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FAMILIES OF FRACTIONAL FANTAPPIÈ TRANSFORMS

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Abstract

Let B_n denote the unit ball in $\mathbb{C}^n$, $n \geq 1$. Given an $\alpha > 0$, let $\mathcal{F}_\alpha(n)$ denote the class of functions defined for $z \in B_n$ by integrating the kernel $(1 - \langle z, w \rangle)^{-\alpha}$ against a complex Borel measure $d\mu(w)$, $w \in B_n$. The family $\mathcal{F}_0(n)$ corresponds to the logarithmic kernel $\log(1/(1 - \langle z, w \rangle))$. Various properties of the spaces $\mathcal{F}_\alpha(n)$, $\alpha \geq 0$, are obtained. In particular, pointwise multiplies for $\mathcal{F}_\alpha(n)$ are investigated.

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1. Introduction

Let $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$, $n \geq 1$. For $\lambda \in B_1$, put

$$k_0(\lambda) = \log \frac{1}{1 - \lambda};$$
$$k_\alpha(\lambda) = \frac{1}{(1 - \lambda)^\alpha}, \quad \alpha > 0.$$

Here and in what follows we use the principal branch of the logarithm.

1.1. Fractional Fantappiè transforms. Let $\mathcal{M}(B_n)$ denote the space of complex-valued Borel measures defined on the ball B_n. Let $\alpha \geq 0$. Given a measure $\mu \in \mathcal{M}(B_n)$, its fractional Fantappiè transform of order α is defined by the identity

$$F_\alpha[\mu](z) = \int_{B_n} k_\alpha(\langle z, w \rangle) \, d\mu(w), \quad z \in B_n.$$

The classical Fantappiè transform in the ball is $F_1[\mu]$. See, for example, [1] for a detailed treatment of the Fantappiè transform.

Note that

$$\frac{1}{(1 - \langle z, w \rangle)^{n+1}}, \quad z, w \in B_n,$$

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is the Bergman kernel in the ball. So, one may consider $F_\alpha[\mu]$ as fractional Bergman transforms of $\mu \in \mathcal{M}(B_n)$.

Let ν_n denote the normalized Lebesgue measure on the ball B_n. In numerous applications one usually considers $F_\alpha[\mu]$, where μ is absolutely continuous with respect to ν_n (see, for example, [12]).

Let $\mathcal{H}ol(B_n)$ denote the space of functions holomorphic on the ball B_n. Consider the families

$$\mathcal{F}_0(n) = \{f \in \mathcal{H}ol(B_n) : f - f(0) = F_0[\mu] \text{ for some } \mu \in \mathcal{M}(B_n)\};$$
$$\mathcal{F}_\alpha(n) = \{F_\alpha[\mu] : \mu \in \mathcal{M}(B_n)\}, \quad \alpha > 0.$$

Note that

$$1 = \int_{B_n} \frac{d\nu_n(w)}{(1 - \langle z, w\rangle)^\alpha} \quad \forall z \in B_n, \ \alpha > 0.$$

Therefore, $1 \in \mathcal{F}_\alpha(n)$ for all $\alpha \geq 0$. Standard arguments show that $\mathcal{F}_\alpha(n)$, $\alpha \geq 0$, are Banach spaces with respect to the following norms:

$$\|f\|_{\mathcal{F}_0(n)} = |f(0)| + \inf\{\|\mu\|_{\mathcal{M}(B_n)} : f - f(0) = F_0[\mu]\}, \quad f \in \mathcal{F}_0(n);$$
$$\|f\|_{\mathcal{F}_\alpha(n)} = \inf\{\|\mu\|_{\mathcal{M}(B_n)} : f = F_\alpha[\mu]\}, \quad f \in \mathcal{F}_\alpha(n), \ \alpha > 0.$$

Assume that $f = F_\alpha[\mu]$, $\alpha > 0$, for some positive measure $\mu \in \mathcal{M}(B_n)$. Let $f = F_\alpha[\rho]$, $\rho \in \mathcal{M}(B_n)$. Then $\|\rho\|_{\mathcal{M}(B_n)} \geq \rho(B_n) = f(0) = \|\mu\|_{\mathcal{M}(B_n)}$. Therefore, $\|f\|_{\mathcal{F}_\alpha(n)} = \|\mu\|_{\mathcal{M}(B_n)}$.

1.2. Multipliers. Assume that $X, Y \subset \mathcal{H}ol(B_n)$ are Banach spaces. A function $g \in \mathcal{H}ol(B_n)$ is called a (pointwise) multiplier from X to Y provided that $fg \in Y$ for all $f \in X$. Let $\mathfrak{M}(X, Y)$ denote the set of all multipliers from X to Y. Put $\mathfrak{M}(X) = \mathfrak{M}(X, X)$ and $\mathfrak{M}_\alpha(n) = \mathfrak{M}(\mathcal{F}_\alpha(n))$. $\mathfrak{M}_\alpha(n)$, $\alpha \geq 0$, is a Banach space with respect to the natural operator norm defined by the identity

$$\|g\|_{\mathfrak{M}_\alpha(n)} = \sup\{\|fg\|_{\mathcal{F}_\alpha(n)} : \|f\|_{\mathcal{F}_\alpha(n)} \leq 1\}, \quad g \in \mathfrak{M}_\alpha(n).$$

1.3. Families $\mathcal{F}_\alpha(n)$ and fractional Cauchy transforms. Let $\mathcal{M}(\partial B_n)$ denote the space of complex-valued Borel measures defined on the sphere ∂B_n. Let $\alpha \geq 0$. Given a measure $\rho \in \mathcal{M}(\partial B_n)$, its fractional Cauchy transform of order α is defined by the identity

$$K_\alpha[\rho](z) = \int_{\partial B_n} k_\alpha(\langle z, \zeta\rangle)\, d\rho(\zeta), \quad z \in B_n.$$

Put

$$\mathcal{K}_0(n) = \{f \in \mathcal{H}ol(B_n) : f - f(0) = K_0[\rho] \text{ for some } \rho \in \mathcal{M}(\partial B_n)\};$$
$$\mathcal{K}_\alpha(n) = \{K_\alpha[\rho] : \rho \in \mathcal{M}(\partial B_n)\}, \quad \alpha > 0.$$

$\mathcal{F}_\alpha(n)$, $\alpha \geq 0$, are Banach spaces with respect to the natural norms

$$\|f\|_{\mathcal{K}_0(n)} = |f(0)| + \inf\{\|\rho\|_{\mathcal{M}(\partial B_n)} : f - f(0) = K_0[\rho]\}, \quad f \in \mathcal{K}_0(n);$$
$$\|f\|_{\mathcal{K}_\alpha(n)} = \inf\{\|\rho\|_{\mathcal{M}(\partial B_n)} : f = K_\alpha[\rho]\}, \quad f \in \mathcal{K}_\alpha(n), \ \alpha > 0.$$

The families $\mathcal{K}_\alpha(n)$ and $\mathfrak{M}(\mathcal{K}_\alpha(n))$ are rather close to $\mathcal{F}_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$, respectively. In particular, $\mathcal{F}_\alpha(n) \subset \mathcal{K}_\alpha(n)$, $\mathcal{F}_\alpha(n) \neq \mathcal{K}_\alpha(n)$ and $\mathfrak{M}_\alpha(n) \subset \mathfrak{M}(\mathcal{K}_\alpha(n))$. The author does not know whether $\mathfrak{M}_\alpha(n)$ coincides with $\mathfrak{M}(\mathcal{K}_\alpha(n))$.

