PROJECTIVE BITOPOLOGICAL SPACES

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1. Introduction

J. C. Kelly [2] introduced the concept of a bitopological space. Lane [3], Patty [4] and Pervin [5] have continued his work. Our purpose in this paper is to identify the projective objects in a suitable category of bitopological spaces after the manner of Gleason [1] and generalize his theorem that in the category of compact Hausdorff topological spaces, the projective spaces are precisely the extremally disconnected ones.

We give new definitions of continuity and of separation in bitopological spaces. These definitions are strictly weaker than the corresponding ones (for bitopological spaces) in the literature. Precisely, our 'quasi-continuity' is weaker than Pervin's continuity [5] and our 'quasi-Hausdorff' bitopological space is more general than a pairwise Hausdorff one [2]. Here we give a definition of 'semi-compactness' for bitopological spaces. (The reason for the nomenclature will be clear from the definition.) All these three concepts reduce to the corresponding conventional concepts, with the prefix 'quasi' or 'semi' removed, when the two topologies of the bitopological structure coincide and we have just a space with a single topology. We also generalize the definition of an extremally disconnected space to bitopological space. Under this setting, we prove the following

MAIN THEOREM. *In the category of semi-compact, quasi-Hausdorff bitopological spaces and quasi-continuous maps, the projective spaces are precisely the extremally disconnected ones.*

Throughout the following, (X, P, Q) stands for a bitopological space with topologies P and Q. If we need two bitopological spaces simultaneously we shall take them as (X_1, P_1, Q_1) and (X_2, P_2, Q_2).

2. Preliminaries on continuity

2.1. DEFINITION. A subset A of (X, P, Q) is said to be *quasi-open* if for every $x \in A$ there exists a P-open neighborhood $U_x \subset A$ or a Q-open neighbourhood $V_x \subset A$.

2.2. PROPOSITION. *Quasi-open sets are precisely the unions of P-open and Q-open sets.*

Proof follows directly from definitions.

REMARK. Every P-open (Q-open) set is quasi-open. That arbitrary union of quasi-open sets is quasi-open follows by 2.2. Finite intersection of quasi-open sets need not be quasi-open. For example, let X be the real line R. Let P be the topology with base $[a, b)$ and Q be the topology with base $(c, d]$. Let $a < b < c$. Then $(a, b]$ and $[b, c)$ are quasi-open sets, but $(a, b] \cap [b, c) = \{b\}$ is not a quasi-open set.

2.3. DEFINITION. A *quasi-closed* set is the complement of a quasi-open set.

REMARK. Every P-closed (Q-closed) set is quasi-closed. Arbitrary intersection of quasi-closed sets is quasi-closed. Finite union of quasi-closed sets need not be quasi-closed. Every quasi-closed set is the intersection of a P-closed set and a Q-closed set.

2.4. DEFINITION. The *quasi-closure* of $A \subset (X, P, Q)$ is $(P\text{-cl}(A)) \cap (Q\text{-cl}(A))$, where P-cl denotes P-closure and Q-cl denotes Q-closure. The quasi-closure of A is denoted by $\bar{A}$.

2.5. PROPOSITION. *If $A \subset (X, P, Q)$, then $\bar{A}$ is the smallest quasi-closed set containing A.*

Proof follows from definitions 2.1, 2.3 and 2.4.

2.6. DEFINITION. Let $f: (X_1, P_1, Q_1) \to (X_2, P_2, Q_2)$. f is said to be *quasi-continuous* if the inverse image of every quasi-open set is quasi-open.

2.7. PROPOSITION. *Let $f: (X_1, P_1, Q_1) \to (X_2, P_2, Q_2)$ be quasi-continuous. Let $G \subset X_1$. Then $f(\bar{G}) \subset \overline{f(G)}$.*

PROOF. Let $x \in (P_1\text{-cl}(G) \cap Q_1\text{-cl}(G))$. Let U be any P_2-neighborhood of $f(x)$ in X_2. Since f is quasi-continuous, $f^{-1}(U)$ is quasi-open in X_1 and $x \in f^{-1}(U)$. Therefore there exists either a P_1-neighborhood or a Q_1-neighborhood W of x such that $W \subset f^{-1}(U)$. W meets G because $x \in (P_1\text{-cl}(G) \cap Q_1\text{-cl}(G))$. Hence $f^{-1}(U)$ meets G and so U meets $f(G)$. Hence $f(x) \in P_2\text{-cl}(f(G))$. Similarly $f(x) \in Q_2\text{-cl}(f(G))$. Thus $f(\bar{G}) \subset \overline{f(G)}$.

