A GENERAL SELECTION PRINCIPLE, WITH APPLICATIONS IN ANALYSIS AND ALGEBRA

Norman M. Rice

(received June 21, 1968)

1. Introduction. The General Selection Principle referred to in the title is really Tychonoff's theorem on products of compact spaces, but in a somewhat disguised form (c.f. Theorem 2.1, below). It is believed that this form is one which lends itself very well to many applications. More specifically, one of the immediate corollaries of the main theorem is a theorem due to Rado (c.f. Cor 2.1.1.), which has been used by Erdos and de Bruijn [2], Luxemburg [7], and the author [9] to give simple proofs of a variety of results.

The purpose of this paper is to show how the theorem and its corollaries can be used in a natural way to give simple proofs of a few old and new results. In § 2, we prove the main theorem and deduce several corollaries; in § 3, we discuss colorings of infinite graphs; in § 4, we give a proof of the Stone representation theorem for Boolean algebras; in § 5, we give a proof of the existence of Haar measure in locally compact groups; in § 6, we give a representation of certain vector lattices as function spaces.

2. The Main Theorem

THEOREM 2.1. *Suppose we are given a family* $\{C_t : t \in T\}$ *of compact sets and a net* $\mathfrak{N} = \{\phi_\alpha : \alpha \in A\}$ *of choice functions (i.e.* $\phi_\alpha : T \to \bigcup_{t\in T} C_t$, *with* $\phi_\alpha(t) \in C_t$ *for each* t). *Then there exists a choice function* $\psi : T \to \bigcup C_t$ *with the following property: if any finite subset* $F_o \subset T$ *is chosen, and for each* $t \in F_o$ *a neighbourhood* $N_{\psi(t)}$ *of* $\psi(t)$ *in* C_t *is chosen, then for any* $\alpha_o \in A$ *there exists* $\beta \in A$, $\beta \geq \alpha_o$, *such that* $\phi_\beta(t) \in N_{\psi(t)}$ *for each* $t \in F_o$.

REMARK This theorem is exactly Tychonoff's theorem on the compactness of the product of compact spaces, using the fact that a space is compact if and only if every net has a limit point. For completeness, however, we give below a paraphrase of Bourbaki's proof of Tychonoff's

Canad. Math. Bull. vol. 11, no. 4, 1968.

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

theorem, avoiding mention of the product topology.

<u>Proof of Theorem 2.1.</u> The net $\mathcal{N}$ in the product space ΠC_t gives rise to a filter $\mathcal{F}$ in ΠC_t in the usual way, i.e. $\mathcal{F}$ is generated by sets of the form $\{\phi_\beta : \beta \geq \alpha\}$. $\mathcal{F}$ is contained in some ultrafilter $\mathcal{U}$. Given any $t_o \in T$, we may define $\psi_{t_o} : \Pi C_t \to C_{t_o}$ by $\psi_{t_o}(\phi) = \phi(t_o)$. Then $\psi_{t_o}(\phi)$ is the base of an ultrafilter $\mathcal{U}_{t_o}$ (in C_{t_o}), which converges to some point in C_{t_o} since C_{t_o} is compact. We will call this point $\psi(t_o)$, thus defining a mapping $\psi : T \to \bigcup C_t$. Since $\mathcal{U}_{t_o}$ converges to $\psi(t_o)$, we have $\mathcal{U}_{t_o} \supset \mathcal{N}_{\psi(t_o)}$ (= neighbourhood system at $\psi(t_o)$).

