SEPARABILITY OF THE L^1-SPACE OF A VECTOR MEASURE

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Let Σ be a σ-algebra of subsets of some set Ω and let $\mu:\Sigma\to[0,\infty]$ be a σ-additive measure. If $\Sigma(\mu)$ denotes the set of all elements of Σ with finite μ-measure (where sets equal μ-a.e. are identified in the usual way), then a metric d can be defined in $\Sigma(\mu)$ by the formula

$$d(E,F)=\mu(E\,\Delta F)=\int_\Omega|\chi_E-\chi_F|\,d\mu\quad(E,F\in\Sigma);\tag{1}$$

here $E\,\Delta F=(E\backslash F)\cup(F\backslash E)$ denotes the symmetric difference of E and F. The measure μ is called *separable* whenever the metric space $(\Sigma(\mu),d)$ is separable. It is a classical result that μ is separable if and only if the Banach space $L^1(\mu)$ is separable [**8**, p. 137]. To exhibit non-separable measures is not a problem; see [**8**, p. 70], for example. If Σ happens to be the σ-algebra of μ-measurable sets constructed (via outer-measure μ^*) by extending μ, defined originally on merely a semi-ring of sets $\Gamma\subseteq\Sigma$, then it is also classical that the countability of Γ guarantees the separability of μ and hence, also of $L^1(\mu)$, [**8**, p. 69].

There arises the natural question of what form such classical results on separability of L^1-spaces should take for vector-valued measures. We aim to formulate such results in this note.

So, suppose that X is a locally convex space (briefly, lcs), always assumed to be Hausdorff and sequentially complete. A σ-additive map $m:\Sigma\to X$, where Σ is a σ-algebra of subsets of some set Ω, is called a (X-valued) *vector measure.* A Σ-measurable function $f:\Omega\to\mathbb{C}$ is called *m-integrable* if it is integrable with respect to the complex measure $\langle m,x'\rangle:E\mapsto\langle m(E),x'\rangle$, for $E\in\Sigma$, for every $x'\in X'$ (the continuous dual space of X), and if, for every $E\in\Sigma$, there exists an element of X, denoted by $\int_E f\,dm$, which satisfies $\langle\int_E f\,dm,x'\rangle=\int_E f\,d\langle m,x'\rangle$, for every $x'\in X'$. The linear space of all m-integrable functions is denoted by $L(m)$. Let $\mathcal{Q}_X$ denote the family of all continuous seminorms in X or, at least enough seminorms to determine the topology of X. Each $q\in\mathcal{Q}_X$ induces a seminorm $q(m)$ in $L(m)$ via the formula

$$q(m):f\mapsto\sup\left\{\int_\Omega|f|\,d\,|\langle m,x'\rangle|;x'\in U_q^0\right\}\quad(f\in L(m)),\tag{2}$$

where $U_q^0\subseteq X'$ denotes the polar of the unit ball $U_q=q^{-1}([0,1])$. The seminorms (2), as q varies through $\mathcal{Q}_X$, define a lc topology $\tau(m)$ in $L(m)$. Since $\tau(m)$ may not be Hausdorff we form the usual quotient space of $L(m)$ with respect to the closed subspace $\bigcap_{q\in\mathcal{Q}_X}q^{-1}(\{0\})$. The resulting Hausdorff space (with topology again denoted by $\tau(m)$) is denoted by $L^1(m)$; it can be identified with equivalence classes of functions from $L(m)$ modulo m-null functions, where a function $f\in L(m)$ is m-null whenever $\int_E f\,dm=0$, for every $E\in\Sigma$. All of the above definitions and further properties of $L^1(m)$ can be found in [**6**].

Let $\Sigma(m)$ denote the subset of $L^1(m)$ corresponding to $\{\chi_E;E\in\Sigma\}\subseteq L(m)$. Of course, elements of $\Sigma(m)$ can also (and will) be identified with equivalence classes of elements from Σ. The formula (1) suggests how to topologize $\Sigma(m)$. Namely, we restrict the $L^1(m)$-topology $\tau(m)$ to $\Sigma(m)$. That is, each seminorm $q(m)$, where $q \in \mathcal{Q}_X$, induces a semi-metric d_q on $\Sigma(m)$ by the formula

$$d_q(\chi_E, \chi_F) = q(m)(\chi_E - \chi_F) \quad (E, F \in \Sigma). \tag{3}$$

Again $\tau(m)$ will denote the uniform structure and topology in $\Sigma(m)$ so defined by the semi-metrics (3) as q varies through $\mathcal{Q}_X$.

1. Main results. Throughout this section X is a Hausdorff, sequentially complete lcs. A vector measure $m : \Sigma \to X$ is called *separable* whenever the topological space $(\Sigma(m), \tau(m))$ is separable. For $X = \mathbb{C}$ (or $\mathbb{R}$) this coincides with the classical definition.