The classical spaces $\mathcal{K}_1(1)$ and $\mathfrak{M}(\mathcal{K}_1(1))$ are investigated in [3]. Various properties of the families $\mathcal{K}_\alpha(1)$ and $\mathfrak{M}(\mathcal{K}_\alpha(1))$ are collected in [8]. Certain results about $\mathcal{K}_\alpha(n)$ and $\mathfrak{M}(\mathcal{K}_\alpha(n))$ are obtained in [4–6] for arbitrary $n \in \mathbb{N}$. To the best of the author's knowledge, the spaces $\mathcal{F}_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$ have not been investigated systematically.

1.4. Organization of the paper. Definitions and auxiliary results are collected in Section 2. In Section 3 embedding properties for the families $\mathcal{F}_\alpha(n)$ and Bergman–Sobolev spaces are investigated. Radial derivatives of functions from $\mathcal{F}_\alpha(n)$ are studied in Section 4. It is shown in Section 5 that $\mathcal{F}_\alpha(n)$ is a proper subset of $\mathcal{K}_\alpha(n)$. The multiplier spaces $\mathfrak{M}_\alpha(n)$ are investigated in the final Sections 6 and 7. The main results are Proposition 7.1 and Theorem 7.7.

2. Preliminaries

2.1. Radial derivatives. Given $f \in \mathcal{H}ol(B_n)$, the radial derivative $\mathcal{R}f$ is defined by the identity

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z), \qquad z \in B_n.$$

It is well known that

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{R}f(tz)}{t}\, dt, \qquad z \in B_n. \tag{2.1}$$

Direct calculations show that

$$\mathcal{R}(fg) = f \cdot \mathcal{R}g + \mathcal{R}f \cdot g \quad \forall f, g \in \mathcal{H}ol(B_n).$$

Further, put

$$R = \mathcal{R} + I : \mathcal{H}ol(B_n) \to \mathcal{H}ol(B_n).$$

Note that

$$R(fg) = f \cdot Rg + \mathcal{R}f \cdot g \quad \forall f, g \in \mathcal{H}ol(B_n). \tag{2.2}$$

2.2. Operators of radial differentiation. Fractional analogs of the operator R are defined in terms of the homogeneous expansions of holomorphic functions. Namely, assume that $s, t \in \mathbb{R}$, $s > -n - 1$ and $s + t > -n - 1$. An invertible operator

$$D_s^t : \mathcal{H}ol(B_n) \to \mathcal{H}ol(B_n)$$

is defined as follows. If $f = \sum_{k=0}^\infty f_k$ is the homogeneous expansion of $f \in \mathcal{H}ol(B_n)$, then

$$D_s^t f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s)\Gamma(n+1+k+s+t)}{\Gamma(n+1+s+t)\Gamma(n+1+k+s)} f_k(z), \qquad z \in B_n. \tag{2.3}$$

Note that, under the above restrictions on s and t, definition (2.3) coincides with those in [12, Section 1.4] and [9].

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We have

$$(D_s^t)^{-1} = D_{s+t}^{-t} : \mathcal{H}ol(B_n) \to \mathcal{H}ol(B_n),$$

where D_{s+t}^{-t} is defined by formula (2.3). Finally, observe that $R = \mathcal{R} + I = D_{-n}^1$.

3. Embedding properties

3.1. Weighted Bergman spaces. Recall that ν_n denotes the normalized Lebesgue measure on the ball B_n. For $p, q > 0$, the weighted Bergman space $A_q^p(B_n)$ consists of those $f \in \mathcal{H}ol(B_n)$ for which

$$\|f\|_{A_q^p(B_n)}^p = \int_{B_n} |f(z)|(1-|z|)^{q-1}\, d\nu_n(z) < \infty.$$

PROPOSITION 3.1. *Suppose that $n \in \mathbb{N}$.*

(i) *If $\alpha > n$, then $A_{\alpha-n}^1(B_n) \subset \mathcal{F}_\alpha(n)$.*

(ii) *If $\beta > \alpha \geq 0$ and $\beta > n$, then $\mathcal{F}_\alpha(n) \subset A_{\beta-n}^1(B_n)$.*

PROOF. (i) If $f \in A_{\alpha-n}^1(B_n)$, $\alpha > n$, then [12, Theorem 2.2] guarantees that

$$f(z) = c_\alpha \int_{B_n} \frac{f(w)(1-|w|^2)^{\alpha-n-1}\, d\nu_n(w)}{(1-\langle z, w\rangle)^\alpha}$$

for all $z \in B_n$. By the assumption,

$$f(w)(1-|w|^2)^{\alpha-n-1}\, d\nu_n(w) \in \mathcal{M}(B_n).$$

Therefore, $A_{\alpha-n}^1(B_n) \subset \mathcal{F}_\alpha(n)$ for $\alpha > n$.

(ii) Assume that $\alpha > 0$, $\mu \in \mathcal{M}(B_n)$ and

$$f(z) = F_\alpha[\mu](z) = \int_{B_n} \frac{d\mu(w)}{(1-\langle z, w\rangle)^\alpha}, \quad z \in B_n.$$

If $\beta > n$ and $\beta > \alpha$, then, changing the order of integration, we obtain

$$\begin{aligned}\int_{B_n} |f(z)|(1-|z|)^{\beta-n-1}\, d\nu_n(z) &\leq \int_{B_n}\int_{B_n} \frac{(1-|z|)^{\beta-n-1}\, d\nu_n(z)}{|1-\langle z, w\rangle|^\alpha}\, d|\mu|(w)\\ &\leq C\int_{B_n} d|\mu|(w) < \infty\end{aligned}$$

by [10, Proposition 1.4.10]. So $\mathcal{F}_\alpha(n) \subset A_{\beta-n}^1(B_n)$. The proof of the embedding $\mathcal{F}_0(n) \subset A_{\beta-n}^1(B_n)$, $\beta > n$, is analogous. $\square$

3.2. Bergman–Sobolev spaces. Suppose that $p, q > 0$ and $j \in \mathbb{Z}_+$. By definition, the Bergman–Sobolev space $A_{q,j}^p(B_n)$ consists of those $f \in \mathcal{H}ol(B_n)$ for which

$$\|f\|_{A_{q,j}^p(B_n)} = \int_{B_n} |R^j f(z)|^p(1-|z|)^{q-1}\, d\nu_n(z) < \infty.$$

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For $p \geq 1$, $A^p_{q,j}(B_n)$ is a Banach space. Note that $A^p_{q,0}(B_n)$ coincides with the weighted Bergman space $A^p_q(B_n)$. Basic properties of the general spaces $A^p_{q,s}(B_n)$, $s \in \mathbb{R}$, are gathered together in [2].