Now suppose we are given $F_o \subset T$, a neighbourhood $N_{\psi(t)}$ for each $t \in F_o$, and $\alpha_o \in A$. For each $t \in F_o$, $\psi_t^{-1}(N_{\psi(t)}) \in \mathcal{U}$, and also $\{\phi_\alpha : \alpha \geq \alpha_o\} \in \mathcal{U}$; hence the set $U = \{\phi_\alpha : \alpha \geq \alpha_o\} \cap$ $\cap \{\psi_t^{-1}(N_{\psi(t)}) : t \in F_o\} \in \mathcal{U}$; and in particular U is non-empty. Thus there exists $\beta \in A$, $\beta \geq \alpha_o$, such that $\phi_\beta \in U$, and it follows that for each $t \in F_o$ we have $\phi_\beta(t) = \psi_t(\phi_\beta) \in N_{\psi(t)}$ (since $\phi_\beta \in \psi_t^{-1}(N_{\psi(t)})$).

COROLLARY 2.1.1 (R. Rado [8]). <u>Given a family</u> $\{K_t : t \in T\}$ <u>of finite sets, and given for each finite subset</u> $F \subset T$ <u>a corresponding choice function</u> $\phi_F : T \to \bigcup K_t$ <u>(with</u> $\phi_F(t) \in K_t$ <u>for each</u> $t \in T$), <u>then there exists a choice function</u> $\psi : T \to \bigcup K_t$ <u>with the following property: if any finite subset</u> $F_o \subset T$ <u>is chosen, then there exists a finite subset</u> $F \subset T$ <u>with</u> $F \supset F_o$ <u>such that</u> $\phi_F(t) = \psi(t)$ <u>for each</u> $t \in F_o$.

<u>Proof</u>. The finite sets K_t become compact under the discrete topology, and the finite subsets of T are directed by inclusion; hence the collection $\{\phi_F : F \subset T, F \text{ finite}\}$ becomes a net of choice functions into a family of compact sets. Let ψ be the mapping which exists by Theorem 2.1.

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

For any finite subset $F_o \subset T$, choose the neighbourhoods $N_{\psi(t)}$ to be the singleton sets $\{\psi(t)\}$ for $t \in F_o$, and take the index α_o to be F_o. Then by Theorem 2.1 there exists a larger index F (i.e. a finite set $F \supset F_o$) such that $\phi_F(t) \in \{\psi(t)\}$ for all $t \in F_o$.

REMARK. Proofs of Rado's theorem have also been given by Gottschalk [4] (using Tychonoff's theorem, but in a different way), and by Luxemburg [7] (using the method of ultrapowers).

For most applications of Rado's theorem it is sufficient to take all the finite sets K_t to be the same. We then get

COROLLARY 2.1.2. Given a set T and finite set K, suppose that for each finite subset $F \subset T$ there is given a corresponding function $\phi_F : T \to K$. Then there exists a function $\psi : T \to K$ with the following property: if any finite subset $F_o \subset T$ is chosen, then there exists a finite subset $F \subset T$ with $F \supset F_o$ such that $\phi_F(t) = \psi(t)$ for each $t \in F_o$.

For some applications of the main theorem, it is necessary to retain a general net of mappings, but is sufficient to take all the compact sets equal to one finite set. Theorem 2.1 then becomes

COROLLARY 2.1.3. Given a set T, a finite set K, and a net $\{\phi_\alpha : \alpha \in A\}$ of mappings, $\phi_\alpha : T \to K$, then there exists a mapping $\psi : T \to K$ such that: given any finite subset $F_o \subset T$ and any $\alpha_o \in A$, there exists $\beta \geq \alpha_o$ such that $\phi_\beta(t) = \psi(t)$ for all $t \in F$.

Finally, we mention the common situation where the compact sets C_t are all compact subsets of the reals R. The neighbourhoods $N_{\psi(t)}$ can then all be taken to be ϵ-intervals, and Theorem 2.1 becomes

COROLLARY 2.1.4. Suppose we are given a family $\{C_t : t \in T\}$ of compact subsets of R, and a net $\{\phi_\alpha : \alpha \in A\}$ of functions $\phi_\alpha : T \to R$ such that $\phi_\alpha(t) \in C_t$ for every $\alpha \in A$. Then there is a function $\psi : T \to R$ with the following property : given any finite subset $F_o \subset T$ and any $0 < \epsilon \in R$, then for any $\alpha_o \in A$ there exists $\beta \geq \alpha_o$ such that $|\phi_\beta(t) - \psi(t)| \leq \epsilon$ for each $t \in F_o$.

