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HOMOLOGICAL DUALITY AND QUASI-HEREDITY

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ABSTRACT. This paper represents a general study of the (Yoneda) Ext-algebra A^* of a finite dimensional K-algebra A. Our motivation lies in the problem of establishing conditions under which (i) the species of A^* coincides with the dual species of A and (ii) the quasi-heredity of A (or A^*) yields the quasi-heredity of A^* (or A, respectively). These questions are closely related to the Kazhdan–Lusztig Theory as presented by [CPS2]. The main results include introducing the concept of a solid algebra and the relevant Theorem 4.5 as well as a rather complete description of the situation in the case of monomial algebras in Section 5.

1. **Introduction. Notation and basic definitions.** Since the introduction of quasi-hereditary algebras by Cline, Parshall and Scott in [CPS1] and [PS], the concept has proved to be instrumental in a number of areas of representation theory. The quasi-hereditary algebras arising in most of these applications enjoy some additional properties. Thus, a Kazhdan–Lusztig theory of Cline, Parshall and Scott ([CPS2]) leads to quasi-hereditary algebras whose homological dual is again a quasi-hereditary algebra. One of the main objectives of the present paper is to find a natural class of such algebras; the resulting concept of a solid algebra with the related Theorem 4.5 is given in Section 4. In the course, we also study the Ext-algebra A^* of an algebra A in general. The essential components of our machinery include the concepts of top embeddings and the subcategory $\mathcal{C} \subseteq$ mod-A in Section 2 and the functor Ext^*: mod-$A \longrightarrow A^*$-mod in Section 3. In Section 5 we deal with monomial algebras; here we present a rather complete description, using and extending results of Green and Zacharia ([GZ]). Finally, the fact that some of the results in the text cannot be strengthened is illustrated by examples in Section 6.

Some of our results are parallel to those of Cline, Parshall and Scott in [CPS2] although our approach and basic assumptions are different. We should also like to refer to the recent study of graded Koszul rings by Beilinson, Ginzburg and Soergel ([BGS]), as well as the lectures of P. Smith and R. Martínez-Villa, presented at the Seventh International Conference on Representation Theory in Mexico in August, 1994.

Let A be a finite dimensional algebra over an arbitrary field K. Let $\{ e_i \mid i \in I \}$ be a fixed complete set of primitive orthogonal idempotents, with the corresponding indecomposable projective (right) modules denoted by $P(i)$, and their simple tops by $S(i)$. Without

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loss of generality we will assume that A is basic, thus $1 = \sum_{i \in I} e_i$. Throughout the paper we shall denote by $\hat{S}$ the direct sum of the simple modules $S(i)$; thus $\hat{S}_A \simeq A/\operatorname{rad} A$. Whenever it is needed, we shall speak about left A-modules, too; the corresponding projective and simple left modules will be denoted by $P^\circ(i)$ and $S^\circ(i)$.

The *Ext-algebra* of A will be denoted by A^*. It is, by definition, the K-algebra whose underlying vector space is

$$\bigoplus_{k \geq 0} \operatorname{Ext}_A^k(\hat{S}, \hat{S}) \simeq \bigoplus_{k \geq 0} \bigoplus_{i,j \in I} \operatorname{Ext}_A^k\big(S(i), S(j)\big)$$

and the multiplication is defined by the Yoneda product of extensions. That is to say, if

$$0 \longrightarrow S(i) \longrightarrow X_{\ell-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow S(j) \longrightarrow 0$$

and

$$0 \longrightarrow S(j) \longrightarrow Y_{m-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow S(k) \longrightarrow 0$$

represent elements of $\operatorname{Ext}_A^\ell\big(S(j), S(i)\big)$ and $\operatorname{Ext}_A^m\big(S(k), S(j)\big)$, respectively, then the corresponding product is represented by the exact sequence

$$0 \longrightarrow S(i) \longrightarrow X_{\ell-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow Y_{m-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow S(k) \longrightarrow 0$$

in $\operatorname{Ext}_A^{\ell+m}\big(S(k), S(i)\big)$.

It is easy to check that in this way one gets an associative K-algebra which is finite dimensional if and only if $\operatorname{gl.dim} A < \infty$. Let us also mention that, in the presence of the standard K-duality, $(A^*)^{op} \simeq (A^{op})^*$, and thus we may confine ourselves to studying Ext-algebras defined in terms of right A-modules alone.

In what follows we list some of the properties of A^*; the proofs are straightforward.

PROPOSITION 1.1. *Let A be a basic finite dimensional K-algebra and A^* its Ext-algebra as defined above.*

(1) *The decomposition* $A^* = \oplus_{k \geq 0} \operatorname{Ext}_A^k(\hat{S}, \hat{S})$ *defines a graded algebra structure on A^*.*

(2) *If* $\operatorname{gl.dim} A < \infty$, *then for $X \in$* mod-A *the correspondence* $\operatorname{Ext}^*(X) = \oplus_{k \geq 0} \operatorname{Ext}_A^k(X, \hat{S})$ *defines a contravariant functor from* mod-A *to* A^*-mod_{gr}, *the category of finitely generated graded left A^*-modules.*

(3) *The elements* $f_i = \operatorname{id}_{S(i)} \in \operatorname{Hom}_A\big(S(i), S(i)\big) \subseteq A^*$ *for $i \in I$ form a complete set of primitive orthogonal idempotents in A^*. Thus the indecomposable left projective A^*-modules may be identified with* $P^{*\circ}(i) = A^* f_i \simeq \operatorname{Ext}_A^*\big(S(i), \hat{S}\big) = \oplus_{k \geq 0} \operatorname{Ext}_A^k\big(S(i), \hat{S}\big)$.

(4) *If A^* is finite dimensional, then* $\operatorname{rad} A^* = \oplus_{k \geq 1} \operatorname{Ext}_A^k(\hat{S}, \hat{S})$ *and* $\operatorname{rad}^\ell A^* \subseteq \oplus_{k \geq \ell} \operatorname{Ext}_A^k(\hat{S}, \hat{S})$.

Notice that in part (4) of the previous statement the containment $\operatorname{rad}^\ell A^* \subseteq \oplus_{k \geq \ell} \operatorname{Ext}_A^k(\hat{S}, \hat{S})$ is very often proper, *i.e.*, the filtration given by the powers of the radical of A^* (radical filtration, for short) will not, in general, coincide with the filtration obtained from the natural grading of A^* mentioned in part (1). One of the key points of our investigation is precisely the question, when will the relation $\operatorname{rad}^{\ell} A^* = \oplus_{k\geq\ell} \operatorname{Ext}_A^k(\hat{S},\hat{S})$ hold for every ℓ.

This question may also be formulated in terms of the species of the algebras involved. Let us recall that for the basic K-algebra A with primitive idempotents $\{ e_i \mid i \in I \}$ the *species* $\mathcal{S}(A)$ of A is the system $(D_i, i \in I; {}_iW_j, i,j \in I)$ of division algebras D_i, finite dimensional over K, and D_i-D_j-bimodules ${}_iW_j$, where $D_i = e_i\big(A/\operatorname{rad} A\big)e_i \simeq \operatorname{End}_A\big(S(i)\big)$ and ${}_iW_j = e_i\big(\operatorname{rad} A/\operatorname{rad}^2 A\big)e_j \simeq D\operatorname{Ext}_A^1\big(S(i),S(j)\big)$. (Here D stands for the standard K-duality.) When all the division algebras D_i are equal to K and the bimodules ${}_iW_j$ are direct sums of copies of the regular bimodule K (for example, when K is algebraically closed) then one may speak about the *quiver* $\Gamma(A)$ of A; hence the complete information is contained in an oriented graph having I as its vertex set and $\dim_K {}_iW_j$ arrows from the vertex i to the vertex j.

Furthermore, if a species $\mathcal{S} = (D_i, i \in I; {}_iW_j, i,j \in I)$ is given, then we may define the *dual species* $D\mathcal{S} = (\tilde{D}_i, i \in I; {}_i\tilde{W}_j, i,j \in I)$ for which the division algebras are $\tilde{D}_i = D_i$, and ${}_i\tilde{W}_j = D({}_jW_i)$ for $i,j \in I$. — Then it is not too difficult to see that the previous question whether the natural grading of A^* gives the radical filtration is equivalent to asking whether $\mathcal{S}(A^*) = D\mathcal{S}(A)$ holds. For quivers the previous condition translates to $\Gamma(A^*) = \Gamma(A^{op})$.

The principal question that we investigate in our paper is the following: given a finite dimensional quasi-hereditary algebra A, when is the Ext-algebra of A also quasi-hereditary? Or more generally: what can one say about A or A^* if one of them is quasi-hereditary?

To this end, let us recall the definition of a quasi-hereditary algebra (*cf.* [CPS1], [DR2]). Let A be a finite dimensional algebra with a fixed ordering $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of a complete set of primitive orthogonal idempotents. Denote by ε_i the idempotent $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$; for convenience let $\varepsilon_{n+1} = 0$. The *trace filtration* of a module M (with respect to the fixed order $\mathbf{e}$) is given by $M = M\varepsilon_1 A \supseteq M\varepsilon_2 A \supseteq \cdots \supseteq M\varepsilon_n A \supseteq 0$. Then we may define the *i-th standard module* $\Delta(i)$ to be the first non-trivial quotient in the trace filtration of the indecomposable projective module $P(i)$. Thus $\Delta(i) \simeq e_iA\big/e_iA\varepsilon_{i+1}A$. Note that $\Delta(i)$ is the largest quotient of $P(i)$ with composition factors $S(j)$ with $j \leq i$. The algebra A is called *quasi-hereditary* with respect to the ordering $\mathbf{e}$ if $\Delta(i)$ is *Schurian*, *i.e.*, $\operatorname{End}_A \Delta(i)$ is a division algebra for $1 \leq i \leq n$ and the trace filtration factors of the regular module A_A are isomorphic to direct sums of standard modules. In the sequel we shall also use the notations $U(i) = \operatorname{rad}\Delta(i)$ and $V(i) = e_iA\varepsilon_{i+1}A$, with the corresponding left modules denoted by $\Delta^\circ(i)$, $U^\circ(i)$ and $V^\circ(i)$. Thus we have the exact sequences $0 \to U(i) \to \Delta(i) \to S(i) \to 0$ and $0 \to V(i) \to P(i) \to \Delta(i) \to 0$. For the basic properties of quasi-hereditary algebras and standard modules we refer the reader to [PS], [DR1], [DR2] and [DK].