PROPOSITION 3.2. *Assume that $n \in \mathbb{N}$, $j \in \{0, 1, \ldots, n\}$ and $\alpha > n - j$.*

(i) *The embedding $A^1_{\alpha-n+j,j}(B_n) \subset \mathcal{F}_\alpha(n)$ holds.*

(ii) *If $\beta > \alpha$, then $\mathcal{F}_\alpha(n) \subset A^1_{\beta-n+j,j}(B_n)$.*

PROOF. If $j = 0$, then Proposition 3.1 applies. We assume that $j \in \{1, \ldots, n\}$.

(i) Let $f \in A^1_{\alpha-n+j,j}(B_n)$. We have

$$D^j_{\alpha-n-1}(A^1_{\alpha-n+j,j}(B_n)) = A^1_{\alpha-n+j}(B_n)$$

by [9, Proposition 3.2]. (Note that in [9], Bergman–Sobolev spaces are called Besov spaces.) Therefore, $D^j_{\alpha-n-1} f \in A^1_{\alpha-n+j}(B_n) \subset \mathcal{F}_{\alpha+j}(n)$ by Proposition 3.1(i). So,

$$D^j_{\alpha-n-1} f(z) = \int_{B_n} \frac{d\mu(w)}{(1 - \langle z, w\rangle)^{\alpha+j}}, \quad z \in B_n,$$

for some $\mu \in \mathcal{M}(B_n)$. We have $\alpha > 0$, hence, the inverse operator $(D^j_{\alpha-n-1})^{-1} = D^{-j}_{\alpha+j-n-1}$ is correctly defined by (2.3). Next, [12, Proposition 1.14] guarantees that

$$D^{-j}_{\alpha+j-n-1} \frac{1}{(1 - \langle z, w\rangle)^{\alpha+j}} = \frac{1}{(1 - \langle z, w\rangle)^{\alpha}}, \quad z, w \in B_n.$$

Therefore,

$$f(z) = D^{-j}_{\alpha+j-n-1} D^j_{\alpha-n-1} f(z) = \int_{B_n} \frac{d\mu(w)}{(1 - \langle z, w\rangle)^{\alpha}}, \quad z \in B_n.$$

In other words, $f \in \mathcal{F}_\alpha(n)$.

(ii) Assume that $j \in \{1, \ldots, n\}$, $\alpha > n - j$ and

$$f(z) = \int_{B_n} \frac{d\mu(w)}{(1 - \langle z, w\rangle)^{\alpha}}, \quad z \in B_n,$$

for some $\mu \in \mathcal{M}(B_n)$. For $z \in B_n$, straightforward calculations show that

$$Rf(z) = \mathcal{R} f(z) + f(z) = \int_{B_n} \frac{\alpha\, d\mu(w)}{(1 - \langle z, w\rangle)^{\alpha+1}} + (1 - \alpha) f(z) \in \mathcal{F}_{\alpha+1}(n) + \mathcal{F}_\alpha(n).$$

By induction,

$$R^j f \in \mathcal{F}_{\alpha+j}(n) + \cdots + \mathcal{F}_\alpha(n). \tag{3.1}$$

Now, suppose that $\beta > \alpha$. Note that $\beta + j > \alpha + j$ and $\beta + j > n$, hence,

$$\mathcal{F}_{\alpha+j}(n) + \cdots + \mathcal{F}_\alpha(n) \subset A^1_{\beta+j-n}(B_n)$$

by Proposition 3.1(ii). So (3.1) guarantees that $f \in A^1_{\beta+j-n,j}(B_n)$. □

For $\alpha = 0$, we have the following analog of Proposition 3.2(ii).

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PROPOSITION 3.3. *Let $n \in \mathbb{N}$. Then $\mathcal{F}_0(n) \subset A^1_{\varepsilon,n}(B_n)$ for any $\varepsilon > 0$.*

PROOF. Let $\mu \in \mathcal{M}(B_n)$ and let

$$f(z) = f(0) + \int_{B_n} \log \frac{1}{1 - \langle z, w\rangle}\, d\mu(w), \quad z \in B_n.$$

For $z \in B_n$, direct calculations show that

$$\begin{aligned} Rf(z) = \mathcal{R}f(z) + f(z) &= \int_{B_n} \frac{1}{1 - \langle z, w\rangle}\, d\mu(w) + C + \log \frac{1}{1 - \langle z, w\rangle}\, d\mu(w) \\ &\in \mathcal{F}_1(n) + \mathcal{F}_0(n). \end{aligned}$$

Therefore,

$$R^n f \in \mathcal{F}_n(n) + \cdots + \mathcal{F}_0(n) \subset A^1_\varepsilon(B_n)$$

by Proposition 3.1(ii) with $\beta = \varepsilon + n > n$. So $f \in A^1_{\varepsilon,n}(B_n)$. □

COROLLARY 3.4. *Let $n \in \mathbb{N}$ and let $0 \le \alpha < \beta$. Then $\mathcal{F}_\alpha(n) \subset \mathcal{F}_\beta(n)$.*

PROOF. Applying Proposition 3.2 with $j = n$ and Proposition 3.3, we obtain $\mathcal{F}_\alpha(n) \subset A^1_{\beta,n}(B_n) \subset \mathcal{F}_\beta(n)$ for $0 \le \alpha < \beta$. □

4. Families $\mathcal{F}_\alpha(n)$ and radial derivatives

The proof of the following proposition is similar to that of [6, Proposition 4.2].

PROPOSITION 4.1. *Suppose that $\alpha \ge 0$, $n \in \mathbb{N}$ and $f \in \mathcal{H}ol(B_n)$. Then $f \in \mathcal{F}_\alpha(n)$ if and only if $\mathcal{R}f \in \mathcal{F}_{\alpha+1}(n)$.*

PROOF. Let $f \in \mathcal{F}_\alpha(n)$. Assume that $\alpha = 0$. Then

$$f(z) = f(0) + \int_{B_n} \log \frac{1}{1 - \langle z, w\rangle}\, d\mu(w), \quad z \in B_n,$$

for some $\mu \in \mathcal{M}(B_n)$. Direct calculations show that

$$\mathcal{R}f(z) = \int_{B_n} \frac{\langle z, w\rangle}{1 - \langle z, w\rangle}\, d\mu(w) = F_1[\rho](z), \quad z \in B_n,$$

where $\rho = \mu - \mu(B_n)\nu_n$. So $\mathcal{R}f \in \mathcal{F}_1(n)$.

Now assume that $\alpha > 0$. Then

$$f(z) = \int_{B_n} \frac{1}{(1 - \langle z, w\rangle)^\alpha}\, d\mu(w), \quad z \in B_n,$$

for some $\mu \in \mathcal{M}(B_n)$. We have

$$\mathcal{R}f(z) = \int_{B_n} \frac{\alpha}{(1 - \langle z, w\rangle)^{\alpha+1}}\, d\mu(w) - \alpha f(z), \quad z \in B_n.$$

Note that $\alpha f \in \mathcal{F}_\alpha(n) \subset \mathcal{F}_{\alpha+1}(n)$ by Corollary 3.4, thus $\mathcal{R}f \in \mathcal{F}_{\alpha+1}(n)$.