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

3. Graph Coloring

Definition. Given a positive integer k, a k-coloring of a graph T is a mapping from T to the set $\{1, 2, \ldots, k\}$ such that adjacent vertices are assigned different integers (colors).

Erdös and de Bruijn [2] noticed that the following theorem about infinite graphs follows easily from Rado's theorem. The proof is omitted since it is similar to that of Theorem 3.3.

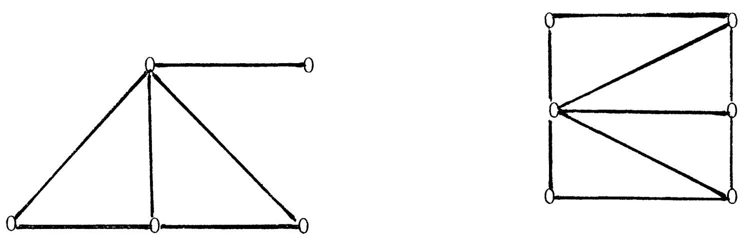
THEOREM 3.1 (Erdös, de Bruijn.) A graph T is k-colorable if and only if every finite subgraph is k-colorable.

Given a graph T, we may ask to what extent T is determined by its colorings. More precisely: given T, we may consider the (possibly empty) collection of k-colorings of T; and conversely, given any point set S together with a collection of k-colorings of the points of S, we may obtain a graph by connecting two points of S if and only if none of the colorings assigns them the same color. We say that a graph T is k-reproducible if the graph generated by the k-colorings of T is T again.

The following lemma is clear.

LEMMA 3.2. A graph T is k-reproducible if and only if for any two non-adjacent vertices there is some k-coloring that assigns them the same color.

Fig. 1(a) below shows a 3-reproducible graph (having two distinct 3-colorings), and Fig. 1(b) shows a graph which is not 3-reproducible (the top left and lower right vertices always get different colors).



(a) (b)

Figure 1

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

THEOREM 3.3. A graph T is k-reproducible if and only if every finite subgraph of T is k-reproducible.

Proof. Suppose T is such that every finite subgraph is k-reproducible. We will show that T is also k-reproducible. (By Lemma 3.2 the converse is clear.) Let t_1 and t_2 be any two fixed non-adjacent vertices of T. By Lemma 3.2 it is sufficient to find a k-coloring ψ which assigns the same color to t_1 and t_2.

Let $K = \{1, 2, \ldots k\}$. For any finite subset $F \subset T$, let $\phi_F : T \rightarrow K$ be a k-coloring of the finite subgraph $F \cup \{t_1, t_2\}$ which assigns the same color to t_1 and t_2. Then the mapping $\psi : T \rightarrow K$ which exists in virtue of Corollary 2.1.2 is the required coloring. For, let t_3 and t_4 be any two adjacent vertices of T, and let $F_o = \{t_1, t_2, t_3, t_4\}$. Then Corollary 2.1.2 states there exists $F \supset F_o$ such that $\phi_F(t) = \psi(t)$ for each $t \in F_o$, and hence $\psi(t_3) = \phi_F(t_3) \neq \phi_F(t_4) = \psi(t_4)$, and $\psi(t_1) = \phi_F(t_1) = \phi_F(t_2) = \psi(t_2)$.

4. Representations of Boolean Algebras. The author noticed recently [9] that Stone's representation theorem for Boolean algebras could be proved simply using Rado's theorem. We have:

THEOREM 4.1 (Stone [11]). Every Boolean algebra T is isomorphic to a Boolean algebra of subsets of some set H.

