory arguments. Note that similar arguments yield the stationary distribution on any two bounded circles in $\boldsymbol{R}^2$.

Remark 1. Consider two unit discs A and B in $\boldsymbol{R}^2$ such that the distance between their centres is $s > 2$ units. Then

$p(\psi)$ = stationary probability that Brownian motion enters B through $d\psi \subset \partial B$

and

$p(\phi)$ = stationary $P\{$Brownian motion enters A through $d\phi \subset \partial A\}$ are given by

$$p(\psi) = \frac{1-u^2}{2\pi(1-2u \operatorname{Cos} \psi + u^2)},\quad p(\phi) = \frac{1-u^2}{2\pi(1-2u \operatorname{Cos} \phi + u^2)}$$

where ψ, ϕ are the angles between the line joining the centres of the circles and $d\psi$, $d\phi$ respectively and $u = \frac{1}{2}s - (\frac{1}{4}s^2 - 1)^{1/2}$.

Remark 2. Consider a unit disc A and a circle B with radius $b > 1$ unit in $\boldsymbol{R}^2$ such that the distance between their centres $s > 1 + b$. Then

$f(\phi, 1)$ = probability density of stationary distribution on ∂A, and
$f(\psi, b)$ = probability density of stationary distribution on ∂B

are given by

$$f(\phi, 1) = \frac{1-U_A^2}{2\pi(1-2U_A \operatorname{Cos} \phi + U_A^2)},\quad f(\psi, b) = \frac{b^2 - U_B^2}{2\pi b(b^2 - 2bU_B \operatorname{Cos} \psi + U_B^2)}$$

where

$$U_A = \frac{1+s^2-b^2}{2s} - \left(\frac{(1+s^2-b^2)^2}{4s^2}-1\right)^{1/2}, \quad U_B = \frac{s^2+b^2-1}{2s} - \left(\frac{(s^2+b^2-1)^2}{4s^2}-b^2\right)^{1/2}$$

and ϕ, ψ are as in Lemma 1.3.

Next define $\Gamma_x(t) = P_x\{U_1 \leqq t\}$, $\Gamma_x^i(t) = P_x\{U_i \leqq t\}$. Then

$$\Gamma_z^i(t) = P_z\{U_i \leqq t\} = \int_{\partial D_1} \Gamma_x(t) p^{i-1}(z, dx),$$

and since $\{U_i\}$ inherits stationarity from $\{a_i\}$,

$\Gamma(t) = \int_{\partial D_1} \Gamma_x(t) p(dx) = \lim_{i\to\infty} \Gamma_z^i(t)$ is the asymptotic distribution of U_i which does not depend on i.

Definition. A strictly stationary sequence $\{U_j\}$ is said to be *uniformly mixing* if for all $D \in M_{k+n}^{\infty}$,

$$|P\{D \mid M_{-\infty}^{k}\} - P(D)| \leqq \phi(n) \downarrow 0 \text{ as } n \to \infty;$$

where the σ-algebra M_{k+n}^{∞} describes the future of the sequence $\{U_j\}$ and is generated by $\{U_{k+n}, U_{k+n+1}, U_{k+n+2}, \ldots\}$ while the σ-algebra $M_{-\infty}^{k}$ is generated by $\{U_1, U_2, \ldots, U_k\}$, and $\phi(n)$ is said to be the mixing coefficient.