It is convenient to prove the converse implication in two steps.

Step 1. $\alpha = m \in \mathbb{N} \cup \{0\}$. By the hypothesis,

$$\mathcal{R}f(z) = \int_{B_n} \frac{1}{(1-\langle z, w\rangle)^{m+1}}\, d\mu(w), \quad z \in B_n.$$

Applying (2.1), we obtain

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{R}f(tz)}{t}\, dt = \int_0^1 \int_{B_n} \frac{1}{t(1-t\langle z, w\rangle)^{m+1}}\, d\mu(w)\, dt, \quad z \in B_n.$$

If $\lambda \in \mathbb{C}$ and $|\lambda| < 1$, then

$$\frac{1}{t(1-t\lambda)^{m+1}} = \frac{1}{t} + \sum_{j=1}^{m+1} \frac{\lambda}{(1-t\lambda)^j}.$$

Put $\lambda = \langle z, w\rangle$. Note that $\mathcal{R}f(0) = 0$, thus $\mu(B_n) = 0$. Hence, changing the order of integration, we obtain

$$\begin{aligned} f(z) - f(0) &= \sum_{j=1}^{m+1} \int_{B_n} \int_0^1 \frac{\langle z, w\rangle}{(1-t\langle z, w\rangle)^j}\, dt\, d\mu(w) \\ &= \int_{B_n} \log \frac{1}{1-\langle z, w\rangle}\, d\mu(w) \\ &\quad + \sum_{j=2}^{m+1} \int_{B_n} \frac{1}{(j-1)(1-\langle z, w\rangle)^{j-1}}\, d\mu(w) \\ &\in \mathcal{F}_0(n) + \cdots + \mathcal{F}_m(n) \subset \mathcal{F}_m(n) \end{aligned}$$

by Corollary 3.4. Recall that $1 \in \mathcal{F}_m(n)$, hence $f \in \mathcal{F}_m(n)$.

Step 2. $\alpha > 0$, $\alpha \notin \mathbb{N}$. Repeating the arguments used in Step 1 and changing the order of integration,

$$\begin{aligned} f(z) - f(0) &= \int_0^1 \int_{B_n} \frac{1}{t(1-t\langle z, w\rangle)^{\alpha+1}}\, d\mu(w)\, dt \\ &= \int_{B_n} \int_0^1 \frac{\langle z, w\rangle}{(1-t\langle z, w\rangle)^{\alpha+1}}\, dt\, d\mu(w) \\ &\quad + \int_0^1 \frac{1}{t} \int_{B_n} \frac{1}{(1-t\langle z, w\rangle)^{\alpha}}\, d\mu(w)\, dt. \end{aligned}$$

The inner integral in the first summand is explicitly calculable. So, consider the second summand. Put $[\alpha] = m \in \mathbb{N} \cup \{0\}$. We have $m + 1 > \alpha$, hence, by Corollary 3.4, there exists a measure $\rho \in \mathcal{M}(B_n)$ such that $F_{m+1}[\rho] = F_\alpha[\mu]$. Also, we have

$$\rho(B_n) = F_{m+1}[\rho](0) = F_\alpha[\mu](0) = \mu(B_n) = 0.$$

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Therefore,

$$f(z) - f(0) = \frac{1}{\alpha}\int_{B_n} \frac{1}{(1-\langle z, w\rangle)^\alpha}\, d\mu(w) + \int_0^1 \int_{B_n} \frac{1}{t(1-t\langle z, w\rangle)^{m+1}}\, d\rho(w)\, dt$$
$$\in \mathcal{F}_\alpha(n) + \mathcal{F}_m(n).$$

The latter property is obtained in Step 1. Recall that $\alpha > m$, hence $f \in \mathcal{F}_\alpha(n) + \mathcal{F}_m(n) \subset \mathcal{F}_\alpha(n)$ by Corollary 3.4. □

THEOREM 4.2. *Assume that $\alpha \geq 0$, $n \in \mathbb{N}$ and $f \in \mathcal{H}ol(B_n)$. Then the following properties are equivalent:*

(i) $f \in \mathcal{F}_\alpha(n)$;
(ii) $\mathcal{R} f \in \mathcal{F}_{\alpha+1}(n)$;
(iii) $R f \in \mathcal{F}_{\alpha+1}(n)$.

PROOF. Proposition 4.1 guarantees that (i) holds if and only if (ii) holds. Next, (i) and (ii) imply (iii). Finally, assume that (iii) holds. Then

$$R f \in \mathcal{F}_{\alpha+1}(n) \subset \mathcal{F}_{\alpha+3/2}(n) \subset A^1_{\alpha+1,n-1}(B_n)$$

by Proposition 3.2(ii) with $j = n-1$. Therefore, $f \in A^1_{\alpha+1,n}(B_n) \subset \mathcal{F}_{\alpha+1}(n)$ by Proposition 3.2(i). So $\mathcal{R} f = R f - f \in \mathcal{F}_{\alpha+1}(n)$, that is, (iii) implies (ii). □

5. Spaces $\mathcal{F}_\alpha(n)$ and $\mathcal{K}_\alpha(n)$

It is shown in [6] that analogs of Proposition 3.2, Corollary 3.4 and Theorem 4.2 hold for the spaces $\mathcal{K}_\alpha(n)$ of fractional Cauchy transforms. Note that the arguments used in Sections 3 and 4 provide alternative proofs of those analogs. Also, the following assertion implies certain embedding properties.

PROPOSITION 5.1. *Let $n \in \mathbb{N}$ and let $\alpha \geq 0$. Then $\mathcal{F}_\alpha(n) \subset \mathcal{K}_\alpha(n)$.*

PROOF. Suppose that $\alpha > 0$, $\mu \in \mathcal{M}(B_n)$ and

$$f(z) = \int_{B_n} \frac{d\mu(w)}{(1-\langle z, w\rangle)^\alpha}, \qquad z \in B_n.$$

Note that $\mu \in \mathcal{M}(\overline{B}_n)$, where $\mathcal{M}(\overline{B}_n)$ denotes the space of complex-valued Borel measures defined on the closed ball $\overline{B}_n$. Without loss of generality, assume that μ is a probability measure. Let δ_ξ denote the point mass at $\xi \in \mathbb{C}^n$. By the Banach–Alaoglu theorem, there exist probability measures $\mu_k = \sum_{j=1}^{J(k)} a_{k,j}\delta_{\xi_{k,j}}$, $\xi_{k,j} \in \overline{B}_n$, such that $\mu_k \to \mu$ in the weak* topology of $\mathcal{M}(\overline{B}_n)$. Therefore,

$$\sum_{j=1}^{J(k)} \frac{a_{k,j}}{(1-\langle z, \xi_{j,k}\rangle)^\alpha} \to f(z), \qquad \text{as } k \to \infty,$$

for all $z \in B_n$.