Proof. Let H be the set of all non-trivial Boolean homomorphisms $h : T \rightarrow \{0, 1\}$. We may then map T into the collection $\mathcal{P}(H)$ of subsets of H by $\rho : T \rightarrow \mathcal{P}(H)$ where $\rho(t) = \{h \in H : h(t) = 1\}$. It is easy to check that ρ is a Boolean homomorphism, and to show that it is an isomorphism, it is only necessary to show that for any $0 \neq t_o \in T$ there is an $h \in H$ such that $h(t_o) \neq 0$ (i.e. that there are enough homomorphisms). Now, it is not hard to see that any finite Boolean algebra does have enough homomorphisms into $\{0, 1\}$, so for any finite subset $F \subset T$ we may let $\phi_F : T \rightarrow \{0, 1\}$ be such that it acts as a homomorphism on the finite subalgebra generated by $F \cup \{t_o\}$ and such that $\phi_F(t_o) = 1$. It can then be seen (c.f. [9]) that the mapping $\psi : T \rightarrow \{0, 1\}$ which exists in virtue of Corollary 2.1.2 is a homomorphism of T, and that $\psi(t_o) = 1$.

We may put a topology on H by taking as a basis for the open sets the collection $\{\rho(t) : t \in T\}$. We note that for any $t \in T$, with

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

complement t', the complementary set $\complement \rho(t) = \rho(t')$ is open, so that the basis sets are both open and closed. The fact that H has enough homomorphisms implies that the space is Hausdorff.

THEOREM 4.2. The topological space H is compact.

Proof. Let $\{h_\alpha : \alpha \in A\}$ be a net in H. Since $h_\alpha : T \to K = \{0, 1\}$, by Corollary 2.1.3 we get a certain mapping $h_o : T \to \{0, 1\}$. Again, it may be verified without difficulty that h_o is a non-trivial homomorphism from T to $\{0, 1\}$, so $h_o \in H$. And in fact h_o is a limit point of the net $\{h_\alpha\}$, for: take any basic neighbourhood of h_o, (i.e. take any $t_o \in T$ such that $h_o(t_o) = 1$ and consider the neighbourhood $\rho(t_o)$), and take any $\alpha_o \in A$. Let $F_o = \{t_o\} \subset T$. Then there exists $\beta \geq \alpha_o$ such that $h_\beta(t_o) = h_o(t_o) = 1$, i.e. $h_\beta \in \rho(t_o)$.

The reader will notice that the topology on H is the weakest topology making the characteristic functions $\chi_{\rho(t)}$, $t \in T$, all continuous. Another example of the compactness of weak topologies will be given in § 6.

5. Existence of Haar Measure. Our object in this section is to use the Main Theorem (actually, Corollary 2.1.4) to prove

THEOREM 5.1. In every locally compact topological group X, there exists at least one regular Haar measure, i.e. a left-translation-invariant Borel measure.

Proof. The proof, in outline, follows that in Halmos [5]. One first observes that it is sufficient to find an invariant content ψ on the compact sets, since then the measure induced by the content will be as required.

Let T be the collection of compact subsets of X, and let T^o be the collection of compact subsets of X with non-empty interior. For $D \in T$ and $U \in T^o$ let $r(D;U)$ = the minimum number of left translations of U needed to cover D. To fix the scale, fix $B \in T^o$; and then for any $U \in T^o$ define $\phi_U : T \to R$ by

$$\phi_U(D) = \frac{r(D;U)}{r(B;U)} \quad \text{for } D \in T.$$

It follows ϕ_U is translation invariant, and subadditive; and if $D_1 \cap D_2 = \phi$, then for U sufficiently small, $\phi_U(D_1 \cup D_2) = \phi_U(D_1) + \phi_U(D_2)$.