One of the main tools in our description is the concept of top submodules or top embeddings. Recall that a submodule $X \subseteq Y$ is said to be a *top submodule* (denoted by $X \overset{t}{\subseteq} Y$) if the embedding of X into Y induces an embedding of $\operatorname{top} X = X/\operatorname{rad} X$ into $\operatorname{top} Y = Y/\operatorname{rad} Y$ ([ADL1]). Or more formally: $X \overset{t}{\subseteq} Y$ if and only if $X \subseteq Y$ and $\operatorname{rad} X =$

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$X \cap \operatorname{rad} Y$. In this case the embedding of X into Y is also called a *top embedding*. A filtration of a module X is called a *top filtration* if all the terms of the filtrations are top submodules of X. An algebra A is called *lean* with respect to an ordering $\mathbf{e}$ of simple A-modules if $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for $1 \le i \le n$. Further properties of top embeddings as well as characterizations of lean algebras can be found in [ADL1] and [ADL2].

2. **The species of an Ext-algebra.**

For an arbitrary module $X \in \text{mod-}A$ let

$$\cdots \xrightarrow{d_{j+1}} \mathcal{P}_j(X) \xrightarrow{d_j} \cdots \xrightarrow{d_2} \mathcal{P}_1(X) \xrightarrow{d_1} \mathcal{P}_0(X) \xrightarrow{d_0} X \longrightarrow 0$$

be a minimal projective resolution of X, with the corresponding syzygies $\Omega_{j+1}(X) = \operatorname{Ker} d_j$ for $j = 0, 1, \ldots$.

DEFINITION 2.1. We say that a module $X \in \text{mod-}A$ belongs to $\mathcal{C}^{(i)} = \mathcal{C}_A^{(i)}$ for some $i \in \mathbb{N}$ if $\Omega_j(X) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_{j-1}(X)$ for $j = 1, 2, \ldots, i$. For convenience define $\mathcal{C}^{(0)} = \text{mod-}A$. Finally, let $\mathcal{C} = \mathcal{C}_A = \bigcap_{i=0}^{\infty} \mathcal{C}^{(i)}$. – Similarly, one may define the subcategory $\mathcal{C}_A^\circ \subset A\text{-mod}$ of left A-modules.

REMARK 2.2. (i) $X \in \mathcal{C}^{(i)}$ does not depend on the particular minimal projective resolution chosen for X. In particular, all projective modules will belong to $\mathcal{C}$.

(ii) $X \in \mathcal{C}^{(i)}$ implies $X \in \mathcal{C}^{(j)}$ for every $j \le i$.

(iii) For any $i \ge 1$ we have that $X \in \mathcal{C}^{(i)}$ if and only if $\Omega_1(X) \in \mathcal{C}^{(i-1)}$ and $\Omega_1(X) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(X)$. If $\operatorname{gl.dim} A = \ell < \infty$ then $\mathcal{C}_A = \mathcal{C}_A^{(\ell)}$.

(iv) From the definition it is clear that A is lean if and only if the right and left standard modules $\Delta(i)$ and $\Delta^\circ(i)$ belong to $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(1)^\circ}$, respectively.

The next three lemmas give some of the closure properties of these subcategories.

LEMMA 2.3. *Let $0 \to X \to Y \to Z \to 0$ be a split exact sequence. Then for given $i \in \mathbb{N}$, we have $Y \in \mathcal{C}^{(i)}$ if and only if $X, Z \in \mathcal{C}^{(i)}$.*

PROOF. Consider the sequence $0 \to \Omega_1(X) \oplus \Omega_1(Z) \to \mathcal{P}_0(X) \oplus \mathcal{P}_0(Z) \to X \oplus Z \to 0$. Then clearly $\Omega_1(X) \oplus \Omega_1(Z)$ maps into $\operatorname{rad}\big(\mathcal{P}_0(X) \oplus \mathcal{P}_0(Z)\big) = \operatorname{rad} \mathcal{P}_0(X) \oplus \operatorname{rad} \mathcal{P}_0(Z)$, and this is a top embedding if and only if $\Omega_1(X) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(X)$ and $\Omega_1(Z) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Z)$. By induction on i and Remark 2.2. (iii) we are done. ∎

LEMMA 2.4. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence with the map $X \to Y$ a top embedding. Then if X and Z both belong to $\mathcal{C}^{(i)}$ then also $Y \in \mathcal{C}^{(i)}$.*

PROOF. We may assume $i \geq 1$. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_1(X) & \longrightarrow & \Omega_1(Y) & \longrightarrow & \Omega_1(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_0(X) & \longrightarrow & \mathcal{P}_0(Y) & \longrightarrow & \mathcal{P}_0(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here $\Omega_1(X), \Omega_1(Z) \in \mathcal{C}^{(i-1)}$, and $\Omega_1(X) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(X)$, $\Omega_1(Z) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Z)$. Note also that the middle row is split, *i.e.*, $\mathcal{P}_0(Y) = \mathcal{P}_0(X) \oplus \mathcal{P}_0(Z)$, since $X \overset{t}{\subseteq} Y$. Thus we get that $\Omega_1(X) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Y)$, hence $\Omega_1(X) \overset{t}{\subseteq} \Omega_1(Y)$. So by induction on i we get that $\Omega_1(Y) \in \mathcal{C}^{(i-1)}$. Also, $\Omega_1(Y)/\Omega_1(X) = \Omega_1(Z) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Z)$ and $\operatorname{rad} \mathcal{P}_0(Z)$ is a direct summand of $\operatorname{rad} \mathcal{P}_0(Y)/\Omega_1(X)$, thus $\Omega_1(Y)/\Omega_1(X) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Y)/\Omega_1(X)$. Now, Lemma 1.1. c) of [ADL1] gives that $\Omega_1(Y) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Y)$, so by Remark 2.2. (iii), $Y \in \mathcal{C}^{(i)}$, as required. ∎

It is easy to show that the converse of Lemma 2.4 does not hold. Actually, examples will be given in the last section showing that $X, Y \in \mathcal{C}$ does not imply that $Z \in \mathcal{C}^{(1)}$ and $Y, Z \in \mathcal{C}$ does not imply that $X \in \mathcal{C}^{(1)}$.

LEMMA 2.5. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence with $X \overset{t}{\subseteq} \operatorname{rad} Y$. Then we have:*

(i) *if $X \in \mathcal{C}^{(i)}$ and $Y \in \mathcal{C}^{(i+1)}$ then $Z \in \mathcal{C}^{(i+1)}$;*
(ii) *if $Z \in \mathcal{C}^{(i+1)}$ and Y is projective then $X \in \mathcal{C}^{(i)}$;*
(iii) *if $Z \in \mathcal{C}^{(i)}$ and X is projective then $Y \in \mathcal{C}^{(i)}$.*

PROOF. (i) Consider the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega_1(Y) & \longrightarrow & \Omega_1(Z) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{P}_0(Y) & \overset{\sim}{\longrightarrow} & \mathcal{P}_0(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}$$

Here $X \subseteq \operatorname{rad} Y$ implies that $\mathcal{P}_0(Y) \simeq \mathcal{P}_0(Z)$, and the Snake Lemma gives us an additional exact sequence $0 \to \Omega_1(Y) \to \Omega_1(Z) \to X \to 0$. Since $\Omega_1(Y) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Y) = \operatorname{rad} \mathcal{P}_0(Z)$ we get that $\Omega_1(Y) \overset{t}{\subseteq} \Omega_1(Z)$. By assumption, $X \in \mathcal{C}^{(i)}$, and by Remark 2.2. (iii), $\Omega_1(Y) \in \mathcal{C}^{(i)}$. Hence by Lemma 2.4 we get that $\Omega_1(Z) \in \mathcal{C}^{(i)}$. Furthermore, $X \overset{t}{\subseteq} \operatorname{rad} Y = \operatorname{rad} \mathcal{P}_0(Y)/\Omega_1(Y)$. Since $\Omega_1(Y) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Y)$, we get by Lemma 1.1. c) of [ADL1] that $\Omega_1(Z) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Y) = \operatorname{rad} \mathcal{P}_0(Z)$. Hence $Z \in \mathcal{C}^{(i+1)}$.

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(ii) This follows from the definition of $\mathcal{C}^{(i)}$ and by Remark 2.2. (i) and (iii).

(iii) Consider the diagram we had in the proof of part (i). Again we get an exact sequence $0 \to \Omega_1(Y) \to \Omega_1(Z) \to X \to 0$ which is in this case split, as X is projective. Since $\Omega_1(Z) \in \mathcal{C}^{(i-1)}$, we get by Lemma 2.3 that $\Omega_1(Y) \in \mathcal{C}^{(i-1)}$. Moreover, $\Omega_1(Y) \overset{t}{\subseteq} \Omega_1(Z) \overset{t}{\subseteq} \operatorname{rad} \mathcal{P}_0(Z) = \operatorname{rad} \mathcal{P}_0(Y)$, hence by Remark 2.2. (iii), $Y \in \mathcal{C}^{(i)}$, as required. ∎

As in the case of Lemma 2.4, it is again easy to construct examples to show that the statements of (ii) and (iii) of the previous lemma cannot be strengthened. Examples will be given in the last section that $Y, Z \in \mathcal{C}$ does not imply that $X \in \mathcal{C}^{(1)}$ and $X, Z \in \mathcal{C}$ does not imply that $Y \in \mathcal{C}^{(1)}$.

The next proposition gives one of the most important homological properties of the elements of $\mathcal{C}^{(i)}$. It turns out (*cf.* Proposition 2.11) that this property almost completely characterizes these modules. The statement generalizes one direction of Theorem 3 of [ADL2].

PROPOSITION 2.6. *If $X \in \mathcal{C}^{(i)}$ for some $i \in \mathbb{N}$, then the natural maps $\operatorname{Ext}_A^k(\operatorname{top} X, S) \to \operatorname{Ext}_A^k(X, S)$ are surjective for every $0 \leq k \leq i$ and every (semi)simple module S.*

PROOF. We shall proceed by induction on i. Clearly, the statement is true for $i = 0$. For $i = 1$ this is just Proposition 2 of [ADL2]; actually for the case $i = 1$, $k = 1$, the converse statement is also proved there.