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Let $k \in \mathbb{N}$. If $\xi_{k,j} \in \partial B_n$, then put $\rho_{k,j} = \delta_{\xi_{k,j}} \in \mathcal{M}(\partial B_n)$. Thus,

$$\frac{1}{(1-\langle z, \xi_{k,j}\rangle)^\alpha} = K_\alpha[\rho_{k,j}](z), \quad z \in B_n.$$

Further, assume that $\xi_{k,j} \in B_n$. Given $z \in B_n$,

$$h(w) = \frac{1}{(1-\langle z, w\rangle)^\alpha} \in \overline{\mathcal{H}ol}(B_n) \cap C(\overline{B}_n), \quad w \in B_n.$$

Hence,

$$\frac{1}{(1-\langle z, w\rangle)^\alpha} = \int_{\partial B_n} \frac{1}{(1-\langle z, \zeta\rangle)^\alpha} \frac{1-|w|^2}{|w-\zeta|^{2n}}\, d\sigma_n(\zeta), \quad z, w \in B_n.$$

In other words, if $\xi_{k,j} \in B_n$, then

$$\frac{1}{(1-\langle z, \xi_{k,j}\rangle)^\alpha} = K_\alpha[\rho_{k,j}](z), \quad z \in B_n,$$

for a probability measure $\rho_{k,j} \in \mathcal{M}(\partial B_n)$. Put

$$\rho_k = \sum_{j=1}^{J(k)} a_{k,j}\rho_{k,j}.$$

Then $\rho_k \in \mathcal{M}(\partial B_n)$ is a probability measure and

$$K_\alpha[\rho_k](z) = \sum_{j=1}^{J(k)} \frac{a_{k,j}}{(1-\langle z, \xi_{j,k}\rangle)^\alpha} \to f(z), \quad \text{as } k \to \infty, \tag{5.1}$$

for all $z \in B_n$. By the Banach–Alaoglu theorem, there exists a subsequence ρ_{k_m} such that $\rho_{k_m} \to \rho$ in the weak* topology of $\mathcal{M}(\partial B_n)$. Therefore,

$$K_\alpha[\rho_{k_m}](z) \to K_\alpha[\rho](z), \quad \text{as } k_m \to \infty,$$

for all $z \in B_n$. Hence, $f = K_\alpha[\rho]$ by (5.1).

The proof for $\alpha = 0$ is analogous. □

Below we show that $\mathcal{F}_\alpha(n) \neq \mathcal{K}_\alpha(n)$ for all $\alpha \geq 0$.

Given $\xi \in \partial B_n$ and $C > 1$, recall that the Korányi approach region $D_C(\xi)$ is defined by the identity

$$D_C(\xi) = \{z \in B_n : |1-\langle z, \xi\rangle| < C(1-|z|)\}.$$

PROPOSITION 5.2. *Suppose that $\alpha > 0$, $n \in \mathbb{N}$, $\mu \in \mathcal{M}(B_n)$, $\xi \in \partial B_n$ and $C > 1$. Then*

$$\lim_{\substack{z\to\xi \\ z\in D_C(\xi)}} (1-\langle z, \xi\rangle)^\alpha F_\alpha[\mu](z) = 0.$$

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PROOF. Assume that $w \in B_n$ and $\xi \in \partial B_n$. Then

$$\lim_{\substack{z \to \xi \\ z \in B_n}} \frac{(1 - \langle z, \xi\rangle)^\alpha}{(1 - \langle z, w\rangle)^\alpha} = 0.$$

If $z \in D_C(\xi)$, then

$$\left|\frac{1 - \langle z, \xi\rangle}{1 - \langle z, w\rangle}\right|^\alpha \le \frac{|1 - \langle z, \xi\rangle|^\alpha}{(1 - |z|)^\alpha} \le C^\alpha.$$

Hence,

$$\lim_{\substack{z \to \xi \\ z \in D_C(\xi)}} (1 - \langle z, \xi\rangle)^\alpha F_\alpha[\mu](z) = \lim_{\substack{z \to \xi \\ z \in D_C(\xi)}} \int_{B_n} \frac{(1 - \langle z, \xi\rangle)^\alpha}{(1 - \langle z, w\rangle)^\alpha}\, d\mu(w) = 0$$

by the dominated convergence theorem. $\square$

Assume that $\alpha > 0$, $n \in \mathbb{N}$, $\rho \in \mathcal{M}(\partial B_n)$, $\xi \in \partial B_n$ and $C > 1$. Then, by [6, Proposition 6.1],

$$\lim_{\substack{z \to \xi \\ z \in D_C(\xi)}} (1 - \langle z, \xi\rangle)^\alpha K_\alpha[\rho](z) = \rho(\{\xi\}).$$

Therefore, if $\rho(\{\xi\}) \ne 0$ for some point $\xi \in \partial B_n$, then $K_\alpha[\rho] \in \mathcal{K}_\alpha(n) \setminus \mathcal{F}_\alpha(n)$. Similar arguments show that $\mathcal{F}_0(n) \ne \mathcal{K}_0(n)$.

6. Multipliers: necessary conditions

PROPOSITION 6.1. *Assume that $n \in \mathbb{N}$, $0 \le \alpha \le \beta$ and $g \in \mathcal{H}ol(B_n)$. Then the following properties are equivalent.*

(i) $g \in \mathfrak{M}(\mathcal{K}_\alpha(n), \mathcal{K}_\beta(n))$.
(ii) $g \in \mathfrak{M}(\mathcal{F}_\alpha(n), \mathcal{K}_\beta(n))$.
(iii) $g(z)k_\alpha(\langle z, w\rangle) \in \mathcal{K}_\beta(n)$ *for all* $w \in B_n$, *and*

$$\sup_{w \in B_n} \|g(z)k_\alpha(\langle z, w\rangle)\|_{\mathcal{K}_\beta(n)} \le C < \infty;$$

if $\alpha = 0$, *then it is also required that* $g \in \mathcal{K}_\beta(n)$.
(iv) $g(z)k_\alpha(\langle z, \zeta\rangle) \in \mathcal{K}_\beta(n)$ *for all* $\zeta \in \partial B_n$, *and*

$$\sup_{\zeta \in \partial B_n} \|g(z)k_\alpha(\langle z, \zeta\rangle)\|_{\mathcal{K}_\beta(n)} \le C < \infty;$$

if $\alpha = 0$, *then it is also required that* $g \in \mathcal{K}_\beta(n)$.

PROOF. By Proposition 5.1, (i) implies (ii). Let (ii) hold. Note that

$$\|k_\alpha(\langle z, w\rangle)\|_{\mathcal{F}_\alpha(n)} \le 1, \quad \forall w \in B_n,$$

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hence (iii) holds by the closed graph theorem. If $\alpha = 0$, then $g \in \mathcal{K}_\beta(n)$, since $1 \in \mathcal{F}_0(n)$.