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

We now apply Corollary 2.1.4. It is easy to show that $0 \leq \phi_U(D) \leq r(D; B)$, hence for all $U \in T^o$, $\phi_U(D)$ is in the compact interval $[0, r(D;B)] = C_D$, say. Also, the sets $U \in T^o$ are directed (down) by inclusion. Hence $\{\phi_U : U \in T^o\}$ is a net of choice functions into compact subsets of R, so by Corollary 2.1.4 we get a certain mapping $\psi: T \to R$.

It is straightforward now to verify that ψ is the required content. We show, as a sample, that ψ is finitely additive: given $D_1 \cap D_2 = \phi$, take any $\epsilon > 0$, and find U_o such that $U \subset U_o$ implies $\phi_U(D_1 \cup D_2) = \phi_U(D_1) + \phi_U(D_2)$. Let $F_o = \{D_1, D_2, D_1 \cup D_2\}$. By Corollary 2.1.4 there exists $U \subset U_o$ such that $|\phi_U(D) - \psi(D)| \leq \epsilon$ for $D \in F_o$; hence $|\psi(D_1 \cup D_2) - \psi(D_1) - \psi(D_2)|$

$$= |\psi(D_1 \cup D_2) - \phi_U(D_1 \cup D_2) - \psi(D_1) + \phi_U(D_1) - \psi(D_2) + \phi_U(D_2)|$$

$$\leq |\psi(D_1 \cup D_2) - \phi_U(D_1 \cup D_2)| + |\psi(D_1) - \phi_U(D_1)| + |\psi(D_2) - \phi_U(D_2)| \leq 3\epsilon .$$

And since ϵ is arbitrary we get $\psi(D_1 \cup D_2) = \psi(D_1) + \psi(D_2)$.

6. Representations of Vector Lattices. Let L be a vector lattice which is Dedekind complete (i.e. conditionally complete: every subset of L which is bounded above has a least upper bound), and which has a strong order unit 1 (i.e. for every $x \in L$ there is a real number α such that $|x| \leq \alpha 1$). L then becomes a normed space by setting $\|x\| = \inf\{\alpha : |x| \leq \alpha 1\}$. Several proofs exist of the fact that L can be represented as the space of all continuous functions on a suitable compact Haudsdorff space (c.f., for instance [1], [6]).

A common technique in representation theory is to take as the base space some collection of linear functionals, and one of the main problems is usually to show that there are enough functionals of the required type. More precisely, we say that a linear functional $\phi : L \to R$ is lattice-preserving if for $x, y \in L$, $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$; and we will use the main theorem (actually Corollary 2.1.4) to prove

THEOREM 6.1. *Let L be a Dedekind complete vector lattice with a strong order unit 1, and let $0 < x_o \in L$. Then there is a lattice-preserving linear functional ψ of norm 1, such that $\psi(x_o) = \|x_o\|$.*

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

We will prove Theorem 6.1 by using a series of lemmas to get approximations to the functional we want, and then applying Corollary 2.1.4. First of all, we say that an element $e \in L$ is unitary if $\inf(e, 1-e) = 0$. Freudenthal proved [3] that if $0 < x \in L$, then there exists a unitary element $e > 0$ and a real number $\alpha > 0$ such that $\alpha e \leq x$. If $x \in L$ and e is unitary, we define the product $x \cdot e$ to be the projection of x into the normal subspace (band) generated by e. It is easy to verify that this "product" has desirable properties such as:

(i) if e_1 and e_2 are unitary, then $e_1 \cdot e_2 = \inf(e_1, e_2)$, and is again unitary;

(ii) $(x \cdot e_1) \cdot e_2 = x \cdot (e_1 \cdot e_2)$;

(iii) $(x + y) \cdot e = x \cdot e + y \cdot e$;

(iv) $x \leq y$ implies $x \cdot e \leq y \cdot e$.