PROPOSITION 1. *Let $m : \Sigma \to X$ be a vector measure.*

(i) *If the measure m is separable, then the lcs $L^1(m)$ is separable.*

(ii) *Let the lcs X be metrizable. Then m is separable if and only if $L^1(m)$ is separable.*

Proof. (i) Let $B \subseteq \Sigma(m)$ be a countable $\tau(m)$-dense set in $\Sigma(m)$. Then the collection $\mathcal{S}(B)$ of all simple functions of the form $\sum_{j=1}^{k} \alpha_j \chi_{F(j)}$ for k a positive integer, α_j a "rational complex number" and $F(j) \in B$, $1 \le j \le k$, is also countable. By the $\tau(m)$-density of the Σ-simple functions in $L^1(m)$, [**6**, Ch. 2], it suffices to show that if $f = \sum_{j=1}^{n} \beta_j \chi_{E(j)}$ is a Σ-simple function and positive numbers ϵ_r are given together with seminorms $q_r \in \mathcal{Q}_X$, $1 \le r \le k$, then there exists an element $h \in \mathcal{S}(B)$ satisfying

$$q_r(m)(f - h) < \epsilon_r \quad (1 \le r \le k). \tag{4}$$

Let $\epsilon = \min\{\epsilon_r; 1 \le r \le k\}$ and $K = \max\{q_r(m)(\chi_\Omega); 1 \le r \le k\}$. Choose "rational complex numbers" $\alpha_j, 1 \le j \le n$, satisfying $|\alpha_j - \beta_j| < \epsilon/(2nK)$ for $1 \le j \le n$. By $\tau(m)$-density of B in $\Sigma(m)$ there exist sets $F(j) \in B$ such that, for every $j \in \{1, 2, \ldots, n\}$ we have

$$d_{q_r}(E(j), F(j)) = q_r(m)(\chi_{E(j)} - \chi_{F(j)}) < \epsilon_j/(2n\beta), \tag{5}$$

for every $1 \le r \le k$, where $\beta = \max\{|\beta_j|; 1 \le j \le n\}$. Let h be the element $\sum_{j=1}^{n} \alpha_j \chi_{F(j)}$ of $\mathcal{S}(B)$. Since

$$|f - h| \le \sum_{j=1}^{n} |\alpha_j - \beta_j|\, \chi_{F(j)} + \sum_{j=1}^{n} |\beta_j| \,.\, |\chi_{F(j)} - \chi_{E(j)}|$$

it follows that

$$|f - h| \le \epsilon(2nK)^{-1} \sum_{j=1}^{n} \chi_{F(j)} + \beta \sum_{j=1}^{n} |\chi_{F(j)} - \chi_{E(j)}|. \tag{6}$$

Since $q_r(m)(\chi_{F(j)}) \le q_r(m)(\chi_\Omega)$, for every $1 \le j \le n$ and $1 \le r \le k$, and $q(m)(g) = q(m)(|g|)$, for every $q \in \mathcal{Q}_X$ and $g \in L^1(m)$—see (2)—it follows from (5), (6) and the definitions of K and ϵ that (4) is satisfied.

(ii) If X is metrizable, then $\mathcal{Q}_X$ can be chosen to be a countable set. It is then clear from the definition of $\tau(m)$ that $L^1(m)$ is also a metrizable lcs. Since $(\Sigma(m), \tau(m))$ is a subset of $L^1(m)$ with the relative topology it follows that $(\Sigma(m), \tau(m))$ is separable whenever $L^1(m)$ is separable [**8**, p. 20]. □

It would seem useful to have available a criterion for determining separability. Given a measure $m:\Sigma\to X$ we recall that Σ is called *m-essentially countably generated* [**6**, p. 32] if there exists a countably generated σ-algebra $\Sigma_0\subseteq\Sigma$ such that $\Sigma(m)=\Sigma_0(m)$.

Proposition 2. *Let $m:\Sigma\to X$ be a vector measure. If Σ is m-essentially countably generated, then m is a separable measure. In particular, $L^1(m)$ is separable.*

The proof of this result relies on the following two facts: the first is straightforward and the second follows from the first and [**3**, III Lemma 8.4].

