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Note on rational approximations of the exponential function at rational points

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Let p, q, u, and v be any four positive integers, and let δ be a number in the interval $0 < \delta \le 2$. In one of his papers, Kurt Mahler, *Bull. Austral. Math. Soc.* 10 (1974), 325-335, proved that if q satisfies the inequalities

$$q \ge e^{\{64(u+v)\}^{10/\delta}}, \quad q \ge e^{8u/\delta v},$$

$$q \ge e^{\left(e^2u/v\right)^{24/\delta}}, \text{ and } q \ge e^{e^{(e\delta)^{-2}}},$$

then

$$\left|e^{u/v} - \frac{p}{q}\right| > q^{-(2+\delta)}.$$

In this note, by a slightly different treatment of some inequalities in Mahler's paper, we easily obtain the same result with q only restricted by the first condition.

1.

Denote by n, v, two positive integers and put

$$P_1(x) = \sum_{k=n-1}^{2n-1} \frac{nk!v^{k-n+1}}{(k-n+1)!(2n-k-1)!}(-x)^{2n-k-1}$$

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and

$$P_2(x) = \sum_{k=n}^{2n-1} \frac{k!v^{k-n+1}}{(k-n)!(2n-k-1)!} (-x)^{2n-k-1} .$$

Furthermore put

$$f_1(x, z) = \frac{x^{n-1}}{(n-1)!} (vx-z)^n \quad \text{and} \quad f_2(x, z) = \frac{vx^n}{(n-1)!} (vx-z)^{n-1} .$$

It follows from the definition of $P_1(x)$, $P_2(x)$ that

$$\sum_{k\geq 0} \frac{\partial^k}{\partial x^k} f_i(x, z)\Big|_{x=0} = P_i(z) \quad (i = 1, 2)$$

and

$$\sum_{k\geq 0} \frac{\partial^k}{\partial x^k} f_1(x, z)\Big|_{x=z/v} = P_2(-z) , \quad \sum_{k\geq 0} \frac{\partial^k}{\partial x^k} f_2(x, z)\Big|_{x=z/v} = P_1(-z) .$$

Then, by Hermite's identity,

$$e^{z/v}P_1(z) - P_2(-z) = e^{z/v} \int_0^{z/v} e^{-t}f_1(t, z)dt =$$

$$= z^{2n}e^{z/v} \int_0^{1/v} \frac{x^{n-1}}{(n-1)!} (vx-1)^n e^{-zx}dx ,$$

and

$$e^{z/v}P_2(z) - P_1(-z) = e^{z/v} \int_0^{z/v} e^{-t}f_2(t, z)dt =$$

$$= z^{2n}e^{z/v} \int_0^{1/v} \frac{x^n}{(n-1)!} (vx-1)^{n-1} e^{-zx}dx .$$

Therefore, the determinant

$$\Delta(z) = P_1(z)P_1(-z) - P_2(z)P_2(-z)$$

is a polynomial in z of the exact degree $2n$ which has at $z = 0$ a zero of order $2n$. Then

$$\Delta(z) \neq 0 \text{ if } z \neq 0 .$$

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2.

Denote by p, q, u, and v four positive integers, and let

$$\theta = |qe^{u/v}-p| .$$

By putting $z = u$ in the preceding formulae, we obtain

$$|qP_2(-u)-pP_1(u)| \le \theta|P_1(u)| + q(e^{u/v}-1) \sup_{0\le t\le u/v} |f_1(t, u)|$$

and

$$|qP_1(-u)-pP_2(u)| \le \theta|P_2(u)| + q(e^{u/v}-1) \sup_{0\le t\le u/v} |f_2(t, u)| .$$

Since

$$P_1(u)P_1(-u) - P_2(u)P_2(-u) \ne 0 ,$$

at least one of the integers

$$qP_2(-u) - pP_1(u) \quad \text{and} \quad qP_1(-u) - pP_2(u)$$

is distinct from zero. It follows that

$$(1) \qquad 1 \le \theta|P_i(u)| + qe^{u/v} \sup_{0\le t\le u/v} |f_i(t, u)|$$

where $i = 1$ or $i = 2$.

We have

$$|P_1(u)| \le \sum_{k=n-1}^{2n-1} \frac{nk!v^{k-n+1}u^{2n-k-1}}{(k-n+1)!(2n-k-1)!} = \sum_{j=0}^{n} \frac{n(n+j-1)!}{j!(n-j)!} v^j u^{n-j} \le \frac{(2n-1)!}{(n-1)!} (u+v)^n$$

and

$$|P_2(u)| \le v \sum_{j=0}^{n-1} \frac{(n+j)!v^j u^{n-1-j}}{j!(n-1-j)!} \le v \frac{(2n-1)!}{(n-1)!} (u+v)^{n-1} .$$

Next, when t lies in the interval $0 \le t \le u/v$,

$$\max\{|f_1(t, u)|, |f_2(t, u)|\} \le \frac{u^{2n-1}}{(n-1)!v^{n-1}} \sup_{0\le t\le 1} (t(1-t))^{n-1} \le$$

$$\le \frac{u^{2n-1}}{(n-1)!(4v)^{n-1}} \le \frac{u^{2n-1}}{(n-1)!4^{n-1}} .$$

Then, by (1), we can write

$$(2) \qquad 1 \le \frac{(2n-1)!}{(n-1)!}(u+v)^n\theta + q\,\frac{e^{u/v}u^{2n-1}}{(n-1)!\,4^{n-1}} .$$

3.