Lemma 1.3. *The sequence $\{U_i\}$ is uniformly mixing.*

Proof. Let X be a measure space and let μ, v be two measures on X such that $|\mu(S) - v(S)| < \varepsilon$ for all $S \subset X$. Let $0 \leqq f(x) \leqq 1$ be a function measurable on X. Then f is the limit of a monotone increasing sequence of non-negative simple functions, so that application of Lebesgue's theorem (see page 121 of (**6**)) to this sequence gives $(\int f d\mu - \int f dv) < \varepsilon$ and $-\varepsilon < (\int f d\mu - \int f dv)$ separately and hence the result $|\int f d\mu(x) - \int f dv(x)| < \varepsilon$. Now ∂D_1 is a measure space on which two measures $p^i(x, S)$, $p(S)$ are such that $|p^i(x, S) - p(S)| < c\rho^i$ for all $S \subset \partial D_1$. Therefore

$$\left|\int_{\partial D_1} \Gamma_z(t) p^i(x, dz) - \int_{\partial D_1} \Gamma_z(t) p(dz)\right| < c\rho^i,$$

for constants $c > 0$, $0 < \rho < 1$.

Define θ as the shift function $\theta(U_1, U_2, \ldots) = (U_2, U_3, \ldots)$. Then for $D \in M_{k+n}^{\infty}$, $\theta^{-n}(D)$ depends on at most $U_k, U_{k+1}, U_{k+2}, \ldots$, and $\theta^{-n-k+1}(D)$ depends on at most $U_1, U_2, \ldots$. By the Markov property of $\{a_i\}$,

$$\{C_1 \leqq P(D \mid a_k = z) \leqq C_2\} \Rightarrow \{C_1 \leqq P(D \mid M_{-\infty}^{k}) \leqq C_2\} \text{ a.s.}$$

Moreover

$$P(D \mid a_k = z) = \int_{\partial D_1} P\{\theta^{-n-k+1}(D) \mid a_1 = y\} p^n(z, dy)$$

by strict stationarity and $P\{\theta^{-n-k+1}(D) \mid a_1 = y\}$ is a fixed function of y since it depends neither on n nor k. Therefore $P\{D \mid a_k = z\}$ depends on z but not on k. If we restart the

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sequence at $a_{k+1}=z_1$ say, D now depends on a sequence starting from z_1 and so depends on $U_{n+1}, U_{n+2}, \ldots$. Therefore replacing k by appropriate suffix corresponding to z_1 gives

$$|P\{D \mid M_{-\infty}^{k}\}-P(D)|=|P_{z_1}(D)-P(D)|=|P_{z_1}(D)-P_{X(\alpha_1)}(D)|.$$

It is easy to show that $|P_{z_1}(D)-P_L(D)|<c'\rho^n$, $0<\rho<1$, where

$$P_L(D)=\lim_{n\to\infty}\int_{\partial D_1} P_{y_1}(D)p^n(z_1, dy_1)=\int_{\partial D_1} P_{y_1}(D)p(dy_1).$$

Similarly $|P_L(D)-P_{X(\alpha_1)}(D)|<c_1\rho^{n+k}$, $0<\rho<1$. Therefore

$$|P\{D \mid M_{-\infty}^{k}\}-P(D)|<c\rho^n, \text{for constants } c>0, 0<\rho<1; \tag{1.4}$$

which completes the proof of Lemma 1.3.

Corollary. *The sequence* $\{V_i\}$ *is uniformly mixing.*

Now define $Y_i=m(D_2)U_i-m(D_1)V_i$.

It is easy to see, using the ergodic theorem for stationary processes (see e.g. (**6**)) that $E(Y_i)\to 0$ as $i\to\infty$.

Also Lemma 1.1 immediately gives

$$P_x\{Y_i>t\}\leqq ce^{-c_1t} \text{ for all } x\in \bar{D}_1. \tag{1.5}$$

Applying the same method as for $\{U_i\}$ we arrive at

Lemma 1.4. *The sequence* $\{Y_i-E_x(Y_i)\}$ *is strictly stationary and uniformly mixing, with mixing coefficient given by* (1.4).

Next we state two useful results.