Lemma 1. (i) *Let Λ be a family of subsets of a set Ω. Then the σ-algebras of subsets of Ω generated by $\Lambda\cup\{\Omega\}$ and by Λ coincide.*

(ii) *Let Σ be a countably generated σ-algebra of subsets of a set Ω. Then there exists a countable algebra of sets $\Sigma_0\subseteq\Sigma$ such that the σ-algebra generated by Σ_0 is precisely Σ.*

Proof of Proposition 2. Let Σ_0 be a countable algebra of subsets of Ω which m-essentially generates Σ. Let m_0 denote the restriction of m to Σ_0. Then m_0 is σ-additive on Σ_0 and has an extension to a σ-additive measure on Σ, namely m. It follows from the equivalence of (i) and (xi) in the Theorem of Extension in [**5**] (the topology $\tau^*(m)$ stated there in (xi) coincides with our $\tau(m)$; see p. 178 of [**5**]) that Σ_0 is $\tau(m)$-dense in $\Sigma\simeq\Sigma(m)$. Accordingly, m is separable. □

Corollary 1. *Let $m:\Sigma\to X$ be a vector measure. If Σ is m-essentially countably generated, then the closed subspace of X generated by the range of m is separable for the relative topology induced by X.*

Proof. The integration map Φ given by $\Phi:f\mapsto\int_\Omega f\,dm$, for $f\in L^1(m)$, is continuous from $(L^1(m),\tau(m))$ into X. Let Y denote the closed subspace of X generated by the range, $m(\Sigma)=\{m(E);E\in\Sigma\}$, of m. By approximating elements of $L^1(m)$ by Σ-simple functions it is clear that $\Phi(L^1(m))\subseteq Y$ and hence, the closure $\overline{\Phi(L^1(m))}\subseteq Y$. But, the formula $m(E)=\Phi(\chi_E)$, for $E\in\Sigma$, shows that actually $\overline{\Phi(L^1(m))}=Y$.

The proof of Proposition 2 showed that there exists a countable algebra of sets B which m-essentially generates Σ and such that B is $\tau(m)$-dense in $\Sigma(m)$. Then the collection $\mathcal{S}(B)$ of "rational" B-simple functions as defined in the proof of Proposition 1(i) is countable and dense in $L^1(m)$. Clearly the Φ-image of the set $\mathcal{S}(B)$ is countable and contained in Y. So, it suffices to show that elements of $\Phi(L^1(m))$ can be approximated (in X) by elements of $\Phi(\mathcal{S}(B))$. That this is the case follows from the density of $\mathcal{S}(B)$ in $L^1(m)$ and the continuity of Φ. □

In many situations, the converse of Proposition 2 is also valid. In order to formulate it we recall some notions from topology. Let Λ be a topological Hausdorff space and $Y\subseteq\Lambda$. Then $[Y]$ denotes the set of all elements in Λ which are the limit of some sequence of points from Y. A set $Y\subseteq\Lambda$ is called *sequentially closed* if $Y=[Y]$. The sequential closure $\bar{Y}_s$, of a set $Y\subseteq\Lambda$, is the smallest sequentially closed subset of Λ which contains Y. Alternatively, let $Y_0=Y$. Let Ω_1 be the smallest uncountable ordinal. Suppose that $0<\alpha<\Omega_1$ and that Y_β has been defined for all ordinals β satisfying $0\le\beta<\alpha$. Define $Y_\alpha=\left[\bigcup_{0\le\beta<\alpha}Y_\beta\right]$. Then $\bar{Y}_s=\bigcup_{0\le\alpha<\Omega_1}Y_\alpha$.

Let $\mathcal{A}$ be a family of subsets of a non-empty set Ω. Then the σ-algebra of subsets generated by $\mathcal{A}$ is denoted by $\mathcal{A}_\sigma$. The cardinality of a set B is denoted by $\#(B)$.

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LEMMA 2. (i) *Let $\mathscr{A}$ be an infinite family of subsets of a non-empty set Ω. Then the algebra of sets generated by $\mathscr{A}$ has cardinality $\#(\mathscr{A})$. Moreover, $\#(\mathscr{A}_\sigma) \leq \#(\mathscr{A})^{\aleph_0}$.*

(ii) *If Y is a subset of a topological space Λ, then $\#(\bar{Y}_s) \leq \#(Y)^{\aleph_0}$.*

(iii) *Let $m:\Sigma \to X$ be a vector measure and $\mathscr{A} \subseteq \Sigma$ be an algebra of sets. Then $\chi(\mathscr{A}_\sigma) = \{\chi_E; E \in \mathscr{A}_\sigma\}$ is contained in the sequential closure of $\chi(\mathscr{A})$ in the topological space $\Sigma(m)$.*

Proof. (i) This can be found on pp. 133–134 of [**4**].

(ii) follows from the transfinite inductive definition of $\bar{Y}_s$ and a modification of the proof of Theorem 10.23 in [**4**].