Now, let (iii) hold and let $\zeta \in \partial B_n$. Fix a sequence $\{w_j\}_{j=1}^{\infty} \subset B_n$ such that $w_j \to \zeta$. Note that

$$g(z)k_\alpha(\langle z, w_j\rangle) \to g(z)k_\alpha(\langle z, \zeta\rangle), \quad \text{as } j \to \infty, \tag{6.1}$$

for all $z \in B_n$. By property (iii), there exist measures $\rho_j \in \mathcal{M}(\partial B_n)$ such that $\|\rho_j\|_{\mathcal{M}(\partial B_n)} \le C$ and

$$g(z)k_\alpha(\langle z, w_j\rangle) = \int_{\partial B_n} k_\beta(\langle z, \zeta\rangle)\, d\rho_j(\zeta), \quad z \in B_n.$$

By the Banach–Alaoglu theorem, there exists a subsequence ρ_{j_m} such that $\rho_{j_m} \to \rho$ in the weak* topology of $\mathcal{M}(\partial B_n)$. Note that $\|\rho\|_{\mathcal{M}(\partial B_n)} \le C$. So

$$g(z)k_\alpha(\langle z, w_{j_m}\rangle) \to \int_{\partial B_n} k_\beta(\langle z, \zeta\rangle)\, d\rho(\zeta), \quad \text{as } j_m \to \infty,$$

for all $z \in B_n$. By (6.1), we obtain

$$\|g(z)k_\alpha(\langle z, \zeta\rangle)\|_{\mathcal{K}_\beta(n)} \le \|\rho\|_{\mathcal{M}(\partial B_n)} \le C.$$

Finally, standard arguments, based on approximation by discrete measures, show that (iv) implies (i) (see [11, Theorem 1], [7, Lemma 2.1]). □

COROLLARY 6.2. *Assume that $n \in \mathbb{N}$ and $0 \le \alpha \le \beta$. Then*

$$\mathfrak{M}(\mathcal{F}_\alpha(n), \mathcal{F}_\beta(n)) \subset \mathfrak{M}(\mathcal{K}_\alpha(n), \mathcal{K}_\beta(n)).$$

In particular, $\mathfrak{M}_\alpha(n) \subset \mathfrak{M}(\mathcal{K}_\alpha(n))$.

PROOF. We have $\mathcal{F}_\beta(n) \subset \mathcal{K}_\beta(n)$, hence

$$\mathfrak{M}(\mathcal{F}_\alpha(n), \mathcal{F}_\beta(n)) \subset \mathfrak{M}(\mathcal{F}_\alpha(n), \mathcal{K}_\beta(n)) = \mathfrak{M}(\mathcal{K}_\alpha(n), \mathcal{K}_\beta(n))$$

by Proposition 6.1. □

Standard arguments guarantee that $\mathfrak{M}_\alpha(n) \subset H^\infty(B_n)$ for all $\alpha \ge 0$. Moreover, if $g \in \mathfrak{M}_\alpha(n)$, $\alpha > 0$, $n \in \mathbb{N}$, then

$$\sup_{\xi \in \partial B_n} \int_0^1 |\mathcal{R}g(r\xi)|\, dr < +\infty$$

by Proposition 6.1 and [6, Proposition 7.3].

COROLLARY 6.3. *Suppose that $n \in \mathbb{N}$, $M \in \{1, \ldots, n\}$, $n \ge \alpha > n - M$ and that $g \in \mathfrak{M}_\alpha(n)$. Then*

$$\sup_{\zeta \in \partial B_n} \int_{B_n} \frac{|R^k g(z)|(1 - |z|)^{\alpha+M-n-1}}{|1 - \langle z, \zeta\rangle|^{\alpha+M-k}} \left(\log \frac{e}{1 - |z|}\right)^{-1-\varepsilon} d\nu_n(z) < \infty \tag{6.2}$$

for $k = 1, \ldots, M$ and any $\varepsilon > 0$.

PROOF. We apply Proposition 6.1 and [5, Theorem 1.2]. □

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7. Multipliers: sufficient conditions

Suppose that $n \in \mathbb{N}$, $\alpha \geq 0$ and $g \in \mathfrak{M}_\alpha(n)$. Note that $g \in H^\infty(B_n)$, hence

$$\sup_{w \in B_n} \|(g(z) - g(w))k_\alpha(\langle z, w\rangle)\|_{\mathcal{F}_\alpha(n)} < \infty \tag{7.1}$$

by the closed graph theorem.

7.1. Spaces $\mathfrak{M}_\alpha(n)$ with $\alpha > n$. The following proposition shows that the necessary condition (7.1) becomes a sufficient one, for $\alpha > n$, when the norm of $\mathcal{F}_\alpha(n)$ is replaced by the norm of the weighted Bergman space $A^1_{\alpha-n}(B_n)$.

PROPOSITION 7.1. *Assume that $n \in \mathbb{N}$, $\alpha > n$ and $g \in H^\infty(B_n)$. Suppose that*

$$\sup_{w \in B_n} \left\| \frac{g(z) - g(w)}{(1 - \langle z, w\rangle)^\alpha} \right\|_{A^1_{\alpha-n}(B_n)} \leq C < \infty. \tag{7.2}$$

Then $g \in \mathfrak{M}_\alpha(n)$.

We will need two auxiliary lemmas.

LEMMA 7.2. *Assume that μ and μ_k, $k \in \mathbb{N}$, are probability measures on B_n. Suppose that $\mu_k \to \mu$ in the weak* topology of $\mathcal{M}(B_n)$. Let $\varepsilon > 0$. Then there exists an $r = r(\varepsilon) \in (0, 1)$ such that $\mu(B_n \backslash rB_n) < \varepsilon$ and $\mu_k(B_n \backslash rB_n) < \varepsilon$ for all $k \in \mathbb{N}$.*

PROOF. We have $\mu(B_n) < \infty$, hence $\mu(B_n \backslash RB_n) < \varepsilon$ for some $R \in (0, 1)$. Put $r = (1 + R)/2$ and consider a function $f \in C_0(rB_n)$ such that $0 \leq f \leq 1$ and $f|_{RB_n} \equiv 1$. By the definition of weak* convergence, we obtain

$$\mu_k(rB_n) \geq \int_{B_n} f \, d\mu_k \to \int_{B_n} f \, d\mu \geq \mu(RB_n) > 1 - \varepsilon.$$

Therefore, $\mu_k(B_n \backslash rB_n) < \varepsilon$ for all sufficiently large k. Hence, the required property holds. □

LEMMA 7.3. *Let $\mu_k \in \mathcal{M}(B_n)$, $k \in \mathbb{N}$. Assume that, for any $\varepsilon > 0$, there exists an $r \in (0, 1)$ such that $|\mu_k|(B_n \backslash rB_n) < \varepsilon$ for all $k \in \mathbb{N}$. Let $\mu_k \to \mu$ in the weak* topology of $\mathcal{M}(B_n)$. Then*

$$\int_{B_n} f \, d\mu_k \to \int_{B_n} f \, d\mu, \quad \text{as } k \to \infty,$$

for any bounded function $f \in C(B_n)$.

PROOF. Assume that $f \in C(B_n)$ and $|f(z)| \leq 1$ for all $z \in B_n$.