Assume now that the statement is proved for $i - 1$. Clearly, the only case to consider is $k = i$, as $X \in \mathcal{C}^{(i)}$ implies $X \in \mathcal{C}^{(i-1)}$ hence by the induction hypothesis we get the surjectivity between the Ext^k-modules for $k < i$.

Thus assume that $i > 1$ and let $\mathcal{E}_1$ be an exact sequence of length $k = i$ between S and X:

$$\mathcal{E}_1 : 0 \to S \to X_{k-1} \to X_{k-2} \to \cdots \to X_1 \to X_0 \to X \to 0.$$

Then $\mathcal{E}_1$ is equivalent to a sequence $\mathcal{E}_2$ via the following diagram:

$$\begin{array}{ccccccccccccccc}
\mathcal{E}_2\colon & 0 & \to & S & \to & X_{k-1} & \to & \cdots & \to & \tilde{X}_1 & \to & \mathcal{P}_0(X) & \to & X & \to & 0 \\
& & & \| & & \| & & & & \downarrow & & \downarrow & & \| & & \\
\mathcal{E}_1\colon & 0 & \to & S & \to & X_{k-1} & \to & \cdots & \to & X_1 & \to & X_0 & \to & X & \to & 0.
\end{array}$$

Thus we may write $\mathcal{E}_2$ as the Yoneda composite of $0 \to S \to X_{k-1} \to \cdots \to \tilde{X}_1 \to \Omega_1(X) \to 0$ and $0 \to \Omega_1(X) \to \mathcal{P}_0(X) \to X \to 0$. By assumption we have $\Omega_1(X) \in \mathcal{C}^{(i-1)}$, hence by induction we get that $\mathcal{E}_2$ is equivalent to an exact sequence $\mathcal{E}_3$ (shown similarly as a Yoneda composite) via:

$$\begin{array}{ccccccccccccc}
\mathcal{E}_2\colon 0 \to S \to \cdots \to & \tilde{X}_1 & \to & \Omega_1(X) & \to 0 \to & \Omega_1(X) & \to & \mathcal{P}_0(X) & \to & X & \to 0 \\
\| & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\
\mathcal{E}_3\colon 0 \to S \to \cdots \to & X_1' & \to & \operatorname{top} \Omega_1(X) & \to 0 \to & \operatorname{top} \Omega_1(X) & \to & \tilde{\mathcal{P}}_0 & \to & X & \to 0.
\end{array}$$

Since $\operatorname{top} \Omega_1(X)$ is semisimple and by $i > 1$, $X \in \mathcal{C}^{(1)}$, we get that the last part of the sequence is the image of an extension of $\operatorname{top} X$ with $\operatorname{top} \Omega_1(X)$:

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{top} \Omega_1(X) & \to & \tilde{\mathcal{P}}_0 & \to & X & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{top} \Omega_1(X) & \to & \tilde{\tilde{\mathcal{P}}}_0 & \to & \operatorname{top} X & \to & 0
\end{array}$$

Hence we get the surjectivity of the required Ext-maps. ∎

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COROLLARY 2.7. *Let $X \in C^{(i)}$ and S semisimple. Then any extension in* $\mathrm{Ext}^k(X,S)$ *for $2 \leq k \leq i$ can be represented by a long exact sequence between S and X where the kernel-cokernel terms in the interior of the sequence are all semisimple.*

PROOF. We shall give a proof by induction on k. Any long exact sequence $\mathcal{E}_1$ between S and X is equivalent to a sequence $\mathcal{E}_2$ via

$$\begin{array}{llllllllllll}
\mathcal{E}_2\colon 0 \to & S & \to & \tilde{X}_{k-1} & \to & \mathcal{P}_{k-2}(X) & \to \cdots \to & \mathcal{P}_0(X) & \to & X & \to 0 \\
 & \| & & \downarrow & & \downarrow & & \downarrow & & \| & \\
\mathcal{E}_1\colon 0 \to & S & \to & X_{k-1} & \to & X_{k-2} & \to \cdots \to & X_0 & \to & X & \to 0.
\end{array}$$

Here the kernel-cokernel terms of $\mathcal{E}_2$ are the syzygies $\Omega_{k-1}(X),\ldots,\Omega_1(X)$. Thus $\mathcal{E}_2$ is the Yoneda product of an extension of S by $\Omega_{k-1}(X)$ and the $k-1$-long canonical exact sequence between $\Omega_{k-1}(X)$ and X. Now, by the assumption it follows that $\Omega_{k-1} \in C^{(1)}$, and thus Proposition 2.6 implies that the first short exact sequence is a lifting of an extension of S by $\mathrm{top}\,\Omega_{k-1}(X)$. Hence $\mathcal{E}_2$ is equivalent to the following Yoneda-composite $\mathcal{E}_3$:

$$\begin{array}{llllllllllll}
\mathcal{E}_2\colon 0\to & S\to & \tilde{X}_{k-1}\to & \Omega_{k-1}(X) & \to 0 \to & \Omega_{k-1}(X) & \to\cdots\to & \mathcal{P}_0(X) & \to & X & \to 0 \\
 & \| & \downarrow & \downarrow & & \downarrow & & \downarrow & & \| & \\
\mathcal{E}_3\colon 0\to & S\to & \tilde{X}'_{k-1}\to & \mathrm{top}\,\Omega_{k-1}(X) & \to 0\to & \mathrm{top}\,\Omega_{k-1}(X) & \to\cdots\to & \tilde{\mathcal{P}}_0 & \to & X & \to 0.
\end{array}$$

Consequently, if $k=2$ then we are done. If $k>2$ then we may apply the inductional hypothesis for the second sequence between the semisimple module $\mathrm{top}\,\Omega_{k-1}(X)$ and X. This gives us the statement. ■

In view of Proposition 1.1. (4) and the remarks following it, the previous observation has the following direct consequence for the species of the Ext-algebra A^*.

COROLLARY 2.8. *If A is an algebra where all simple modules belong to C_A then the species $\mathcal{S}(A^*)$ of A^* is equal to the dual of the species of A, that is $\mathcal{S}(A^*) = D\mathcal{S}(A)$.*

PROOF. We have to show only that for $k \geq 2$, every extension in $\mathrm{Ext}^k_A(\hat{S},\hat{S})$ is equivalent to a sequence which is the Yoneda product of k short exact sequences with semisimple outer terms. But this is precisely the statement of Corollary 2.7. ■

The converse of the previous statement is also true.

PROPOSITION 2.9. *Let A be an algebra such that $\mathcal{S}(A^*) = D\mathcal{S}(A)$. Then every simple module over A belongs to C_A.*

PROOF. Assume that there is a simple module S for which $S \in C^{(i-1)} \setminus C^{(i)}$; and assume i is minimal in this respect. Note that $i \geq 2$, as $S \in C^{(1)}$ always holds. Thus $\Omega_{i-1}(S) \notin C^{(1)}$, more precisely $\Omega_i(S) \nsubseteq \mathrm{rad}\,\mathcal{P}_{i-1}(S)$, hence there exists a simple module S' and a short exact sequence $\mathcal{E}'\colon 0 \to S' \to \tilde{P}_{i-1} \to \Omega_{i-1}(S) \to 0$ which is the pushout of the sequence $0 \to \Omega_i(S) \to \mathcal{P}_{i-1}(S) \to \Omega_{i-1}(S) \to 0$ and for which $S' \subseteq \mathrm{rad}^2\,\tilde{P}_{i-1}$. By Lemma 1 of [ADL2], this implies that one cannot obtain $\mathcal{E}'$ as a pullback of an extension of a semisimple module with S'. Choose now the element $\mathcal{E}$ of $\mathrm{Ext}^i(S,S')$, corresponding to $\mathcal{E}' \in \mathrm{Ext}^1\big(\Omega_{i-1}(S),S'\big)$. If it would be possible to split this sequence into the product of short exact sequences with semisimple outer terms (*i.e.*, if $\mathcal{E}$ would be equivalent to such a sequence) then we would get a lifting from the first such sequence to $\mathcal{E}'$, a contradiction. Thus $\mathcal{S}(A^*) \neq D\mathcal{S}(A)$.

Note that by the minimality of i and by Corollary 2.7, for arbitrary simple modules S and S' and for arbitrary $k < i$ the elements of $\operatorname{Ext}^k(S, S')$ can always be factored into the product of shorter exact sequences with semisimple outer terms. Since $\mathcal{E}$ cannot be factored properly in this way, $\mathcal{E} \notin \operatorname{rad}^2 A^*$. ∎

Hence we may formulate the following theorem about the species of the Ext-algebra of an algebra A.

THEOREM 2.10. *The following statements are equivalent for an algebra A.*

(i) $S \in \mathcal{C}_A$ *for every simple right module* S;

(ii) $S^\circ \in \mathcal{C}_A^\circ$ *for every simple left module* S°;

(iii) $\mathcal{S}(A^*) = D\mathcal{S}(A)$.

PROOF. The equivalence of (i) and (iii) is just Corollary 2.8 and Proposition 2.9. For the equivalence of (ii) and (iii) let us observe first that if $\mathcal{S}(A) = (D_i, i \in I; {}_iW_j, i,j \in I)$ then $\mathcal{S}(A^{op}) = (D_i^{op}, i \in I; {}_jW_i, i,j \in I)$ with the modules ${}_jW_i$ being D_i^{op}-D_j^{op}-bimodules in a natural way. Then from the fact that $(A^{op})^* = (A^*)^{op}$ we get that condition (iii) is equivalent to the dual condition $\mathcal{S}\big((A^{op})^*\big) = D\mathcal{S}(A^{op})$. Now we may apply the equivalence of (i) and (iii) to the algebra A^{op}. ∎

Actually, as we mentioned earlier, under the assumption that the simple modules belong to $\mathcal{C}$, one can prove the converse of Proposition 2.6, too.