Lemma 1.5. *A strictly stationary sequence* $\{U_j\}$ *with* $E(U_j)=0$, *satisfying the uniform mixing condition, obeys the law of the iterated logarithm if the following conditions are fulfilled*:

(i) $E|U_j|^{2+\delta}<\infty$, $\delta>0$;

(ii) $\sum_{n=1}^{\infty}\{\phi(n)\}^{1/2}<\infty$, *where* $\phi(n)$ *is the mixing coefficient*;

(iii) $0\neq\sigma^2=E[U_1^2]+2\sum_{j=2}^{\infty}E[U_1\cap U_j]$. *See* (**10**).

Lemma 1.6. *Suppose the strictly stationary sequence* $\{U_j\}$ *satisfies the uniform mixing condition. If the random variables* τ, η *are measurable with respect to* $M_{-\infty}^{k}$ *and* M_{k+n}^{∞} *respectively, and if* $E(|\tau|^p)<\infty$, $E(|\eta|^q)<\infty$ *with* $p, q>1$ *and* $1/p+1/q=1$, *then*

$$|E(\tau\eta)-E(\tau)E(\eta)|\leqq 2\{\phi(n)\}^{1/p}E^{1/p}(|\tau|^p)E^{1/q}(|\eta|^q)$$

where $\phi(n)$ *is the mixing coefficient for* $\{U_j\}$. *See* (**5**).

That conditions (i) and (ii) of Lemma 1.5 are satisfied by $\{Y_i-E_x(Y_i)\}$ follows from (1.5) and (1.4) above respectively. Moreover, by the strong Markov property and the uniformity of the bounds in Lemma 1.1, $P_x\{Y_i>a\}\geqq C_0e^{-ca}(1-C'e^{-c_2a})$ for $a>0$, $x\in\bar{D}_1$. Therefore for sufficiently large a, $P_x\{Y_i>a\}\geqq\varepsilon'>0$ for all i, $x\in\bar{D}_1$. Similarly, $P_x\{Y_i<-a\}\geqq\varepsilon_1>0$ for all i, $x\in\bar{D}_1$. Therefore there is an $\varepsilon(=a^2(\varepsilon'+\varepsilon_1))$ such that for all integers i, the variance $\sigma_x^2(Y_i)$ of Y_i starting at x is at least ε. Hence the limiting variance of Y_i starting at x, $\int\sigma_x^2p(dx)>0$. Conditions (i) and (ii) of Lemma 1.5 are clearly satisfied for the sequence obtained by requiring that the initial point be random with distribution p, from (1.5) and (1.4) above respectively. For condition (iii) we need Lemma 1.6. The conditions of Lemma 1.6 are satisfied by $Y_j-E_x(Y_j)$, $j\geqq 2$ and $Y_1-E_x(Y_1)$ for $p=q=2$. Therefore

$$2\left|\sum_{j=2}^{\infty}E_x([Y_1-E_x(Y_1)][Y_j-E_x(Y_j)])\right|\leqq 4\sum_{j=2}^{\infty}\{\phi(j-1)\}^{1/2}E_x([Y_1-E_x(Y_1)]^2)$$
$$\leqq\frac{4(c\rho)^{1/2}}{1-\rho^{1/2}}E_x([Y_1-E_x(Y_1)]^2);\ c>0,\ 0<\rho<1.$$

Since $\int\sigma_x^2p(dx)$ is positive, condition (iii) is satisfied if

$$2\left|\sum_{j=2}^{\infty}E_x([Y_1-E_x(Y_1)][Y_j-E_x(Y_j)])\right|<E_x([Y_1-E_x(Y_1)]^2), \tag{1.6}$$

so that (1.6) holds if

$$\rho<1/(1+4c^{1/2})^2;\ c>0,\ 0<\rho<1. \tag{1.7}$$

Since the exact values of c and ρ are unknown it is not possible to determine whether or not (1.7) is satisfied. A way out of this difficulty is to consider the subsequences $\{Y_{ki+j})-E_x(Y_{ki+j})\}$, $i=0,1,2,\ldots$; for a FIXED integer k whose value will be determined later. There are k such subsequences of $\{Y_i-E_x(Y_i)\}$ namely

$$\{Y_j-E_x(Y_j)\},\{Y_{k+j}-E_x(Y_{k+j})\},\{Y_{2k+j}-E_x(Y_{2k+j})\},\ldots\ \text{for}\ j=1,2,\ldots k;$$

such that

$$\sum_{j=1}^{k}\sum_{i=0}^{n-1}(Y_{ki+j}-E_x(Y_{ki+j}))=\sum_{i=1}^{nk}(Y_i-E_x(Y_i)).$$