It follows from Freudenthal's result that if $x \not\leq 0$, then there exists $e > 0$ such that $x \cdot e > 0$; for since $x^+ > 0$ we may find $e > 0$ and $\alpha > 0$ such that $0 < \alpha e \leq x^+$; and since x^+ is disjoint from x^-, so is e so that $x^- \cdot e = 0$; and hence $0 < \alpha e = \alpha e \cdot e \leq x^+ \cdot e = x^+ \cdot e - x^- \cdot e = x \cdot e$.

For any $e > 0$, $x \geq 0$ in L, define

$$\rho(e, x) = \sup \{\beta \in R : \beta e \leq x\} .$$

LEMMA 6.2. (i) For any $x \geq 0$ and any unitary $e > 0$,

$$\rho(e, x) \leq \|x \cdot e\| \leq \|x\| .$$

(ii) For any $x \geq 0$ and any unitary $e > 0$, there exists $0 < e' \leq e$ such that whenever $0 < e'' < e'$ then

$$\|x e''\| - \epsilon \leq \|x \cdot e\| - \epsilon \leq \rho(e'', x) .$$

Proof. (i) $\rho(e, x)e \leq x$, so $\rho(e, x)e = \rho(e, x)e \cdot e \leq x \cdot e \leq \|x \cdot e\| 1$, and hence $\rho(e, x)e \leq \|x \cdot e\| e$, so that $\rho(e, x) \leq \|x \cdot e\|$. $\|x \cdot e\| \leq \|x\|$ follows from $0 \leq x \cdot e \leq x$.

(ii) Let $\alpha = \|x \cdot e\| - \epsilon$. Since $x \cdot e \not\leq \alpha 1$, therefore $(x \cdot e - \alpha 1) \not\leq 0$, and hence by the remarks above there exists a unitary $e_1 > 0$ such that $0 < (x \cdot e - \alpha 1) \cdot e_1$. It follows that $e \cdot e_1 \neq 0$, so let $0 < e' = e \cdot e_1 \leq e$, and then whenever $0 < e'' \leq e'$

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

we have $\alpha e'' \leq x \cdot e'' \leq x$, so that $\rho(e'', x) \geq \alpha = \|x \cdot e\| - \epsilon$. The other inequality of (ii) is immediate since $0 \leq x \cdot e'' \leq x \cdot e$, so that $\|x \cdot e''\| \leq \|x \cdot e\|$.

LEMMA 6.3. For any $0 < x \in L$, $0 < \alpha \in R$, and unitary $e > 0$, $\rho(e, \alpha x) = \alpha \rho(e, x)$.

Proof. $\beta e \leq \alpha x$ if and only if $\alpha^{-1} \beta e \leq x$; hence $\rho(e, \alpha x)$ $= \sup \{\beta : \beta e \leq \alpha x\} = \sup \{\beta : \alpha^{-1} \beta e \leq x\} = \alpha \cdot \sup\{\alpha^{-1}\beta : \alpha^{-1}\beta e < x\}$ $= \alpha \rho(e, x)$.

LEMMA 6.4. Given $x, y \geq 0$, $\epsilon > 0$, and $e > 0$, there exists $0 < e' \leq e$ such that whenever $0 < e'' \leq e'$ then

(i) $\rho(e'', x) + \rho(e'', y) \leq \rho(e'', x + y) \leq \rho(e, x) + \rho(e, y) + \epsilon$,

(ii) $\rho(e'', x \vee y) \leq \rho(e'', x) \vee \rho(e'', y) \leq \rho(e'', x \vee y) + \epsilon$,

(iii) $\rho(e'', x \wedge y) \leq \rho(e'', x) \wedge \rho(e'', y) \leq \rho(e'', x \wedge y) + \epsilon$.

Proof. By Lemma 6.2 there exists $0 < e' \leq e$ such that whenever $0 < e'' \leq e'$ then $\|x \cdot e''\| - \epsilon/2 \leq \rho(e'', x) \leq \|x \cdot e''\|$

and $\|y \cdot e''\| - \epsilon/2 \leq \rho(e'', y) \leq \|y \cdot e''\|$.