Fix an $\varepsilon > 0$. We have $|\mu|(B_n) < \infty$, thus, applying the hypotheses of the lemma, choose $r \in (0, 1)$ such that $|\mu|(B_n \backslash rB_n) < \varepsilon$ and $|\mu_k|(B_n \backslash rB_n) < \varepsilon$ for

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all $k \in \mathbb{N}$. Consider a function $f_0 \in C_0(B_n)$ such that $f_0(z) = f(z)$ for all $z \in rB_n$, and $|f(z)| \le 1$ for all $z \in B_n$. By the definition of the weak* topology of $\mathcal{M}(B_n)$,

$$\left| \int_{B_n} f_0 \, d\mu_k - \int_{B_n} f_0 \, d\mu \right| < \varepsilon$$

for all $k \ge k_0$. Also note that

$$\int_{B_n} |f - f_0| \, d(|\mu_k| + |\mu|) \le 2 \int_{B_n \setminus rB_n} d(|\mu_k| + |\mu|) < 4\varepsilon.$$

Therefore,

$$\left| \int_{B_n} f \, d\mu_k - \int_{B_n} f \, d\mu \right| < 5\varepsilon$$

for all $k \ge k_0$. □

Proof of Proposition 7.1. Let $f = F_\alpha[\mu]$, $\alpha > n$. We have to prove that $fg \in \mathcal{F}_\alpha(n)$. Without loss of generality, assume that $\mu \in \mathcal{M}(B_n)$ is a probability measure. Applying the Banach–Alaoglu theorem, select a sequence of probability measures $\mu_k = \sum_{j=1}^{J(k)} a_{k,j} \delta_{w_{k,j}}$, $w_{k,j} \in B_n$, such that $\mu_k \to \mu$ in the weak* topology of $\mathcal{M}(B_n)$.

If $z \in B_n$, then $(1 - \langle z, \cdot \rangle)^{-\alpha} \in C(\overline{B}_n)$. Hence, Lemmas 7.2 and 7.3 guarantee that

$$g(z) \sum_{j=1}^{J(k)} \frac{a_{k,j}}{(1 - \langle z, w_{k,j} \rangle)^\alpha} \to g(z) \int_{B_n} \frac{d\mu(w)}{(1 - \langle z, w \rangle)^\alpha} = g(z) f(z) \tag{7.3}$$

for all $z \in B_n$. Consider the measures $\rho_k = \sum_{j=1}^{J(k)} a_{k,j} g(w_{k,j}) \delta_{w_{k,j}}$. Note that $g \in H^\infty(B_n)$ and the measures $\mu_k \in \mathcal{M}(B_n)$ satisfy the hypotheses of Lemma 7.3, thus the measures $\rho_k \in \mathcal{M}(B_n)$ also satisfy the hypotheses of Lemma 7.3.

Remark that $\|\rho_k\| \le \|g\|_{H^\infty(B_n)}$, thus, by the Banach–Alaoglu theorem, there exists a subsequence ρ_{k_m} which converges in the weak* topology. Without loss of generality, assume that $\rho_k \to \rho$ in the weak* topology of $\mathcal{M}(B_n)$. Lemma 7.3 guarantees that

$$\sum_{j=1}^{J(k)} \frac{a_{k,j} g(w_{k,j})}{(1 - \langle z, w_{k,j} \rangle)^\alpha} \to \int_{B_n} \frac{d\rho(w)}{(1 - \langle z, w \rangle)^\alpha}, \quad \text{as } k \to \infty, \tag{7.4}$$

for all $z \in B_n$. Put

$$h_k(z) = \sum_{j=1}^{J(k)} \frac{a_{k,j}(g(z) - g(w_{k,j}))}{(1 - \langle z, w_{k,j} \rangle)^\alpha}. \tag{7.5}$$

Recall that $a_{k,j} > 0$ and $\sum_{j=1}^{J(k)} a_{k,j} = 1$, hence $\|h_k\|_{A^1_{\alpha-n}(B_n)} \le C$ by (7.2). Therefore, the sequence $\{h_k\}_{k=1}^\infty$ is uniformly bounded on compact subsets of the ball B_n. Thus, there exists a subsequence which converges to a holomorphic function uniformly on compact subsets. Without loss of generality, assume that $h_k(z) \to h(z) \in \mathcal{H}ol(B_n)$ for all $z \in B_n$. Fatou's theorem guarantees that $h \in A^1_{\alpha-n}(B_n)$.

Consider the limit as $k \to \infty$ in identity (7.5). Applying (7.3) and (7.4), we obtain

$$f(z)g(z) = \int_{B_n} \frac{d\rho(w)}{(1-\langle z, w\rangle)^\alpha} + h(z), \quad z \in B_n.$$

Recall that $A^1_{\alpha-n}(B_n) \subset \mathcal{F}_\alpha(n)$ by Proposition 3.1(i), thus $fg \in \mathcal{F}_\alpha(n)$. □

Let $\delta > 0$. The holomorphic Lipschitz space $\Lambda_\delta(B_n)$ consists of those $g \in \mathcal{H}ol(B_n)$ for which

$$|R^j g(z)| \le C(1-|z|)^{\delta-j}, \quad z \in B_n, \tag{7.6}$$

where j is the least integer such that $j > \delta$. It is well known that, replacing R by $\mathcal{R}$ in estimate (7.6), we obtain an equivalent definition of the space $\Lambda_\delta(B_n)$. In particular, if $0 < \delta < 1$, then the space $\Lambda_\delta(B_n)$ is defined by the following property:

$$|\mathcal{R}g(z)| \le C(1-|z|)^{\delta-1}, \quad z \in B_n.$$

We will need the following lemma.

LEMMA 7.4 [4, Lemma 2.7]. *Assume that $n \in \mathbb{N}$, $0 < \delta < 1$ and $g \in \Lambda_\delta(B_n)$. Then g extends continuously to the closed ball $\overline{B}_n$. Moreover, if $0 < \tau < \min\{1/2, \delta\}$, then there exists a constant $C > 0$ such that*

$$|g(z) - g(w)| \le C|1-\langle z, w\rangle|^\tau \quad \forall z, w \in \overline{B}_n. \tag{7.7}$$

COROLLARY 7.5. *Suppose that $n \in \mathbb{N}$, $\alpha > n$ and $\delta > 0$. Then $\Lambda_\delta(B_n) \subset \mathfrak{M}_\alpha(n)$.*

PROOF. Without loss of generality, assume that $0 < \delta < 1/2$. Let $g \in \Lambda_\delta(B_n)$. Lemma 7.4 guarantees that inequality (7.7) holds for some $\tau \in (0, 1/2)$. Hence,

$$\begin{aligned}\sup_{w\in B_n} \int_{B_n} &\frac{|g(z)-g(w)|}{|1-\langle z, w\rangle|^\alpha}(1-|z|)^{\alpha-n-1}\, d\nu_n(z)\\ &\le \sup_{w\in B_n} \int_{B_n} \frac{(1-|z|)^{\alpha-n-1}}{|1-\langle z, w\rangle|^{\alpha-\tau}}\, d\nu_n(z)\\ &\le C\end{aligned}$$

by [10, Proposition 1.4.10]. It remains to apply Proposition 7.1. □

7.2. Spaces $\mathfrak{M}_\alpha(n)$ with $0 \le \alpha \le n$.

PROPOSITION 7.6. *Assume that $n \in \mathbb{N}$, $\alpha \ge 0$, $\beta > \max\{\alpha, n\}$ and $g \in \mathcal{H}ol(B_n)$. Suppose that*

$$\sup_{w\in B_n} \|g(z)k_\alpha(\langle z, w\rangle)\|_{A^1_{\beta-n}(B_n)} < \infty. \tag{7.8}$$

If $\alpha = 0$, then suppose also that $g \in A^1_{\beta-n}(B_n)$. Then $g \in \mathfrak{M}(\mathcal{F}_\alpha(n), A^1_{\beta-n}(B_n))$.