Denote by m_0 the smallest integer which satisfies

$$(3) \qquad 2qe^u u^{2m_0+1} \le m_0!\,4^{m_0} .$$

From the definition of m_0, it follows that

$$(4) \qquad (m_0-1)!\,4^{m_0-1} < 2qe^u u^{2m_0-1} .$$

Since

$$\frac{(2m_0+1)!}{m_0!} = \binom{2m_0+1}{m_0} \cdot (m_0+1)! < 2^{2m_0+1}(m_0+1)! ,$$

we have, by (2), (3), and (4), with $n = m_0 + 1$,

$$1 \le 2.2^{2m_0+1}(u+v)^{m_0+1}\theta.2qe^u u^{2m_0-1} m_0(m_0+1)4^{1-m_0} .$$

Note that

$$m_0(m_0+1) \le 2^{2m_0-1} , \quad u + v > u , \text{ and } m_0 \ge 1 ;$$

then

$$(5) \qquad 1 \le \theta qe^u\big(4(u+v)\big)^{3m_0} .$$

4.

Now, we require an upper estimate for m_0. By (4), we have

$$m_0!\,4^{m_0-1} < 2qe^u u^{2m_0-1} m_0 .$$

Since

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$$m_0! \geq m_0^{m_0+\frac{1}{2}} e^{-m_0}\sqrt{2\pi}\ , \quad m_0^{\frac{1}{2}} \leq \frac{2^{m_0/2}}{\sqrt{2}}\ , \text{ and } \frac{\sqrt{2}e}{4} < 1\ ,$$

it follows that

$$m_0^{m_0} < qe^u u^{2m_0-1} \cdot \frac{2e^{m_0} m_0^{\frac{1}{2}}}{4^{m_0-1}\sqrt{2\pi}} \leq qe^u u^{2m_0} \frac{4}{\sqrt{\pi}} \left(\frac{\sqrt{2}e}{4}\right)^{m_0} < 3qe^u u^{2m_0}\ .$$

Put

(6) $$b = \frac{1}{u^2}\log\left(3qe^u\right) \quad \text{and} \quad x = \frac{m_0}{u^2}\ .$$

Hence

(7) $$x^x < e^b\ .$$

Suppose that

$$b \geq 27 > e^e\ .$$

Then, the condition

$$x \geq \frac{b}{\log(b/\log b)}$$

implies

$$x \log x \geq b + \frac{b\log\log\log b}{\log(b/\log b)} \geq b\ .$$

Hence, by (7),

(8) $$x < \frac{b}{\log(b/\log b)} \quad \text{and} \quad m_0 < \frac{bu^2}{\log(b/\log b)}\ ,$$

provided $b \geq 27$.

5.

We can now prove the

THEOREM. *Let δ be a constant in the interval $0 < \delta \leq 2$, and let p, q, u , and v be four positive integers. Assume that*

$$q \geq e^{\left(4(u+v)\right)^{10/\delta}}$$

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Then

$$\left|e^{u/v} - \frac{p}{q}\right| \geq q^{-(2+\delta)} .$$

Proof. We have

$$\log q \geq \left(4(u+v)\right)^{10/\delta} > u^{10/\delta} \geq u^5 .$$

Then

$$b = \frac{1}{u^2} \log\left(3qe^u\right) > \frac{\log q}{u^2} \geq (\log q)^{3/5} .$$

Since

$$5e^{y/5} \geq 3y \quad \text{for} \quad y \geq 5 \log 5 \text{ , and } \log q \geq 8^{10/\delta} \geq 8^5 ,$$

it follows that

$$\frac{b}{\log b} > \frac{5}{3} \frac{(\log q)^{3/5}}{\log\log q} \geq (\log q)^{2/5} \geq \left(4(u+v)\right)^{4/\delta} .$$

Since

$$b > (\log q)^{3/5} \geq 8^3 > 27 ,$$

we deduce from (8),

$$3m_0 \log\left(4(u+v)\right) < \frac{3bu^2 \log\left(4(u+v)\right)}{\log(b/\log b)} \leq \frac{3\delta}{4} \log\left(3qe^u\right) .$$

On substituting this upper estimate in (5), it follows that

$$1 < \theta q e^u \left(3qe^u\right)^{3\delta/4} .$$

Finally, since

$$e^u\left(3e^u\right)^{3\delta/4} \leq e^{6u} \leq e^{(\delta/4)(4u)^{10/\delta}} \leq q^{\delta/4} ,$$

we find that

$$\theta \geq q^{-1-\delta} .$$

This completes the proof.

6.

Note that the condition for q in this theorem can be easily replaced

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by a weaker one, for example if we suppose $0 < \delta \leq 1$.

Compare also related results by Bundschuh [1].

References

[1] Peter Bundschuh, "Irrationalitätsmaße für e^a , $a \neq 0$ rational oder Liouville-Zahl", *Math. Ann.* 192 (1971), 229-242.

[2] Kurt Mahler, "On rational approximations of the exponential function at rational points", *Bull. Austral. Math. Soc.* 10 (1974), 325-335.

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