For (i) we observe that $\rho(e'', x)e'' \leq x$ and $\rho(e'', y)e'' \leq y$, so that $[\rho(e'', x) + \rho(e'', y)]e'' < x + y$; and hence $\rho(e'', x) + \rho(e'', y) \leq \rho(e'', x + y)$. Also, $\rho(e'', x + y) \leq \|(x + y) \cdot e''\|$ $= \|x \cdot e'' + y \cdot e''\| \leq \|x \cdot e''\| + \|y \cdot e''\| \leq \rho(e'', x) + \epsilon/2 + \rho(e'', y) + \epsilon/2$.

The proofs of (ii) and (iii) are exactly similar.

Proof of Theorem 6.1. We have $x_o > 0$, and we want to find a lattice-preserving linear functional ψ of norm 1 such that $\psi(x_o) = \|x_o\|$. For any finite subset $F \subset L^+$ and any $\epsilon > 0$, there exists by the preceding lemmas a unitary $e > 0$ such that

(i) $0 < \rho(e, x) \leq \|x\|$ for all $x \in L^+$

(ii) $\|x_o\| - \epsilon \leq \rho(e, x_o) \leq \|x_o\|$

(iii) $\rho(e, x) + \rho(e, y) \leq \rho(e, x + y) \leq \rho(e, x) + \rho(e, y) + \epsilon$ for all $x, y \in F$.

(iv) $\rho(e, \alpha x) = \alpha \rho(e, x)$ for $0 < \alpha \in R$ and $x \in L^+$

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

(v) $\rho(e, x \vee y) \le \rho(e,x) \vee \rho(e,y) \le \rho(e, x \vee y) + \epsilon$ for all $x, y \in F$

(vi) $\rho(e, x \wedge y) \le \rho(e,x) \wedge \rho(e,y) \le \rho(e, x \wedge y) + \epsilon$ for all $x, y \in F$.

Let $S = \{F, \epsilon\}$, and define $\phi_S : L^+ \to R$ by $\phi_S(x) = \rho(e, x)$. Then ϕ_S is an approximation (at least on F) to the required functional ψ. Note that by (i), $\phi_S(x)$ is in the compact interval $[0, \|x\|] \subset R$; also, the collection $\{S : S = \{F, \epsilon\},\ F \text{ finite},\ F \subset L^+,\ 0 < \epsilon \in R\}$ is directed by the relation $\ge$ where $\{F_2, \epsilon_2\} \ge \{F_1, \epsilon_1\}$ if $F_2 \supset F_1$ and $\epsilon_2 \le \epsilon_1$. Hence by Corollary 2.1.4 there is a mapping $\psi : L^+ \to R$ such that

(i) $\psi(x) \in [0, \|x\|]$ for $x \in L^+$, and

(ii) given any finite subset $F_o \subset L^+$ and any $\epsilon > 0$, then (letting $S_o = \{F_o, \epsilon\}$), there exists $S = \{F, \epsilon'\} \ge \{F_o, \epsilon\}$ (i.e. $F \supset F_o$ and $\epsilon' \le \epsilon$) such that $|\phi_S(x) - \psi(x)| \le \epsilon$ for all $x \in F_o$.

One can now verify that ψ acts on L^+ as the required functional. For example, we show that ψ is additive. Let $y, z \in L^+$, and let $\epsilon > 0$, and take $S_o = \{\{y, z, y+z\}, \epsilon\}$. Then for an appropriate $S = \{F, \epsilon\} \ge S_o$ we have

$$|\psi(y+z) - \psi(y) - \psi(z)| \le |\psi(y+z) - \phi_S(y+z)| + |\psi(y) - \phi_S(y)| +$$
$$|\psi(z) - \phi_S(z)| + |\phi_S(y+z) - \phi_S(y) - \phi_S(z)| \le \epsilon + \epsilon + \epsilon + \epsilon' \le 4\epsilon.$$

And ϵ is arbitrary, so $\psi(y+z) = \psi(y) + \psi(z)$. The other properties are verified similarly.