PROPOSITION 2.11. *Assume that the conditions of Theorem 2.10 are satisfied, i.e., every simple A-module S is in $\mathcal{C}_A$. Then the following statements are equivalent for a module X:*

(i) $X \in \mathcal{C}^{(i)}$;

(ii) *the natural map* $\operatorname{Ext}_A^k(\operatorname{top} X, S) \longrightarrow \operatorname{Ext}_A^k(X, S)$ *is surjective for every* $0 \leq k \leq i$ *and for every simple module S.*

PROOF. We have to show only that (ii) implies (i) under the assumption on the simple modules. Let us note first that for $i = 1$ the equivalence of these two conditions is stated in Proposition 2 of [ADL2]. Thus we may assume $i > 1$.

Let us recall from the proof of Proposition 2.9 that if $X \in \mathcal{C}^{(j-1)} \setminus \mathcal{C}^{(j)}$ for some $1 \leq j \leq i$, then there exists an exact sequence $\mathcal{E} \in \operatorname{Ext}^j(X, S)$, with S simple so that $\mathcal{E}$ is the Yoneda product of some sequence $\mathcal{E}_1 \in \operatorname{Ext}^1\big(\Omega_{j-1}(X), S\big)$ and the canonical sequence $\mathcal{E}_2 \in \operatorname{Ext}^{j-1}\big(X, \Omega_{j-1}(X)\big)$; moreover the sequence $\mathcal{E}_1$ is not in the image of the map $\operatorname{Ext}^1\big(\operatorname{top}\Omega_{j-1}(X), S\big) \longrightarrow \operatorname{Ext}^1\big(\Omega_{j-1}(X), S\big)$. On the other hand, if $\mathcal{E}$ were a lifting of a sequence $\mathcal{E}' \in \operatorname{Ext}^j(\operatorname{top} X, S)$, then by the assumption on the simple modules, and by Corollary 2.7, $\mathcal{E}'$ could be factored as the product of short exact sequences with semisimple outer terms. This would also result in a lifting of a sequence in $\operatorname{Ext}^1\big(\operatorname{top}\Omega_{j-1}(X), S\big)$ to $\mathcal{E}_1$, a contradiction. ∎

It is clear that the assumption that all simple modules belong to $\mathcal{C}$ is indeed necessary for the equivalence, as otherwise one could choose for X a simple module which is not in $\mathcal{C}^{(i)}$ but trivially satisfying condition (ii).

3. **The functor Ext*.** We shall assume in this section that the Ext-algebra A^* of the finite dimensional algebra A is itself finite dimensional, *i.e.*, $\operatorname{gl.dim} A < \infty$.

Let us recall first that the functor $\operatorname{Ext}^*\colon \text{mod-}A \longrightarrow A^*\text{-mod}_{gr}$ defined in Proposition 1.1 is the direct sum of the functors $\operatorname{Ext}^k_A(-,\hat{S})$ for $k \geq 0$. That is, for an arbitrary module $X \in \text{mod-}A$ let $\operatorname{Ext}^*(X) = \operatorname{Ext}^*_A(X,\hat{S}) = \oplus_{k\geq 0} \operatorname{Ext}^k(X,\hat{S})$ and similarly define the action on morphisms. For a module $X \in A^*\text{-mod}_{gr}$, let $X[j]$ stand for the shifted graded module, *i.e.*, the one for which $X[j]_i = X_{i-j}$.

Here are some of the basic properties of Ext*.

LEMMA 3.1.

(i) *If* $X \in \mathcal{C}_A$, *then we have* $\operatorname{rad}\operatorname{Ext}^*(X) = \oplus_{k\geq 1} \operatorname{Ext}^k_A(X,\hat{S}) = \operatorname{Ext}^*\big(\Omega_1(X)\big)[1]$ *and in general for arbitrary* $\ell \geq 1$ *we have* $\operatorname{rad}^\ell \operatorname{Ext}^*(X) = \oplus_{k\geq \ell} \operatorname{Ext}^k_A(X,\hat{S}) = \operatorname{Ext}^*\big(\Omega_\ell(X)\big)[\ell]$.

(ii) *If* $\hat{S} \in \mathcal{C}_A$, *that is, if all simple A-modules belong to* $\mathcal{C}_A$ *then the condition* $X \in \mathcal{C}_A$ *is equivalent to the condition that* $\operatorname{rad}\operatorname{Ext}^*(X) = \oplus_{k\geq 1} \operatorname{Ext}^k_A(X,\hat{S})$.

PROOF. The first statement follows from Proposition 2.6 and Corollary 2.7, since $\operatorname{rad}^\ell \operatorname{Ext}^*(X) = (\operatorname{rad}^\ell A^*)X$. The second statement is just a reformulation of Proposition 2.11. ■

The next two lemmas show that $\operatorname{Ext}^*\colon \text{mod-}A \longrightarrow A^*\text{-mod}_{gr}$ preserves certain exact sequences.

LEMMA 3.2. *Let* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be exact with the map* $X \longrightarrow Y$ *a top embedding. If* $X \in \mathcal{C}$, *then the sequence* $0 \longrightarrow \operatorname{Ext}^*(Z) \longrightarrow \operatorname{Ext}^*(Y) \longrightarrow \operatorname{Ext}^*(X) \longrightarrow 0$ *is also exact. If in addition* $Z \in \mathcal{C}$, *then the embedding* $\operatorname{Ext}^*(Z) \longrightarrow \operatorname{Ext}^*(Y)$ *is also a top embedding.*

PROOF. We have to show that the sequences $0 \longrightarrow \operatorname{Ext}^i_A(Z,\hat{S}) \longrightarrow \operatorname{Ext}^i_A(Y,\hat{S}) \longrightarrow \operatorname{Ext}^i_A(X,\hat{S}) \longrightarrow 0$ are exact for $i \geq 0$. Let us assume that the exactness is proved for indices smaller than i and we shall prove it for i. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{top} X & \longrightarrow & \operatorname{top} Y & \longrightarrow & \operatorname{top} Z & \longrightarrow & 0. \end{array}$$

Here the bottom row is exact by the assumption $X \overset{t}{\subseteq} Y$. Applying now the functor $\operatorname{Ext}^i_A(-,\hat{S})$, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(Z,\hat{S}) & \longrightarrow & \operatorname{Ext}^i_A(Y,\hat{S}) & \longrightarrow & \operatorname{Ext}^i_A(X,\hat{S}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Ext}^i_A(\operatorname{top} Z,\hat{S}) & \longrightarrow & \operatorname{Ext}^i_A(\operatorname{top} Y,\hat{S}) & \longrightarrow & \operatorname{Ext}^i_A(\operatorname{top} X,\hat{S}) & \longrightarrow & 0. \end{array}$$

Here the bottom row is exact, since the original sequence was split. Also the beginning of the top row is exact by the inductional assumption. Finally, since $X \in \mathcal{C}$, the last vertical map is surjective by Proposition 2.6, hence the top row is exact also at the last step. The second part of the statement follows from Lemma 3.1.(i). ■

LEMMA 3.3. *Let* $0 \to X \to Y \to Z \to 0$ *be exact with* $X \subseteq \operatorname{rad} Y$. *If* $Y \in \mathcal{C}$, *then the sequence* $0 \to \operatorname{Ext}^*(X)[1] \to \operatorname{Ext}^*(Z) \to \operatorname{Ext}^*(Y) \to 0$ *is also exact. If in addition* $Z \in \mathcal{C}$, *then* $\operatorname{Ext}^*(X)[1] \subseteq \operatorname{rad}\operatorname{Ext}^*(Z)$, *while adding the condition* $X \in \mathcal{C}$ *implies that* $\operatorname{Ext}^*(X)[1] \overset{t}{\subseteq} \operatorname{rad}\operatorname{Ext}^*(Z)$.

PROOF. Let us notice first that $\operatorname{top} Y \simeq \operatorname{top} Z$ and the morphism $Y \to \operatorname{top} Y$ may be factored through Z via $Y \to Z \to \operatorname{top} Z \simeq \operatorname{top} Y$. Proposition 2.6 implies that the maps $\operatorname{Ext}^i_A(\operatorname{top} Y, \hat{S}) \to \operatorname{Ext}^i_A(Y, \hat{S})$ are surjective, hence the maps $\operatorname{Ext}^i_A(Z, \hat{S}) \to \operatorname{Ext}^i_A(Y, \hat{S})$ are also surjective, and the kernel of this map is just $\operatorname{Ext}^{i-1}(X, \hat{S})$. Thus the first part of the statement follows. The second part now follows from Lemma 3.1. (i) since $\operatorname{Ext}^*(X)[1] \subseteq \oplus_{k\geq 1} \operatorname{Ext}^k_A(Z, \hat{S}) = \operatorname{rad}\operatorname{Ext}^*(Z)$. Finally, if $X \in \mathcal{C}$ is assumed besides the original condition that $Y \in \mathcal{C}$, then Lemma 2.5. (i) implies that $Z \in \mathcal{C}$. Hence by the previous considerations and by applying Lemma 3.1. (i) for the module X, we get that $\operatorname{Ext}^*(X)[1] \overset{t}{\subseteq} \operatorname{rad}\operatorname{Ext}^*(Z)$, as required. ∎

Let us now denote by $S^{*\circ}(i)$ and $P^{*\circ}(i)$ the corresponding simple and indecomposable projective left A^*-modules. Based on the previous lemmas, we easily get the following statements that we shall need in the sequel.

PROPOSITION 3.4.
(i) $\operatorname{Ext}^*\big(S(i)\big) = P^{*\circ}(i)$.
(ii) $\operatorname{Ext}^*\big(P(i)\big) = S^{*\circ}(i)$.
(iii) $\operatorname{Ext}^*\big(\operatorname{rad} P(i)\big)[1] = \operatorname{rad} P^{*\circ}(i)$.

PROOF. The statements of (i) and (ii) are trivial, while (iii) follows from (i), (ii) and Lemma 3.3. ∎

The next statement establishes a connection between the categories $\mathcal{C}_A$ and $\mathcal{C}^\circ_{A^*}$.

PROPOSITION 3.5. *If* $X, \operatorname{rad} X \in \mathcal{C}_A$, *then* $\operatorname{Ext}^*(X) \in \mathcal{C}^{\circ(1)}_{A^*}$. *Thus if* $\operatorname{rad}^i X \in \mathcal{C}$ *for every* i *then* $\operatorname{Ext}^*(X) \in \mathcal{C}^\circ_{A^*}$.