(iii) Recall that $\mathscr{A}_\sigma$ can be constructed as follows (see [**5**, p. 180], for example): beginning with $\mathscr{A}$, let $\mathscr{A}_i$ be the system of all sets expressible as the union of increasing sequences of elements from $\mathscr{A}$; then construct the system $\mathscr{A}_{id}$ of intersections of decreasing sequences in $\mathscr{A}_i$, then construct $\mathscr{A}_{idi}$, and so on by transfinite induction all the way to Ω_1. At each stage of this procedure the monotone convergence theorem for m, [**6**, Ch. II, §4], guarantees that the next family of sets belongs to $\overline{\chi(\mathscr{A})}_s$. □

We can now formulate a partial converse to Proposition 2 which is applicable to a large class of vector measures; see Remark 1 below.

PROPOSITION 3. *Let $m:\Sigma \to X$ be a vector measure such that its range $m(\Sigma)$ is metrizable for the relative topology from X.*

(i) *Let $\mathscr{A} \subseteq \Sigma$ be an algebra of sets. Then $\mathscr{A}_\sigma = \bar{\mathscr{A}}_s$, meaning that $\chi(\mathscr{A}_\sigma)$ and $\overline{\chi(\mathscr{A})}_s$ coincide as subsets of $\Sigma(m)$. In particular, $\bar{\mathscr{A}}_s$ is a σ-algebra of sets.*

(ii) *The measure m is separable if and only if Σ is m-essentially countably generated.*

Proof. (i) The inclusion $\mathscr{A}_\sigma \subseteq \bar{\mathscr{A}}_s$ follows from Lemma 2(iii). To establish the reverse inclusion it suffices to show that $\mathscr{A}_\sigma$ is sequentially closed. So, let $E(n)$, $n = 1, 2, \ldots$, be elements of $\mathscr{A}_\sigma$ such that $\chi_{E(n)} \to f$ in $L^1(m)$. Since $\Sigma(m)$ is $\tau(m)$-complete by [**7**, Proposition 1], it follows that $f = \chi_E$ for some $E \in \Sigma$. An examination of the proof of Proposition 1 in [**7**] shows that there exists a sequence of continuous seminorms $\{q_k\}_{k=1}^\infty$ in X such that corresponding semi-metrics $\{d_{q_k}\}_{k=1}^\infty$ given by (3) induce the metrizable topology on $m(\Sigma)$. Arguing as in the proof of [**6**, II Section 1, Corollary 2] it follows that there exists a finite positive measure λ on Σ, with the same null sets as m, satisfying $\lambda(F) \to 0$ whenever $q_k(m)(F) \to 0$ for each $k = 1, 2, \ldots$. Accordingly, $\lambda(E(n)\,\Delta E) \to 0$ as $n \to \infty$. Then there is a subsequence $\{E(n_r)\}_{r=1}^\infty$ of $\{E(n)\}_{n=1}^\infty$ such that $\chi_{E(n_r)} \to \chi_E$, λ-a.e. and hence m-a.e. It follows that $E \in \mathscr{A}_\sigma$.

(ii) One direction is clear from Proposition 2. Conversely, suppose that m is separable. Then $\Sigma(m)$ has a countable dense set. By Lemma 2(i) the algebra of sets that it generates, say $\mathscr{A}$, is also countable (and still dense). By part (i), $\mathscr{A}_\sigma = \bar{\mathscr{A}}_s$. Since $\Sigma(m)$ is metrizable the sequential closure of $\mathscr{A}$ coincides with its $\tau(m)$-closure. Accordingly, $\Sigma(m) = \mathscr{A}_\sigma$ and so $\Sigma(m)$ is m-essentially countably generated. □

REMARK 1. Many lc spaces X, themselves not necessarily metrizable, have the property that their bounded sets are metrizable; see [**7**], for example. In such spaces, every vector measure $m:\Sigma \to X$ necessarily has metrizable range and hence is separable if and only if Σ is m-essentially countably generated.

REMARK 2. An essential ingredient in the proof of Proposition 3(i) was the existence of a finite, non-negative measure λ with the same null sets as m and having the property that $\chi_{E(n)} \to \chi_E$ in $\Sigma(m)$ implies $\lambda(E(n) \Delta E) \to 0$. There are other instances when such a measure λ exists *without* the range $m(\Sigma)$ being metrizable. For example, let X be a Banach space and $L_s(X)$ be the space of all continuous linear operators of X into itself, equipped with the strong operator topology. Let $P: \Sigma \to L_s(X)$ be a spectral measure, that is, a σ-additive measure satisfying $P(\Omega) = I$ (the identity operator on X) and $P(W \cap F) = P(E)P(F)$, for every $E, F \in \Sigma$. If X is non-separable then, except for trivial cases, $P(\Sigma)$ is not metrizable for the strong operator topology. Suppose that a separating vector $x \in X$ exists for P; that is, $P(E)x = 0$ implies $P(E) = 0$. For example, cyclic vectors are always separating. By a classical result of W. Bade [**1**, Theorem 3.1] there exists $x' \in X'$ such that the (finite) measure $\lambda = \langle P(.)x, x' \rangle$ is non-negative on Σ and satisfies $P(E)x = 0$ whenever $\lambda(E) = 0$. Since x is separating we have $P(E) = 0$ if and only if $\lambda(E) = 0$. Moreover, if $q(P)(E(n)) \to 0$, for all continuous seminorms q in $L_s(X)$, then also $P(E(n)) \to 0$ in $L_s(X)$ from which it is clear that $\lambda(E(n)) \to 0$. So, Proposition 3(i) holds for any spectral measure P with a separating vector.