ψ can now be extended linearly to the required functional on L.

One may now get a representation of L as a real function space as follows. Let Φ be the collection of all lattice-preserving linear functionals of norm 1, and define $\tau : L \to \mathcal{F}(\Phi, R)$ by $(\tau(x))(\phi) = \phi(x) \in R$. It is easy to verify that τ is a vector-lattice-homomorphism into $\mathcal{F}(\Phi, R)$, and Theorem 6.1 shows that it is an isomorphism.

Φ may be given the weakest topology under which all the functions in $\tau(L)$ are continuous, i.e. a typical neighbourhood of $\phi_o \in \Phi$ is $\{\phi \in \Phi : |\phi(x_i) - \phi_o(x_i)| \le \epsilon,\ x_i \in L,\ i = 1, \ldots, n\}$. This topology is Hausdorff, and we can use Corollary 2.1.4 to see that it is compact: indeed, suppose $\{\phi_\alpha : \alpha \in A\}$ is a net in Φ;

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

since $\|\phi_\alpha\| = 1$, therefore $\phi_\alpha(x) \in [-\|x\|, \|x\|]$ for every $\alpha \in A$, so there exists by Corollary 2.1.4 a certain mapping $\psi: L \to R$, and it follows easily that $\psi \in \Phi$ and is a limit point of the net $\{\phi_\alpha\}$.

The reader will notice that Corollary 2.1.4 can be used in a very similar way to prove the compactness of the unit ball in the norm-dual X' of a normed linear space X, in the weak topology $\omega(X', X)$. This theorem is, of course, already very close to Tychonoff's theorem.

With the topology thus defined on Φ, $\tau(L)$ becomes the space of all continuous functions on the compact Handsorff space Φ. This is a function algebra, so a multiplication can be induced on L itself by using the isomorphism τ. It is interesting to note, therefore, that in fact there is a natural way of defining a multiplication implicitly in L, which coincides with this induced multiplication (c.f. [10]). Under this multiplication in L, the functionals $\phi \in \Phi$ are exactly the multiplicative linear functionals.

REFERENCES

1. N. Bourbaki, Integration (Act. Sc. et Ind. 1175, Hermann, Paris, 1952).

2. N.G. de Bruijn and P. Erdos, A color problem for infinite graphs and a problem in the theory of relations, Nederl. Akad. Wetensch. (A) 54 (1951) 371-373.

3. H. Freudenthal, Teilweise geordnete Moduln, Proc. Acad. Sci. Amsterdam 39 (1936) 641-651.

4. W.H. Gottschalk, Choice Functions and Tychonoff's theorem, Proc. Amer. Math. Soc. 2 (1951) 172.

5. P.R. Halmos, Measure Theory (Van Nostrand, New York, 1950).

6. R.V. Kadison, A representation theory for commutative topological algebra, Memoirs Amer. Math. Soc. 7 (1951).

7. W.A.J. Luxemburg, A remark on a paper by N.G. de Bruijn and P. Erdos, Nederl. Akad. Wetensch. (A) 65 (1962) 343-345.

8. R. Rado, Axiomatic treatment of rank in infinite sets, Canad. J. Math. 1 (1949) 337-343.

9. N.M. Rice, Stone's representation theorem for Boolean algebras, Amer. Math. Monthly 6 (1968) 503-504.

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press

10. N.M. Rice, Multiplication in vector lattices, Canad. J. Math., 20 (1968) 1136-1149.

11. M.H. Stone, The theory of representations for Boolean algebras, Trans. Amer. Math. Soc. 40 (1936) 37-111.

Queen's University,
Kingston,
Ontario

https://doi.org/10.4153/CMB-1968-069-4 Published online by Cambridge University Press