PROOF. Take the exact sequence $0 \to \operatorname{rad} X \to X \to \operatorname{top} X \to 0$. Then Lemma 3.3 implies that the following sequence is also exact:

$$0 \to \operatorname{Ext}^*(\operatorname{rad} X)[1] \to \operatorname{Ext}^*(\operatorname{top} X) \to \operatorname{Ext}^*(X) \to 0\,.$$

Here the middle term is projective by Proposition 3.4. (i). Since $\operatorname{rad} X \in \mathcal{C}$ by assumtion, Lemma 3.3 implies that $\operatorname{Ext}^*(\operatorname{rad} X)[1] \overset{t}{\subseteq} \operatorname{rad}\operatorname{Ext}^*(\operatorname{top} X)$. Hence $\operatorname{Ext}^*(X) \in \mathcal{C}^{\circ(1)}_{A^*}$.

The second statement now follows by straightforward induction. ∎

4. **Quasi-hereditary algebras with special filtrations.** In this section we give a sufficient condition for an algebra to have a quasi-hereditary Ext-algebra. The condition is in terms of the existence of certain filtrations. The canonical constructions from [ADL1] will result in algebras satisfying these conditions. In particular we get that shallow, left medial, right medial and replete algebras will have replete, right medial, left medial and shallow Ext-algebras, respectively.

Let us take an algebra A with a fixed ordering $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of its primitive orthogonal idempotents, and let us consider A^* with the "opposite order", *i.e.*, with $\mathbf{f} = (f_n, f_{n-1} \ldots, f_1)$, where $f_i = \mathrm{id}_{S(i)}$. Take $\varphi_i = f_i + f_{i-1} + \cdots + f_1$ and $\varphi_0 = 0$. Let us denote by $\Delta^{*\circ}(i)$ the corresponding standard left A^*-modules (with respect to this opposite order), furthermore let $U^{*\circ}(i)$ and $V^{*\circ}(i)$ stand for the radical and the first syzygy of $\Delta^{*\circ}(i)$. Then similarly to the results of Proposition 3.4, one may identify the left standard A^*-modules.

PROPOSITION 4.1. *Assume that $(A, \mathbf{e})$ is quasi-hereditary with $\Delta(i) \in \mathcal{C}_A$ and $U(i) \in \mathcal{C}_A$. Then the left standard module $\Delta^{*\circ}(i)$ of A^* is Schurian; moreover,* $\mathrm{Ext}^*\big(\Delta(i)\big) = \Delta^{*\circ}(i)$, $\mathrm{Ext}^*\big(U(i)\big)[1] = V^{*\circ}(i)$ *and* $\mathrm{Ext}^*\big(V(i)\big)[1] = U^{*\circ}(i)$.

PROOF. By Lemma 3.1, the condition $\Delta(i) \in \mathcal{C}_A$ implies that $\mathrm{rad}\,\mathrm{Ext}^*\big(\Delta(i)\big) = \oplus_{k\geq 1} \mathrm{Ext}_A^k\big(\Delta(i), \hat{S}\big)$. Hence we get that $\mathrm{top}\,\mathrm{Ext}^*\big(\Delta(i)\big) \simeq \mathrm{Hom}_A\big(\Delta(i), \hat{S}\big) \simeq \mathrm{Hom}_A\big(S(i), S(i)\big)$ is simple and of type $S^{*\circ}(i)$. Moreover, since $\Delta(i)$ has no extensions with simple modules $S(j)$ for $j < i$, and $\mathrm{Ext}_A^k\big(\Delta(i), S(i)\big) \neq 0$ if and only if $k = 0$ (*cf.* for example [DR2]), we get that the composition factors of $\mathrm{Ext}^*\big(\Delta(i)\big)$ (with the exception of the top factor) are all of type $S^{*\circ}(j)$ with $j > i$. Thus $\mathrm{Ext}^*\big(\Delta(i)\big)$ is a homomorphic image of $\Delta^{*\circ}(i)$.

On the other hand, consider the sequence $0 \to U(i) \to \Delta(i) \to S(i) \to 0$. By Lemma 3.3 we get that the following sequence is also exact:

$$0 \to \mathrm{Ext}^*\big(U(i)\big)[1] \to \mathrm{Ext}^*\big(S(i)\big) \to \mathrm{Ext}^*\big(\Delta(i)\big) \to 0.$$

Moreover, $\mathrm{Ext}^*\big(U(i)\big)[1] \subseteq \mathrm{rad}\,\mathrm{Ext}^*\big(S(i)\big)$, since $\mathrm{Ext}^*\big(S(i)\big) \simeq P^{*\circ}(i)$ by part (i). Finally, using again the fact that $U(i) \in \mathcal{C}_A$, we get by Lemma 3.1.(i) that $\mathrm{Ext}^*\big(U(i)\big)[1] \subseteq A^*\varphi_{i-1}P^{*\circ}(i)$ hence $\Delta^{*\circ}(i)$ is a homomorphic image of $\mathrm{Ext}^*\big(\Delta(i)\big)$. Comparing the two results we get the statement for $\mathrm{Ext}^*\big(\Delta(i)\big)$. We also get that $\Delta^{*\circ}(i)$ is Schurian, since the simple factor $S^{*\circ}(i)$ appears only once as a composition factor of $\Delta^{*\circ}(i)$.

The statement that $\mathrm{Ext}^*\big(U(i)\big)[1] = V^{*\circ}(i)$ follows immediately from the previous considerations. Furthermore, by Lemma 3.1. (i) we have also $U^{*\circ}(i) = \mathrm{rad}\,\Delta^{*\circ}(i) = \mathrm{rad}\,\mathrm{Ext}^*\big(\Delta(i)\big) = \oplus_{k\geq 1} \mathrm{Ext}_A^k\big(\Delta(i), \hat{S}\big) = \oplus_{k\geq 0} \mathrm{Ext}_A^k\big(V(i), \hat{S}\big)[1] = \mathrm{Ext}^*\big(V(i)\big)[1]$. The proof is now complete. ∎

In order to find a subclass of quasi-hereditary algebras which is closed under taking Ext-algebras, we introduce now the main concept of this section.

DEFINITION 4.2. An algebra A with a complete sequence of primitive orthogonal idempotents $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is called *solid* if it satisfies the following conditions for $1 \leq i \leq n$:

(1) $\Delta(i)$ is Schurian;

(2) $V(i) \overset{t}{\subseteq} \mathrm{rad}\,P(i)$;

(3) $U(i)$ has a top filtration by $S(j)$'s and $\Delta(j)$'s for $j < i$;

(4) $V(i)$ has a top filtration by $\Delta(j)$'s and $P(j)$'s for $j > i$.

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PROPOSITION 4.3. *If the algebra* $(A, \mathbf{e})$ *is solid then it is a lean quasi-hereditary algebra with* $S(i), \Delta(i), U(i) \in \mathcal{C}_A$ *for* $1 \leq i \leq n$.

PROOF. To prove the quasi-heredity, observe first, that from the filtration conditions on $V(i)$ it is easy to see that the module A_A has a Δ-filtration with standard modules which are, by assumption, Schurian. Hence the algebra $(A, \mathbf{e})$ is quasi-hereditary.

To prove that the algebra A is lean, it is enough to show that it satisfies condition (2) of Theorem 2.1 in [ADL1], *i.e.*, that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and the trace filtration of $U(i)$ is a top filtration for every i. The first part is included in the definition of a solid algebra. For the second part, it is enough to prove the following observation: A module X has top trace filtration if and only if it has a top filtration where the quotients of the consecutive terms are homomorphic images of the standard modules.

One direction of the statement is obvious since a top trace filtration can be refined to a top filtration where all the quotients of the filtration are local modules. For the opposite direction, assume that X has a filtration with quotients being homomorphic images of standard modules. If ℓ is the largest index for which an image of $\Delta(\ell)$ will occur as a quotient in the filtration then clearly $X\varepsilon_{\ell+1}A = 0$; moreover $X\varepsilon_\ell A \overset{t}{\subseteq} X$. The rest will now follow by downward induction and from the fact that the natural image of a top filtration of X when factoring out with the trace submodule $X\varepsilon_\ell A$ remains a top filtration. (It is worth mentioning that, unless we factor out with the trace of a projective module, it is not true in general that the natural image of a top submodule is necessarily a top submodule of the image.) Hence $(A, \mathbf{e})$ is lean.

Next we shall prove by downward induction on i that $\Delta(i) \in \mathcal{C}_A$. The statement is clear for $\Delta(n)$ since it is projective. So assume the statement for indices larger than i. Then the induction hypothesis and Lemma 2.4 imply that $V(i) \in \mathcal{C}_A$, hence by Remark 2.2. (iii) we get that $\Delta(i) \in \mathcal{C}_A$. Thus the statement is proved for $\Delta(i)$, $1 \leq i \leq n$. In particular, $S(1) = \Delta(1) \in \mathcal{C}_A$. By induction on i and using Lemma 2.4 we get that $U(i) \in \mathcal{C}_A$ and hence by Lemma 2.5. (i) we have that $S(i) \in \mathcal{C}_A$. This proves the statement. ∎

PROPOSITION 4.4. *If A is solid with respect to the sequence* $\mathbf{e} = (e_1, e_2, \ldots, e_n)$*, then the algebras* $A/A\varepsilon_{i+1}A$ *and* $\varepsilon_i A \varepsilon_i$ *for* $1 \leq i \leq n$ *are also solid.*

PROOF. Clearly, in both cases we may apply induction. Thus it is enough to prove that the algebras $A/A\varepsilon_n A$ and $\varepsilon_2 A \varepsilon_2$ are solid. For the second statement one only has to observe that if $e \in A$ is an idempotent element, $X \overset{t}{\subseteq} Y$ in mod-A and $X = XeA$, then $Xe \overset{t}{\subseteq} Ye$ in mod-eAe. Thus the top filtrations of $U(i)$ and $V(i)$ will be inherited from A to the algebra $\varepsilon_2 A \varepsilon_2$. The rest is easy, as is the case of the algebra A/Ae_nA. ∎

Let us mention, that the filtration conditions (3) and (4) for $A/A\varepsilon_{i+1}A$ and $\varepsilon_i A \varepsilon_i$ are precisely those that come by natural restrictions from the filtrations for A.

The main result of this section is the following theorem.