REMARK 3. An essential ingredient in the proof of Proposition 3(ii) was the fact that the sequential closure $\bar{\mathscr{A}}_s$ was all of $\Sigma(m)$. Here, the metrizability of $\Sigma(m)$ was used. The following example shows that this condition cannot be removed in general.

EXAMPLE 1. Let $X = \mathbb{C}^{[0,1]}$ denote the vector space of all $\mathbb{C}$-valued functions on $\Omega = [0, 1]$ equipped with pointwise operations. For each $\omega \in \Omega$, define a seminorm $q_\omega : f \mapsto |f(\omega)|$, for $f \in X$. The seminorms $q_\omega(\omega \in \Omega)$ determine a complete lc Hausdorff topology on X. Bounded subsets of X are not necessarily metrizable.

Let Σ denote the σ-algebra of all subsets of Ω. Then the set function $m: \Sigma \to X$ defined by

$$m(E) = \chi_E, \qquad E \in \Sigma, \tag{7}$$

is σ-additive. Moreover, every function $\psi: \Omega \to \mathbb{C}$ belongs to $L^1(m)$. Indeed, $\int_E \psi \, dm = \chi_E \psi$, $E \in \Sigma$. It is routine to check that the topology $\tau(m)$ is precisely that of X and hence, $(L^1(m), \tau(m))$ is isomorphic to X.

Now, the space $(\Sigma(m), \tau(m))$ can be identified with $\{0, 1\}^\Omega$ equipped with its product topology. Since $\#(\Omega) = c$ the space $\Sigma(m) \simeq \{0, 1\}^\Omega$ is separable. Let $\mathscr{A}$ be any countable algebra of sets whose $\tau(m)$ closure is $\Sigma(m)$. By Lemma 2(ii) we have $\#(\overline{\chi(\mathscr{A})_s}) \le \aleph_0^{\aleph_0} < \#(\Sigma(m))$ and so the sequential closure $\bar{\mathscr{A}}_s$ cannot be all of $\Sigma(m)$.

REMARK 4. Example 1 is of interest for other reasons. We say that an algebra $\mathscr{A}$ of subsets of $\Omega = [0, 1]$ separates points of Ω if, whenever u and v are distinct points of Ω there is a set $A \in \mathscr{A}$ such that $A \cap \{u, v\}$ is a singleton. The following observation is straightforward to check.

FACT 1. *Let X be the lcs of Example 1. Let Σ be any σ-algebra of subsets of $\Omega = [0, 1]$ and $m: \Sigma \to X$ be the vector measure given by (7). Then $L^1(m)$ is precisely the space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω, equipped with the relative topology from X.*

(i) *If there exists a countable subalgebra of Σ which separates points of Ω, then $(\Sigma(m), \tau(m))$ is separable; that is, m is a separable measure.*

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(ii) *Suppose that* $\Sigma(m)$ (*which equals* Σ *as there are no non-trivial m-null sets*) *is* $\tau(m)$-*separable. If* Σ *contains all finite subsets of* Ω, *then there is a countable algebra* $\mathscr{A} \subseteq \Sigma$ *such that* $\mathscr{A}$ *separates points of* Ω.

Using Fact 1 it is clear, when Σ is the σ-algebra of Borel subsets of Ω, or the Lebesgue measurable subsets of Ω, or the universally measurable subsets of Ω, that the measure m given by (7) is always separable and hence, so is $L^1(m)$. However, if Σ is the σ-algebra of countable and co-countable subsets of Ω, then it can be shown that Σ is not countably generated, m is not separable and $L^1(m)$ is not separable (use the fact that if $f \in L^1(m)$, then f is constant on the complement of some countable set). Another example of this phenomenon occurs for the measure P of Example 2 below.

A further feature of the class of measures given by (7) is the following observation (which does not follow from Proposition 3(i)).