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PROOF. If $\alpha > 0$, then it suffices to repeat the arguments used in the proof of Proposition 7.1, putting $\rho_k = \rho = 0$. Now, let $\alpha = 0$ and let $f = f(0) + F_0[\mu]$, $\mu \in \mathcal{M}(B_n)$. As in the case $\alpha > 0$, we have $F_0[\mu] \cdot g \in A^1_{\beta-n}(B_n)$. It remains to remark that $f(0)g \in A^1_{\beta-n}(B_n)$. □

THEOREM 7.7. *Let $n \in \mathbb{N}$ and let $g \in \Lambda_\delta(B_n)$ for some $\delta > 0$. Assume that $M \in \{1, \ldots, n\}$ and $n \geq \alpha > n - M$, or assume that $M = n + 1$ and $\alpha = 0$. Suppose that*

$$\sup_{w \in B_n} \int_{B_n} \frac{|R^k g(z)|(1-|z|)^{\alpha+M-n-1}}{|1-\langle z, w\rangle|^{\alpha+M-k}}\, d\nu_n(z) < \infty, \qquad k = 1, \ldots, \min\{n, M\}. \tag{7.9}$$

If $\alpha = 0$ and $M = n + 1$, then suppose also that

$$\int_{B_n} |R^{n+1} g(z)| \log \frac{e}{1-|z|}\, d\nu_n(z) < \infty. \tag{7.10}$$

Then $g \in \mathfrak{M}_\alpha(n)$.

PROOF. Let $f \in \mathcal{F}_\alpha(n)$. Theorem 4.2 guarantees that $\mathcal{R}^M f \in \mathcal{F}_{\alpha+M}(n)$. We have $g \in \Lambda_\delta(B_n) \subset \mathfrak{M}_{\alpha+M}(n)$ by Corollary 7.5, hence

$$\mathcal{R}^M f \cdot g \in \mathcal{F}_{\alpha+M}(n). \tag{7.11}$$

Further, let $k \in \{1, \ldots, \min\{n, M\}\}$. Then $\mathcal{R}^{M-k} f \in \mathcal{F}_{\alpha+M-k}(n)$ by Theorem 4.2. Note that $\alpha + M > \max\{n, \alpha + M - k\}$, hence property (7.9) and Proposition 7.6 guarantee that

$$\mathcal{R}^{M-k} f \cdot R^k g \in A^1_{\alpha+M-n}(B_n) \subset \mathcal{F}_{\alpha+M}(n) \tag{7.12}$$

by Proposition 3.1(i).

If $\alpha = 0$ and $M = n + 1$, then (7.10) implies (7.8) for $R^{n+1} g$, $\alpha = 0$ and $\beta = n + 1$. Also, property (7.10) guarantees that $R^{n+1} g \in A^1_1(B_n)$. Hence, applying Proposition 7.6 with $\beta = n + 1$, we obtain property (7.12) for $\alpha = 0$ and $k = M = n + 1$.

Now, properties (2.2), (7.11) and (7.12) guarantee that $R^M(fg) \in \mathcal{F}_{\alpha+M}(n)$. Finally, $fg \in \mathcal{F}_\alpha(n)$ by Theorem 4.2. □

Assume that $g \in \mathcal{H}ol(B_n)$, $M \in \{1, \ldots, n\}$ and $n \geq \alpha > n - M$. Note that condition (7.9) is equivalent to the following property:

$$\sup_{\zeta \in \partial B_n} \int_{B_n} \frac{|R^k g(z)|(1-|z|)^{\alpha+M-n-1}}{|1-\langle z, \zeta\rangle|^{\alpha+M-k}}\, d\nu_n(z) < \infty, \qquad k = 1, \ldots, M. \tag{7.13}$$

Indeed, (7.9) implies (7.13) by Fatou's theorem. On the other hand, if $\lambda \in B_1$ and $r \in [0, 1)$, then $|1 - \lambda| \leq 4|1 - r\lambda|$. Hence, $|1 - \langle z, \zeta\rangle| \leq 4|1 - \langle z, r\zeta\rangle|$ for all $z \in B_n$, $\zeta \in \partial B_n$ and $r \in [0, 1)$. Therefore, (7.13) implies (7.9).

Note that the sufficient condition in Theorem 7.7, for $\alpha > 0$, is not too far from the necessary condition in Corollary 6.3. Indeed, dropping the logarithmic term in (6.2), we obtain (7.13) or, equivalently, (7.9).

Several explicit conditions, sufficient for (7.13), are given in [4, 5] when $n > \alpha > 0$. As an illustration, consider the holomorphic Lipschitz spaces $\Lambda_\delta(B_n)$, $\delta > 0$.

COROLLARY 7.8. *Let $n \in \mathbb{N}$. Assume that $0 \le \alpha \le n$ and $g \in \Lambda_\delta(B_n)$ for some $\delta > n - \alpha$. Then $g \in \mathfrak{M}_\alpha(n)$.*

PROOF. If $0 < \alpha < n$ and $g \in \Lambda_\delta(B_n)$ for some $\delta > n - \alpha$, then, as shown in the proof of [5, Corollary 4.2], property (7.13) holds. Hence, Theorem 7.7 is applicable.

Further, let $\alpha = 0$ and let $g \in \Lambda_\delta(B_n)$ for some $\delta \in (n, n + 1)$. Put $M = n + 1$. Then

$$|R^{n+1} g(z)| \le C(1 - |z|)^{\delta - n - 1}, \qquad z \in B_n.$$

Therefore, property (7.10) holds. For $k = 1, \ldots, n$, we have $R^k g \in H^\infty(B_n)$, thus

$$\sup_{w \in B_n} \int_{B_n} \frac{|R^k g(z)|(1 - |z|)^{\alpha + M - n - 1}}{|1 - \langle z, w\rangle|^{\alpha + M - k}}\, d\nu_n(z) \le \sup_{w \in B_n} \int_{B_n} \frac{d\nu_n(z)}{|1 - \langle z, w\rangle|^{n+1-k}} \le C$$

by [10, Proposition 1.4.10]. So, $g \in \mathfrak{M}_0(n)$ by Theorem 7.7.

Finally, let $\alpha = n$ and let $g \in \Lambda_\delta(B_n)$ for some $\delta \in (0, 1)$. Put $M = 1$. Then

$$\sup_{w \in B_n} \int_{B_n} \frac{|Rg(z)|(1 - |z|)^{\alpha + M - n - 1}}{|1 - \langle z, w\rangle|^{\alpha + M - 1}}\, d\nu_n(z) \le C \int_{B_n} \frac{(1 - |z|)^{\delta - 1}}{|1 - \langle z, w\rangle|^n}\, d\nu_n(z) \le C$$

by [10, Proposition 1.4.10]. So, $g \in \mathfrak{M}_n(n)$ by Theorem 7.7. □

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