THEOREM 4.5. *Let* $(A, \mathbf{e})$ *be a solid algebra. Then:*
(a) $\big((A^*)^{op}, \mathbf{f}\big)$ *is solid (hence quasi-hereditary);*
(b) $\mathcal{S}(A^*) = D\mathcal{S}(A)$;

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(c) $\dim_K A^{**} = \dim_K A$;

(d) $(\varepsilon_i A \varepsilon_i)^* \simeq A^* \,/\, (A^* \varphi_{i-1} A^*)$. *(Note that the isomorphisms* $\big(A/(A\varepsilon_i A)\big)^* \simeq \varphi_{i-1} A^* \varphi_{i-1}$ *hold for any quasi-hereditary algebra* $(A, \mathbf{e})$*).*

PROOF. Propositions 4.1 and 4.3 imply that $(A^{*op}, \mathbf{f})$ satisfies condition (1) for solid algebras. Condition (2) for $(A^{*op}, \mathbf{f})$ will follow from Proposition 3.4. (iii), Proposition 4.1 and Lemma 3.2. Conditions (3) and (4) for $(A^{*op}, \mathbf{f})$ are the consequences of Proposition 4.1 and Lemma 3.2. Thus $(A^{*op}, \mathbf{f})$ is solid, proving part (a).

Part (b) is an immediate consequence of Proposition 4.3 and the general Theorem 2.10.

To prove (c), let us recall an earlier observation that the functor Ext^* establishes a bijection between the factors of the top filtrations of $U(i)$ and $V(i)$ and the top filtrations of $V^{*\circ}(i)$ and $U^{*\circ}(i)$; in this bijection simple modules correspond to projective modules, standard modules correspond to standard modules and projective modules to simple modules. Repeating the process, we get that A^{**} has the same type of filtrations for the modules $U^{**}(i)$ and $V^{**}(i)$ as A does. It is clear that $\dim_K S(i) = \dim_K S^{**}(i)$, hence it is enough to show that the indecomposable projective modules over A and A^{**} have the same composition factors. But this is easily proved first by induction for $\Delta(i)$ and $U(i)$ and then by downward induction for $P(i)$ and $V(i)$. Hence $\dim_K A = \dim_K A^{**}$.

Finally we prove (d). Evidently, the exact functor $\mathrm{Hom}_A(\varepsilon_i A, -)\colon \text{mod-}A \longrightarrow \text{mod-}\varepsilon_i A \varepsilon_i$ defines a homomorphism $\Phi\colon A^* \longrightarrow (\varepsilon_i A \varepsilon_i)^*$ whose kernel satisfies $\mathrm{Ker}\,\Phi \supseteq A^* \varphi_{i-1} A^*$. Actually, this is an epimorphism. To see this, let us observe first that by Proposition 4.3 the algebra A is lean, hence $\mathrm{Ext}^1_A\big(S(j), S(\ell)\big) = \mathrm{Ext}^1_{\varepsilon_i A \varepsilon_i}\big(S(j), S(\ell)\big)$ for any $j, \ell \geq i$. Next, Propositions 4.4 and 4.3 imply that $S(j) \in \mathcal{C}_{\varepsilon_i A \varepsilon_i}$ for any $j \geq i$, hence by Corollary 2.7 every element of $(\varepsilon_i A \varepsilon_i)^*$ can be represented by exact sequences with semisimple kernel-cokernel terms in the interior of the sequence. Since the short exact sequences with semisimple outer terms do appear in the image of Φ, we get that Φ is indeed an epimorphism.

Thus to prove (d), it is enough to show that the K-dimensions of $A^*/\big(A^* \varphi_{i-1} A^*\big)$ and $(\varepsilon_i A \varepsilon_i)^*$ are equal. Now the remark following Proposition 4.4 implies that these two algebras satisfy the same filtration conditions, hence an easy induction argument gives the equality of dimensions. ∎

COROLLARY 4.6. *If the algebra* $(A, \mathbf{e})$ *is shallow (left medial, right medial or replete) then* $(A^*, \mathbf{f})$ *is replete (right medial, left medial or shallow, respectively) on the dual species.*

PROOF. The statement follows from the previous theorem and the characterizations of shallow, left medial, right medial and replete algebras given in Section 3 of [ADL1]. ∎

5. **The case of monomial algebras.** In this section we shall be dealing with the case of monomial algebras. Thus we shall assume that $A = K\Gamma/I$, with $\Gamma = (\Gamma_0, \Gamma_1)$ a graph, where $\Gamma_0 = \{1, 2, \ldots, n\}$ is the set of vertices, also thought of as paths of length 0, and Γ_1 is the set of arrows; the corresponding idempotents of A will be denoted by $e_1, e_2, \ldots, e_n$. Denote by Γ_2 the set of minimal 0-paths, *i.e.*, paths belonging to I such that

they have no proper initial or terminal segments in I. The assumption that A is *monomial* means that $I = \langle \Gamma_2 \rangle$. We call a path *right-minimal* 0*-path* if it belongs to I and it has no initial segment in I.

Having defined $\Gamma_0, \Gamma_1, \Gamma_2$, we define the set Γ_k for $k = 3, 4, \ldots$ as follows:

$$\Gamma_k = \{p_1p_2 \ldots p_k \text{ path} \mid p_1 \in \Gamma_1, p_j \text{ is a path} \notin I \text{ for } 1 \leq j \leq k, \\ p_jp_{j+1} \text{ is a right-minimal 0-path for } 1 \leq j \leq k-1\}.$$

Note that in the definition of Γ_k the decomposition $p = p_1p_2 \ldots p_k$ is unique, and we will refer to it as the *canonical decomposition of* p. Finally take $\tilde{\Gamma} = \bigcup_{k=0}^{\infty} \Gamma_k$ and $\tilde{\Gamma}^+ = \bigcup_{k=1}^{\infty} \Gamma_k$.

Then (*cf.* [GZ]) the Ext-algebra A^* of A is isomorphic to the K-algebra whose multiplicative basis is $\tilde{\Gamma}$ and the multiplication is defined by:

$$p \cdot p' = \begin{cases} pp' & \text{if } pp' \in \tilde{\Gamma} \\ 0 & \text{otherwise.} \end{cases}$$

At this isomorphism a path in Γ_k from i to j corresponds to an extension in $\operatorname{Ext}_A^k\big(S(i), S(j)\big)$. In the sequel we shall identify these two algebras.

Let us start with a technical lemma about the multiplicative structure of A^*.

LEMMA 5.1.

(i) *If $p = p_1p_2 \ldots p_k$ is a canonical decomposition and $p_i \in \Gamma_1$ then $p_1 \ldots p_{i-1}$ and $p_i \ldots p_k$ both belong to $\tilde{\Gamma}$ with these canonical decompositions, and thus p is the product of these paths in A^*.*

(ii) *If $p_1p_2 \ldots p_{i-1}$ and $p_i \ldots p_k$ are canonical decompositions of two paths in $\tilde{\Gamma}$, and $p_{i-1}p_i$ is a 0-path then $p_1p_2 \ldots p_{i-1}p_i \ldots p_k \in \tilde{\Gamma}$.*

(iii) *If $p = p'\alpha\beta p'' \in \tilde{\Gamma}$ with $\alpha, \beta \in \Gamma_1$ and $\alpha\beta \in \Gamma_2$, then the paths $p'\alpha$ and $\beta p''$ both belong to $\tilde{\Gamma}$, thus $p = p'\alpha \cdot \beta p''$ in A^*.*

(iv) *Suppose that A is lean with respect to the given order and has Schurian standard modules. If $p = p'\alpha\beta p'' \in \tilde{\Gamma}$ with $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ and $j \leq i, k$, then $p'\alpha, \beta p'' \in \tilde{\Gamma}$; in particular $p \in A^* f_j A^*$.*

PROOF. (i) is obvious from the definition of $\tilde{\Gamma}$. In (ii) we only need to notice that $p_{i-1}p_i$ is right-minimal as a 0-path, since $p_{i-1} \notin I$ and p_i consists only of one arrow. In order to prove (iii) we observe that $\alpha\beta$ cannot be a part of any canonical component of p, thus β is an initial segment of a component. Then the right-minimality condition implies that β itself is a canonical component, so we can apply (i). Finally, we note that the conditions on A in (iv) imply that any 2-long path $\alpha\beta$ whose middle vertex is minimal belongs to Γ_2, thus we can apply (iii). ■

Let us first prove a general statement about the graph of the Ext-algebra of a monomial algebra A. The implication (3) $\Rightarrow$ (2) was also proved by Green and Zacharia (see [GZ]).

THEOREM 5.2. *Let $A \simeq K\Gamma / I$ be a monomial algebra. Then the following are equivalent:*

(1) *$S(i) \in \mathcal{C}_A$ for $1 \leq i \leq n$;*

(2) *A and A^{*op} have the same graph;*

(3) *A is quadratic;*

(4) $\mathrm{Ext}^2_A(\hat{S},\hat{S}) \subseteq \mathrm{rad}^2(A^*)$.

If $(A,\mathbf{e})$ *is, in addition, lean with Schurian standard modules, then conditions* (1)–(4) *are all equivalent to:*

(5) $\Delta(i) \in \mathcal{C}$, $\Delta^\circ(i) \in \mathcal{C}^\circ$ *for* $1 \leq i \leq n$.

PROOF. The equivalence of (1) and (2) was proved in Theorem 2.10 in a more general setting, for arbitrary algebras.

Next, the implication (2) $\Rightarrow$ (4) is clear, since the assumption on the graph of A^* means that the stronger condition of $\mathrm{Ext}^i_A(\hat{S},\hat{S}) \subseteq \mathrm{rad}^2 A^*$ for $i \geq 2$ is satisfied (*cf.* the remark following Proposition 1.1).

To prove that (4) $\Rightarrow$ (3) suppose that A is not quadratic, *i.e.*, Γ_2 contains a path p longer than 2. Recall that the K-linear span of Γ_2 is in bijective correspondence with $\mathrm{Ext}^2_A(\hat{S},\hat{S})$. Thus condition (4) would imply that p is the product of two elements of $\tilde{\Gamma}^+$. On the other hand it is clear that $p \notin \Gamma_1 \cdot \Gamma_1$. Moreover, by the minimality of the elements of Γ_2, p cannot contain a proper zero subpath. Hence $p \notin \mathrm{rad}^2 A^*$, a contradiction.