FACT 2. *Let* X *be the lcs of Example* 1. *Let* Σ *be any* σ-*algebra of subsets of* $\Omega = [0, 1]$ *and* $m : \Sigma \to X$ *be the vector measure given by* (7). *Then the sequential closure* $\bar{\Sigma}_s$ (*taken in* X) *is actually a* σ-*algebra.*

Proof. By Lemma 2(iii) it follows that $\Sigma \subseteq \Sigma_s$. Conversely, suppose that $\{E(n)\}_{n=1}^{\infty}$ is a sequence of sets from Σ which converges to f in $L^1(m)$; that is, $\chi_{E(n)} \to f$ in $\mathbb{C}^{\Omega}$. It is then clear that $f = \chi_E$ for some set $E \subseteq \Omega$. But, the pointwise limit of a sequence of Σ-measurable functions is Σ-measurable and so $E \in \Sigma$. Continuing this argument via the transfinite inductive definition of $\bar{\Sigma}_s$ we conclude that $\bar{\Sigma}_s \subseteq \Sigma$. $\square$

As a simple consequence, let Σ be the Lebesgue measurable sets in $[0, 1]$. With X as in Example 1 and $m : \Sigma \to X$ given by (7) we have seen that $\Sigma(m) = \Sigma$ is separable. Let $\mathscr{A}$ be a countable algebra of sets in Σ whose $\tau(m)$-closure is Σ. By Fact 2 we note that $\bar{\mathscr{A}}_s$ (taken in X) is a σ-algebra. However, $\bar{\mathscr{A}}_s \neq \Sigma$. This follows from Lemma 2(ii) and the fact that $\#(\Sigma) = 2^c$.

REMARK 5. The results of this section suggest the following two natural questions.

(i) Do there exist a lcs X, a measure $m : \Sigma \to X$ and an algebra of sets $\mathscr{A} \subseteq \Sigma$ such that $\bar{\mathscr{A}}_s$ is not sequentially closed in $\Sigma(m)$?

(ii) Do there exist a lcs X and a measure $m : \Sigma \to X$ such that m is not a separable measure but $L^1(m)$ is separable?

It may be worth noting that Example 1 does not answer Question (ii). For, if X is the lcs given there and Σ is any σ-algebra of subsets of $\Omega = [0, 1]$, then $m : \Sigma \to X$ (given by (7)) is a separable measure if and only if $L^1(m)$ is separable. Indeed, suppose that $L^1(m)$ is separable. Considering only $\mathbb{R}$-valued functions, let $\mathscr{F} \subseteq L^1(m)$ be a countable dense set and, for $f \in \mathscr{F}$, set $E(f) = \{\omega \in \Omega; f(\omega) > \frac{1}{2}\}$. It turns out that $D = \{\chi_{E(f)}; f \in \mathscr{F}\}$ is a (countable) dense set in $\Sigma(m)$. The case for $\mathbb{C}$-valued functions then follows. The converse claim follows from Proposition 1(i). The fact that Ω is the interval $[0, 1]$ is not important. Indeed, if Ω is any non-empty set and $X = \mathbb{C}^{\Omega}$ (with the pointwise convergence topology), then a similar argument shows that a vector measure $m : \Sigma \to X$ (with Σ a σ-algebra of subsets of Ω) of the form (7) is separable if and only if $L^1(m)$ is separable.

2. Operator-valued measures. Let X be a lcs and $T \in L(X)$ be a scalar-type spectral operator. Such an operator T has a unique (equicontinuous) spectral measure

$P_T:\mathscr{B}(\mathbb{C})\to L_s(X)$, such that the identity function λ (on $\mathbb{C}$) is P_T-integrable and $T=\int_{\mathbb{C}}\lambda\,dP_T$. Here $\mathscr{B}(\mathbb{C})$ is the σ-algebra of Borel subsets of $\mathbb{C}$. The measure P_T is called the resolution of the identity for T. So, $L^1(P_T)$ is always $\tau(P_T)$-separable. Under certain completeness assumptions on the lc spaces $L_s(X)$ and $L^1(P_T)$ it turns out that $L^1(P_T)$ is isomorphic to the strong operator closed algebra in $L_s(X)$ generated by the range of the resolution of the identity for T; see **[2]**. Accordingly, this algebra of operators is necessarily separable for the strong and hence also the weak operator topology. It may be worth noting that for X a Banach space it is known that $L^1(P_T)$ coincides with $L^\infty(P_T)$ as a vector space. Accordingly, $L^\infty(P_T)$ is always $\tau(P_T)$-separable. Of course, it is rarely separable for the P_T-essential sup-norm topology given by

$$\|f\|_\infty=\inf\{\|f\chi_E\|_\infty;\,E\in\Sigma,\,P_T(E)=I\},\quad \text{for } f\in L^\infty(P_T).$$

To treat operator algebras generated by arbitrary complete and σ-complete Boolean algebras of projections (by realizing the Boolean algebra as the range of a spectral measure) it is necessary to consider σ-algebras more general than $\mathscr{B}(\mathbb{C})$. The results of this section are formulated for arbitrary operator-valued measures, not just spectral measures.