For a typical subsequence, condition (iii) of Lemma 1.5 is satisfied if

$$\rho^k<1/(1+4c^{1/2})^2. \tag{1.8}$$

Because c, ρ are fixed for $\{Y_i-E_x(Y_i)\}$ we can choose k such that k is the smallest integer for which (1.8) holds. For this value of k, it is clear that all three conditions of Lemma 1.5 hold. Therefore the sequence $\{Y_{ki+j}-E_x(Y_{ki+j})\}$ for k given as above, obeys the law of the iterated logarithm. The tail of the distribution of Y_i starting at x has a negative exponential upper bound (application of Lemma 1.1). Moreover $E_x\{Y_i\}\to 0$ as $i\to\infty$. Theorem 1.1 therefore implies

$$\left|\sum_{i=1}^{N}Y_i\right|=\left|\sum_{i=1}^{nk}Y_i\right|\leqq\sum_{j=1}^{k}\left|\sum_{i=0}^{n-1}Y_{ki+j}\right|<k(cN\sigma^2\operatorname{loglog}N)^{1/2} \tag{1.9}$$

for sufficiently large N, where $\sigma^2=\max\{\sigma_j^2;\ j=1,2,\ldots,k\}$ where σ_j^2 is equivalent to σ^2 in Lemma 1.5 for each subsequence $\{Y_{ki+j}-E_x(Y_{ki+j})\}$. If

(a) $\beta_N\leqq T<\alpha_{N+1}$, $f(T,\omega)=\sum_{i=1}^{N}Y_i-m(D_1)R_{D_2}^N$, $0\leqq R_{D_2}^N<V_N$,

(b) $\alpha_{N+1} \leqq T < \beta_{N+1},\, f(T, \omega) = \sum_{i=1}^{N} Y_i + m(D_2) R_{D_1}^N,\, 0 \leqq R_{D_1}^N < U_{N+1},$

where $(N-1)$ is the number of new entries to D_1 from D_2 up to time T. Since the tail of the distribution of $m(D_2)U_{N+1}$ has a negative exponential upper bound, there exists an N_0 such that $m(D_2)U_{N+1} \leqq N^{1/2}$ for all $N \geqq N_0$. Then in both cases (a) and (b) above we have, from (1.9), that

$$|f(T, \omega)| = O((N \operatorname{loglog} N)^{1/2}) \text{ a.s. as } N \to \infty. \tag{1.10}$$

But

$$\lim_{N\to\infty} \frac{1}{N} \int_0^T \chi_{D_i}(X_x(t, \omega))\, dt = c' \text{ a.s.}$$

(see e.g. (**8**)). Therefore with probability one,

$$\frac{f(T, \omega)}{\int_0^T \chi_{D_i}(X_x(t, \omega))\, dt} = O\left(\left(\frac{\operatorname{loglog} N}{N}\right)^{1/2}\right) \to 0 \text{ as } N \to \infty;$$

where $(N-1)$ is the number of new entries to D_1 after hitting D_2. The fact that $N \to \infty$ as $T \to \infty$ completes the proof of Theorem 1.2.

Remark 3. The same result in $\boldsymbol{R}^1$ is substantially easier because the hitting point of the interval is unique which implies that the sequence $\{Y_i\}$ of random variables are independent and identically distributed. This allows an application of the standard law of iterated logarithm.

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University of Ibadan
Ibadan
Nigeria