Finally, we get that (3) $\Rightarrow$ (2), since if Γ_2 consists of paths of length 2 only, then by Lemma 5.1. (i) Γ_i consists of paths of length i, and $\Gamma_i \subseteq \mathrm{rad}^i A^*$.

Assume now that the algebra A is in addition lean with Schurian standard modules. Let us show that (1) $\Rightarrow$ (5). It is easy to see that $U(i)$ is the K-linear span of those non-zero paths for which the first arrow is $i \xrightarrow{\alpha} j$ with $j < i$, while $V(i)$ is the span of paths with first arrow $i \xrightarrow{\beta} \ell$ and $i < \ell$. Thus $\mathrm{rad}\, P(i) = U(i) \oplus V(i)$. Now condition (1) implies that $\mathrm{rad}\, P(i) = \Omega_1\big(S(i)\big) \in \mathcal{C}_A$ and hence by Lemma 2.3 we get that $V(i) \in \mathcal{C}_A$. Since by assumption $V(i) \overset{t}{\subseteq} \mathrm{rad}\, P(i)$, we get that $\Delta(i) \in \mathcal{C}_A$. Using the equivalence of conditions (i) and (ii) of Theorem 2.10, we get that (1) $\Rightarrow$ (5).

To verify that (5) $\Rightarrow$ (1)–(4), we will show that if A is not quadratic, then $\Delta(i) \notin \mathcal{C}_A$ or $\Delta^\circ(i) \notin \mathcal{C}^\circ_A$ for some $1 \leq i \leq n$. If A is not quadratic, then there is a path $p \in \Gamma_2$ of length greater than 2. Let $p = i_0 \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} i_k$. Now, it cannot happen that $i_0 > i_1$ and $i_k > i_{k-1}$ since otherwise the minimal vertex i_j would appear in the interior of the path p, hence by Lemma 5.1. (iv) we would get that p is not a minimal 0-path. Thus, we may assume by the left-right symmetry of condition (5) that $i_0 < i_1$. But then we get that $\Delta(i_0) \notin \mathcal{C}^{(2)}_A$, since the path $i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} i_k$ is non-zero. This completes the proof. ∎

Let us now turn to the question of quasi-heredity. For the idempotents in Γ_0 as elements of A^* we shall also use the notation $f_1, f_2, \ldots, f_n$ to conform with the general notation in this paper. Recall that $\mathbf{f} = (f_n, f_{n-1}, \ldots, f_1)$ and $\varphi_i = f_i + f_{i-1} + \cdots + f_1$, with $\varphi_0 = 0$.

The main result of this section is a necessary and sufficient condition for the quasi-heredity of A^*. In order to handle the concept of quasi-heredity efficiently in this setting, we need the following technical lemma for algebras with a multiplicative basis.

LEMMA 5.3. *Suppose the algebra* $(A, \mathbf{e})$ *has a multiplicative basis* $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ *such that* $\mathcal{B}_1 = \{e_1, e_2, \ldots, e_n\}$ *is a complete set of primitive orthogonal idempotents and* $\mathcal{B}_2$ *is a basis of* $\operatorname{rad} A$. *Then* A *is quasi-hereditary with respect to the given order* $\mathbf{e}$ *if and only if for any* $b, b', c, c' \in \mathcal{B}$,

(i) $be_ic = b'e_ic'$ *implies that either* $be_ic \in A\varepsilon_{i+1}A$ *or* $be_i = b'e_i$ *and* $e_ic = e_ic'$;

(ii) $be_i, e_ic \notin A\varepsilon_{i+1}A$ *implies that* $be_ic \notin A\varepsilon_{i+1}A$.

PROOF. Let us notice first that for any $b \in \mathcal{B}$ there is precisely one pair of indices (i,j) such that $e_ibe_j \neq 0$ and for this pair $b = e_ibe_j$.

Since the image of $\mathcal{B}(e_1 + \cdots + e_{n-1})$ gives a multiplicative basis for A/Ae_nA, it is enough to prove that the conditions (i) and (ii) formulated for $i = n$ hold if and only if Ae_nA is a heredity ideal, *i.e.*, Ae_nA is a direct sum of copies of $\Delta(n) = P(n)$ with $\Delta(n)$ Schurian.

Assume first that Ae_nA is a heredity ideal. Then we have $Ae_nA = \sum_{b\in\mathcal{B}} be_nA$ where the non-zero summands have simple top isomorphic to $S(n)$. On the other hand the images of these summands are independent in $Ae_nA/\operatorname{rad}(Ae_nA)$. Namely, suppose that $0 \neq ce_n \in \sum_{b\in\mathcal{B}\setminus\{c\}} be_nx_b + Ae_n \operatorname{rad} A$ for some $c \in \mathcal{B}$ and $x_b \in A$. Then $ce_n \in \sum_{b\in\mathcal{B}\setminus\{c\}} be_nx_be_n + Ae_n \operatorname{rad} Ae_n = \sum_{b\in\mathcal{B}\setminus\{c\}} \lambda_b e_n + Ae_n \operatorname{rad} Ae_n$ for some $\lambda_b \in K$. But the Schurian property implies that $e_n \operatorname{rad} Ae_n = 0$, hence $c = ce_n = \sum_{b\in\mathcal{B}\setminus\{c\}} \lambda'_b b$, a contradiction. Thus the number of non-zero summands in $\sum_{b\in\mathcal{B}} be_nA$ is equal to the number of direct summands of top Ae_nA, hence to the number of (indecomposable projective) direct summands of Ae_nA. Hence a dimension argument shows that $\sum_{b\in\mathcal{B}} be_nA$ is a direct sum, with non-zero components isomorphic to e_nA. So the different summands have disjoint bases, implying that $be_nc \neq b'e_nc'$ whenever $b \neq b'$ and $be_nc, b'e_nc' \neq 0$; this gives half of condition (i). On the other hand, the natural homomorphism $e_nA \longrightarrow be_nA$ (for $be_n \neq 0$) is injective, hence it maps different basis elements to different basis elements. This implies the other half of condition (i) and also condition (ii).

Now assume that conditions (i) and (ii) are satisfied for $(A, \mathbf{e})$ with $i = n$. We want to show that Ae_nA is a heredity ideal. First, $\Delta(n)$ is Schurian, *i.e.*, $e_n \operatorname{rad} Ae_n = 0$. Otherwise, there would be an element $b \in \mathcal{B}_2$ such that $e_nbe_n \neq 0$, so $e_n \cdot e_n \cdot be_n = e_nb \cdot e_n \cdot e_n = e_nbe_n \neq 0$, contradicting condition (i), since $e_n \neq e_nb$. Next we show that Ae_nA is a direct sum of copies of $P(n)$. As above, we can write $Ae_nA = \sum_{b\in\mathcal{B}} be_nA$, where the non-zero summands are independent by (i). Furthermore, each summand is either 0 or isomorphic to e_nA by conditions (i) and (ii). ∎

THEOREM 5.4. *Let* $A = K\Gamma/I$ *be a monomial algebra with* $\operatorname{gl.dim} A < \infty$. *Then* $(A^*, \mathbf{f})$ *is quasi-hereditary if and only if* $(A, \mathbf{e})$ *is lean with Schurian standard modules.*

PROOF. Assume first that $(A, \mathbf{e})$ is lean with Schurian standard modules. According to Lemma 5.3, all we have to prove is that for every i and for every $p, p', q, q' \in \tilde{\Gamma}$:

(i) $p \cdot f_i \cdot q = p' \cdot f_i \cdot q' \notin A^*\varphi_{i-1}A^*$ implies that $p = p'$ and $q = q'$;

(ii) $p \cdot f_i, f_i \cdot q \notin A^*\varphi_{i-1}A^*$ implies that $p \cdot f_i \cdot q \notin A^*\varphi_{i-1}A^*$.

To prove (i), assume $s = p \cdot f_i \cdot q = p' \cdot f_i \cdot q' \notin A^*\varphi_{i-1}A^*$. Then by Lemma 5.1. (iv) the vertex i is minimal in the path s. If $p \neq p'$, say, p is a proper subpath of p' then

Lemma 5.1. (iv) implies that $s = p \cdot f_i \cdot x \cdot f_i \cdot q'$ with some $x \in \tilde{\Gamma}^+$. Now it follows from the leanness and the Schurian property, together with Lemma 5.1. (ii) that $x, x \cdot x, x \cdot x \cdot x, \ldots$ all belong to $\tilde{\Gamma}$, contradicting the condition that $\operatorname{gl.dim} A < \infty$.

In proving condition (ii) we use the same argument as above, showing first that i is a minimal vertex in both $p \cdot f_i$ and $f_i \cdot q$ and then again Lemma 5.1. (ii) implies (together with leanness and the Schurian property) that $p \cdot f_i \cdot q$ is a non-zero product in A^* and the path contains no vertices smaller than i. Hence $p \cdot f_i \cdot q \notin A^* \varphi_{i-1} A^*$.

Assume now that $(A^*, \mathbf{f})$ is quasi-hereditary. To prove that the monomial algebra $(A, \mathbf{e})$ is lean with Schurian standard modules, it is enough to show that Γ has no loops and for any arrows $i \xrightarrow{\alpha} j$ and $j \xrightarrow{\beta} k$ with $j \leq i, k$, the path $\alpha\beta$ is a 0-path in A, *i.e.*, $\alpha\beta \in \Gamma_2$. The former statement follows immediately from the Schurian property of A^*, while the latter follows from condition (ii) of the quasi-heredity of $(A^*, \mathbf{f})$, as described above. ∎

Another type of relationship between leanness and quasi-heredity is given in the following theorem.

THEOREM 5.5. *Let $A = K\Gamma/I$ be a monomial algebra. If $(A, \mathbf{e})$ is quasi-hereditary then either $(A^*, \mathbf{f})$ is lean with Schurian standard modules or the graph of A^* has loops.*

PROOF. Suppose $(A, \mathbf{e})$ is quasi-hereditary and the graph of A^* has no loops. Let $p \in \Gamma_k$, $k \geq 2$ be a path going through the vertices $v_0, v_1, \ldots, v_\ell$. By induction on k we prove that $v_1, \ldots, v_{\ell-1} < \max\{v_0, v_\ell\}$.