An operator-valued measure is any set function $P:\Sigma\to L_s(X)$, with domain a σ-algebra of subsets of some set Ω, which is σ-additive. The topology of $L_s(X)$ is generated by the seminorms

$$q_x:T\mapsto q(Tx),\qquad T\in L(X),\tag{8}$$

for every $x\in X$ and $q\in\mathscr{Q}_X$. The continuous dual space of $L_s(X)$ consists of all finite linear combinations of functionals of the form

$$\xi_{x,x'}:T\mapsto\langle Tx,x'\rangle,\qquad T\in L(X),\tag{9}$$

for arbitrary $x\in X$ and $x'\in X'$. For each $x\in X$, let $Px:\Sigma\to X$ denote the vector measure $Px:E\mapsto P(E)x$, for $E\in\Sigma$.

The main question is the connection between the separability of the operator-valued measure P and that of the family of (generally simpler) X-valued measures $Px, x\in X$, from which P is synthesized.

PROPOSITION 4. *Let X be a lcs and $P:\Sigma\to L_s(X)$ be a measure. If P is separable, then each induced X-valued measure $Px:\Sigma\to X$, $x\in X$, is also separable. If, in addition, X is metrizable, then each space $L^1(Px)$, $x\in X$, is $\tau(Px)$-separable.*

Proof. Fix $x\in X$. Since each P-null set is also Px-null, it follows that the natural map $\Phi:\Sigma(P)\to\Sigma(Px)$ which sends the P-equivalence class, $[E]_P$, of $E\in\Sigma$, to the Px-equivalence class $[E]_{Px}$ is well-defined and onto. Since the continuous image of a separable space is separable it suffices to show that Φ is continuous.

A typical $\tau(Px)$ semi-metric is of the form

$$d_q(E,F)=\sup\{|\langle Px,x'\rangle|\,(E\,\Delta F);\,x'\in U_q^0\},\qquad E,F\in\Sigma,\tag{10}$$

where $q\in\mathscr{Q}_X$ and $\langle Px,x'\rangle$ is the $\mathbb{C}$-valued measure $\langle Px,x'\rangle:E\mapsto\langle P(E)x,x'\rangle$, for $E\in\Sigma$. A direct calculation shows that, for every $x'\in U_q^0$, the functional $\xi_{x,x'}$ given by (9) belongs to the polar set $U_{q_x}^0$, where q_x is the seminorm (8). Since the measures $\langle P,\xi_{x,x'}\rangle$ and $\langle Px,x'\rangle$ coincide, it follows that the right-hand-side of (10) does not exceed $\sup\{|\langle P,\xi\rangle|\,(E\,\Delta F);\,\xi\in U_{q_x}^0\}=d_{q_x}(E,F)$; that is, we have the inequalities $d_q(E,F)\le$

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$d_{q_x}(E,F)$, for $E, F \in \Sigma$, for every $q \in \mathcal{Q}_X$. Since each q_x is continuous it follows that d_{q_x} is one of the semi-metrics generating $\tau(P)$. This shows that Φ is continuous.

The statement concerning the separability of each space $L^1(Px)$, $x \in X$ (in the event that X is metrizable), follows from Proposition 1. □

In practice, a converse statement to that of Proposition 4 would be more useful. Unfortunately, no such statement is valid, even for X a "nice" space (eg. a Hilbert space).

Example 2. We exhibit an operator-valued measure $P:\Sigma \to L(X)$ such that

(i) P is not a separable measure,

(ii) $L^1(P)$ is not a separable space, but

(iii) each measure $Px, x \in X$, is separable and each space $L^1(Px)$ is separable.

Indeed, let $X = l^2(\Omega)$, where Ω is a set with $c < \#(\Omega)$, and let Σ be the σ-algebra of all subsets of Ω. For each $E \in \Sigma$, let $P(E) \in L(X)$ denote the operator in X of pointwise multiplication by χ_E; here we interpret elements of X as functions $x:\Omega \to \mathbb{C}$ such that $\sum_{\omega\in\Omega} |x(\omega)|^2 < \infty$. Then $P:\Sigma \to L(X)$, so defined, is an operator-valued (even spectral) measure.