Let us first take the case $k = 2$. Suppose there is $1 \leq i \leq \ell - 1$ such that $v_i \geq v_j$ for $0 \leq j \leq \ell$ and $p = p'p''$ where the endpoint of p' is v_i. By the minimality of the 0-path p the subpaths p' and p'' are not 0 in A. Since A is monomial, the non-zero paths of Γ form a multiplicative basis of A, satisfying the requirements of Lemma 5.3. The maximality of v_i implies that $p', p'' \notin A\varepsilon_{i+1}A$, hence we may apply condition (ii) of Lemma 5.3 to get that $p = p'p''$ is a non-zero path. This contradicts to the assumption that $p \in \Gamma_2$.

Suppose the statement has been proved for indices up to k, $k \geq 2$; now we shall prove it for $k + 1$. Let $p = p_1p_2 \ldots p_kp_{k+1}$ be the canonical decomposition of p. Let $p' = p_1p_2 \ldots p_k \in \Gamma_k$, $p_k = rs$ such that $p'' = sp_{k+1} \in \Gamma_2$. Thus by the induction hypothesis no internal vertex of p can be maximal as it would be an internal vertex of p' or p''.

Now, to prove that A^* is lean with Schurian standard modules, by Theorem 2.1 in [ADL1] it is enough to show that $f_i \operatorname{rad}^2 A^* f_j \subseteq f_i \operatorname{rad} A^* \varphi_M \operatorname{rad} A^* f_j$ for $1 \leq i,j \leq n$ and $M = \max\{i,j\}$. So let $p = p' \cdot p'' \in \tilde{\Gamma}$ be a path from i to j with $p', p'' \in \tilde{\Gamma}^+$. Then, as we proved above, $M = \max\{i,j\}$ is bigger than any internal vertex of p, in particular than the endpoint of p'. So $p \in f_i \operatorname{rad} A^* \varphi_{M-1} \operatorname{rad} A^* f_j$, proving that $(A^*, \mathbf{f})$ is lean. Notice that this also proves the Schurian property (*i.e.*, that $f_i \operatorname{rad} A^* f_i \subseteq f_i A^* \varphi_{i-1} A^* f_i$) if we take into account the assumption that the graph of A^* has no loops. ∎

The previous two theorems yield the following corollary.

COROLLARY 5.6. *Let $A = K\Gamma/I$ be a monomial algebra. If $(A, \mathbf{e})$ is lean and quasi-hereditary, then so is $(A^*, \mathbf{f})$.*

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PROOF. Since A is quasi-hereditary, gl. dim $A < \infty$ and hence by Theorem 5.4 $(A^*, \mathbf{f})$ is quasi-hereditary. This, on the other hand, implies that the graph of A^* cannot have loops, so by Theorem 5.5 $(A^*, \mathbf{f})$ is also lean. ∎

6. **Examples.**

EXAMPLE 6.1. Let $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$, and consider

$$0 \longrightarrow \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \xrightarrow{\alpha} \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \longrightarrow 0.$$

Here α is a top embedding and the first two terms of the sequence are in $\mathcal{C}_A$ but the last one is not in $\mathcal{C}_A^{(1)}$ (*cf.* Lemma 2.4).

EXAMPLE 6.2. Let $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$, and consider

$$0 \longrightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \xrightarrow{\alpha} \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} \longrightarrow 3 \longrightarrow 0.$$

Here α is a top embedding and the last two terms of the sequence are in $\mathcal{C}_A$, but the first is not in $\mathcal{C}_A^{(1)}$ (see again Lemma 2.4).

EXAMPLE 6.3. Let $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 1 & & 3 \\ 2 & & \\ & 1 & \end{smallmatrix}$, and consider

$$0 \longrightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \xrightarrow{\beta} \begin{smallmatrix} 4 \\ 1 \\ 2 \end{smallmatrix} \longrightarrow 4 \longrightarrow 0.$$

Here β is a top embedding into the radical of the second term, the last two terms of the sequence are in $\mathcal{C}_A$, but the first one is not in $\mathcal{C}_A^{(1)}$ (see Lemma 2.5).

EXAMPLE 6.4. Let $A_A = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus 3$, and consider

$$0 \longrightarrow 2 \xrightarrow{\beta} \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \longrightarrow 0.$$

Here β is again an embedding into the top of the radical of the second term, the first and the last term of the sequence are in $\mathcal{C}_A$ but the middle term is not in $\mathcal{C}_A^{(1)}$ (*cf.* Lemma 2.5).

EXAMPLE 6.5. Let $A_A = \begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$. Here $\Delta(1) \in \mathcal{C}, \Delta(1)^{\circ} \in \mathcal{C}^{\circ}$ but $S(1) \notin \mathcal{C}$. Thus the assumption in Theorem 5.2 on the Schurian property of $\Delta(i)$ was really needed to prove the implication (5) $\Rightarrow$ (1). (Let us mention here that for this example only the projective modules are in $\mathcal{C}_A$.)

EXAMPLE 6.6. Let us consider the following algebra:

$$A_A = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} & 2k & \\ 2k{+}1 & & 2k{+}2 \\ & 2k{+}3 & \end{smallmatrix} \oplus \begin{smallmatrix} 2k{+}1 \\ 2k{+}3 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2\ell{+}2 \\ 2\ell{+}3 \\ 2\ell{+}4 \end{smallmatrix} \oplus \begin{smallmatrix} 2\ell{+}3 \\ 2\ell{+}4 \end{smallmatrix} \oplus 2\ell{+}4 .$$

Then $S(1) \in \mathcal{C}^{(\ell+1)} \setminus \mathcal{C}^{(\ell+2)}$ and all the other simple modules are in $\mathcal{C}_A$. Hence observe that $\mathrm{Ext}^k(\hat{S}, \hat{S}) \subseteq \mathrm{rad}^2 A^*$ for all $2 \leq k \leq \ell + 1$, thus the graph of A^* is increased only by an element from $\mathrm{Ext}^{\ell+2}(\hat{S}, \hat{S})$. This cannot happen for monomial algebras by Theorem 5.2 and Theorem 2.10.

EXAMPLE 6.7. Take the following algebra:

$$A_A = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 3 & & \\ 2 & & 5 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 1 \end{smallmatrix}.$$

Then $\Delta(i) \in \mathcal{C}_A, \Delta(i)^\circ \in \mathcal{C}_A^\circ$ for $1 \leq i \leq n$ but $U(4) \notin \mathcal{C}_A$ (*cf.* Proposition 4.1). Let us also observe that

$${}_{A^*}A^* = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 3 & & 5 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 1 \end{smallmatrix}$$

is not lean with respect to the opposite order (*cf.* also Proposition 3.5) although A is lean quasi-hereditary; for monomial algebras this cannot happen by Corollary 5.6.

EXAMPLE 6.8. Let A be given by

$$A_A = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5.$$

Then we have:

$${}_{A^*}A^* = \begin{smallmatrix} 1 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5.$$

Thus A is lean quasi-hereditary but A^* is not quasi-hereditary (with respect to the opposite order), something that cannot happen for monomial algebras by Corollary 5.6. — Notice also that here for $C = \varepsilon_2 A \varepsilon_2$ we have

$$C_C = \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5 \quad \text{and}$$

$${}_{C^*}C^* = \begin{smallmatrix} & 2 & \\ 3 & & 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5.$$

On the other hand for $B^* = A^*/A^*\varphi_1 A^*$ we have

$${}_{B^*}B^* = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5.$$

Hence $(\varepsilon_2 A \varepsilon_2)^* \not\simeq A^*/A^*\varphi_1 A^*$, something that cannot happen for solid algebras (see Theorem 4.5).

EXAMPLE 6.9. Let $A_A = \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$. Then ${}_{A^*}A^* = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 3 \end{smallmatrix}$. Hence A is quasi-hereditary, monomial, and the graph of A^* contains a loop, hence the second possibility in the implication of Theorem 5.5 actually may occur.

EXAMPLE 6.10. Let $A_A = \begin{smallmatrix} 1 \\ 3 \\ 4 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}$. Then ${}_{A^*}A^* = \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}$. Here A is monomial, lean and not quasi-hereditary, but A^* is lean and quasi-hereditary. Hence the converse of Theorem 5.5 and Corollary 5.6 is not true.

EXAMPLE 6.11. Let us consider the following algebra:

$$A_A = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \\ 6 \\ 7 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2k \\ 2k+1 \\ 2k+2 \\ 2k+3 \end{smallmatrix} \oplus \begin{smallmatrix} 2k+1 \\ 2k+2 \\ 2k+3 \end{smallmatrix} \oplus \begin{smallmatrix} 2k+2 \\ 2k+3 \\ 2k+4 \end{smallmatrix} \oplus \begin{smallmatrix} 2k+3 \\ 2k+4 \end{smallmatrix} \oplus 2k+4 \ .$$

Then the Ext-algebra is given by:

$$_{A^*}A^* = \begin{smallmatrix} & & 1 & & \\ 2 & 4 & 6 & \cdots & 2k+4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & & 3 & \\ 4 & 6 & \cdots & 2k+4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2k+2 \\ 2k+3 \end{smallmatrix} \oplus \begin{smallmatrix} 2k+3 \\ 2k+4 \end{smallmatrix} \oplus 2k+4 \ .$$

Thus A is a monomial algebra, and $\mathrm{Ext}_A^{k+2}(\hat{S}, \hat{S}) \not\subseteq \mathrm{rad}^2 A^*$; hence some new elements of the species of A^* come from high Ext's. Of course, by Theorem 5.2 there are elements of $\mathrm{Ext}_A^2(\hat{S}, \hat{S})$ which get into the species, too.

EXAMPLE 6.12. Let A be given by

$$A_A = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 \\ 1 & 5 \\ & 4 \\ & 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} .$$

Then we have:

$$_{A^*}A^* = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & & 2 & \\ & 1 & & 5 \\ 3 & & & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \quad \text{and}$$

$$A^{**}_{A^{**}} = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & & \\ 1 & & 5 & \\ & & & 4 \\ & & & 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} .$$

Here A is replete but $A \not\simeq A^{**}$. Thus in Theorem 4.5 although we have equality for the dimensions of A and A^{**}, in general we cannot state that these two algebras would be isomorphic.

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