Fix $x \in X$. Then Px is the X-valued measure $Px: E \mapsto x\chi_E$, for $E \in \Sigma$. The space $L^1(Px)$ can be identified with all functions $\psi:\Omega \to \mathbb{C}$ for which the product function ψx belongs to X, with integrals given by $\int_E \psi\, dPx = \psi x \chi_E$, for $E \in \Sigma$. Since $\mathcal{Q}_X$ consists of a single norm, denoted by $\|.\|_2$, the space $L^1(Px)$ is a normed space with norm $\|.\|_x$ given by

$$\|\psi\|_x = \sup\left\{\int_\Omega |\psi|\, d\,|\langle Px, x'\rangle|;\ \|x'\|_2 \le 1\right\}, \qquad \psi \in L^1(Px). \tag{11}$$

We have used the formula (2) and identified X' with X. But, a direct calculation shows that $|\langle Px, x'\rangle|$ is the measure $E \mapsto \Sigma_{\omega\in\Omega}\, \chi_E(\omega)\,|x(\omega)x'(\omega)|$, for $E \in \Sigma$, where the sum consists of countably many terms (since $xx' \in l^1(\Omega)$ implies that there are only countably many points $\omega \in \Omega$ for which $x(\omega)x'(\omega) \neq 0$). It follows from this fact and (11) that $\|\psi\|_x = \|x\psi\|_2$, for $\psi \in L^1(Px)$. Now, $x \in X$ implies that $Z(x) = \{\omega \in \Omega; x(\omega) \neq 0\}$ is a countable set. Furthermore, the metric d_x determining the topology $\tau(Px)$ is given by

$$d_x(E,F) = \|\chi_E - \chi_F\|_x = \|(\chi_E - \chi_F)x\|_2, \qquad E, F \in \Sigma.$$

Since any function $h:\Omega \to \mathbb{C}$ such that $Z(h) \cap Z(x) = \phi$ is Px-null, the set (of equivalence classes) $\Sigma(Px)$ is countable. Accordingly, Px is a separable measure. By Proposition 1 also $L^1(Px)$ is separable. This establishes (iii).

As a linear space we can identify $L^1(P)$ with the space $l^\infty(\Omega)$. Indeed, every $f \in l^\infty(\Omega)$ has indefinite integral given by $E \mapsto \int_E f\, dP$, $E \in \Sigma$, where $\int_E f\, dP \in L(X)$ is the operator in X of pointwise multiplication by $\chi_E f$. We consider an equivalent family of seminorms generating the topology $\tau(P)$; see [**6**;, Ch. II Sections 1–2]. Recalling that $\mathcal{Q}_X = \{\|.\|_2\}$ this family of seminorms can be specified as

$$\|f\|_{x,2} = \sup\left\{\left\|\left(\int_E f\, dP\right)x\right\|_2;\ E \in \Sigma\right\} = \|xf\|_2, \qquad f \in L^1(P), \tag{12}$$

for every $x \in X$.

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Let $Y = l^\infty(\Omega)$, equipped with the topology of pointwise convergence on Ω. This is a lcs with topology determined by the seminorms $\varphi \mapsto |\varphi(\omega)|$, $\varphi \in Y$, for each $\omega \in \Omega$. Given $\omega \in \Omega$, the element $e_\omega = \chi_{\{\omega\}}$ of X satisfies

$$\|e_\omega f\|_2^2 = \sum_{s \in \Omega} |e_\omega(s) f(s)|^2 = |f(\omega)|^2, \qquad f \in L^1(P),$$

and so (12) shows that the identity map from $L^1(P)$ onto Y is continuous. Now, Y is a dense subspace of the lcs $Z = \mathbb{C}^\Omega$, equipped with the pointwise convergence topology on Ω or, equivalently, the product topology. Since Z is non-separable (as $c < \#(\Omega)$) it follows that Y, hence also $L^1(P)$, cannot be separable. This is (ii). Then (i) follows from Proposition 1. □

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Addendum. Shortly before the proofs of this paper arrived Dr D. Fremlin (private communication) showed that the answer to Question (ii) of Remark 5 is negative. Indeed, the seminorms $q(m)$, given by (2), are Riesz seminorms. Accordingly, if $\mathcal{F}$ is dense in $L^1(m)$ and $A(f) = \{\omega; |f(\omega) - 1| \le \frac{1}{2}\}$ then, for $\mathbb{R}$-valued f, we have

$$q(m)(\chi_E - \chi_{A(f)}) \le 2q(m)(\chi_E - f)$$

for every $E \in \Sigma$, $f \in \mathcal{F}$ and continuous seminorm q in X. It follows (even for $\mathbb{C}$-valued f) that $\{\chi_{A(f)}; f \in \mathcal{F}\}$ is dense in $\Sigma(m)$. Accordingly, Proposition 1 can be improved as follows.

Proposition 1A. *Let X be a Hausdorff, sequentially complete lcs. Then m is separable if and only if $L^1(m)$ is separable.*

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