REMARK. Every quasi-continuous map is continuous for the upper bound topology. The converse need not be true as is shown by the following example.

Let X_1 be the real line R and d be a metric defined by $d(x, y) = \min. \{1, |x-y|\}$. Define conjugate quasi-pseudo-metrics p and q on X_1 by putting, for $(x, y) \in R$,

$$p(x, y) = \begin{cases} d(x, y) & \text{if } x \leqq y \\ 1 & \text{if } x > y \end{cases}$$

and $q(x, y) = p(y, x)$.

Let P_1 and Q_1 be the topologies on X_1 determined by p and q respectively. Let X_2 be the real line R. Define $u(x, x) = 0$ for each $x \in R$. If $x \neq y$, define $u(x, y) = 1$ if x is rational and $u(x, y) = |x-y|/(1+|x-y|)$ if x is irrational. Define v by $v(x, y) = u(y, x)$. Let P_2 and Q_2 be the topologies on X_2 determined by u and v respectively.

Consider the identity map $I: (X_1, P_1, Q_1) \to (X_2, P_2, Q_2)$. I is continuous in the upper bound topology. I is not quasi-continuous. For, consider the open interval (a, b) without rational points. It is quasi-open in X_2 but is not quasi-open in X_1.

3. Semi-compact bitopological spaces

3.1. Definition. $A \subset (X, P, Q)$ is said to be *semi-open* if it is open in the upper bound topology of P and Q.

Remark. Consequently we have the natural notion of a semi-closed set and of semi-closure.

3.2 Definition. A set $A \subset (X, P, Q)$ is said to be *semi-compact* if it is compact in the upper bound topology of P and Q; in other words A is semi-compact if and only if, given any covering of A by semi-open subsets of X there exists a finite subcovering.

Remark. Every semi-closed (and therefore every P-closed, Q-closed and quasi-closed) subset of a semi-compact bitopological space is semi-compact.

3.3. Proposition. *The quasi-continuous image of a semi-compact bitopological space is semi-compact.*

Proof. Let f be a quasi-continuous mapping of a semi-compact space (X_1, P_1, Q_1) onto an arbitrary space (X_2, P_2, Q_2). We shall show that (X_2, P_2, Q_2) is semi-compact.

Let $v = (U_\alpha)_{\alpha \in A}$ be a covering of X_2, where each U_α is semi-open and so of the form $\bigcup_{i,j} (V_{\alpha_i} \cap W_{\alpha_j})$ where V_{α_i} is P-open and W_{α_j} is Q-open. Then $f^{-1}(U_\alpha) = f^{-1}(\bigcup_{i,j} (V_{\alpha_i} \cap W_{\alpha_j})) = \bigcup_{i,j} (f^{-1}(V_{\alpha_i}) \cap f^{-1}(W_{\alpha_j}))$. V_{α_i} is quasi-open, so $f^{-1}(V_{\alpha_i})$ is quasi-open and hence semi-open. Similarly, $f^{-1}(W_{\alpha_j})$ is semi-open. Hence $f^{-1}(U_\alpha)$ is semi-open. Therefore $(f^{-1}(U_\alpha))_{\alpha \in A}$ is a semi-open covering of X_1. Since X_1 is semi-compact, there exists a finite subset $A_1 \subset A$ such that $\bigcup_{\alpha \in A_1} f^{-1}(U_\alpha)$ covers X_1. Therefore $\bigcup_{\alpha \in A_1} U_\alpha$ is a finite subcovering of v which covers X_2. Therefore X_2 is semi-compact.

3.4. Definition. (X, P, Q) is said to be *quasi-Hausdorff* if given $x_1 \neq x_2$ there exist quasi-open sets U_1, U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Remark. Kelly [2] calls (X, P, Q) pairwise Hausdorff if given $x_1 \neq x_2$ there exists a P-open set U_1 and a Q-open set U_2 containing x_1 and x_2 respectively such that $U_1 \cap U_2 = \emptyset$. So every pairwise Hausdorff space is also quasi-Hausdorff. But the converse need not be true. For example, let X be the real line R and d be a metric defined by $d(x, y) = \min. \{1, |x-y|\}$. Define, for $(x, y) \in R$, $u(x, y) = 0$ if $x \leqq y$, $= d(x, y)$ if $x > y$ and $v(x, y) = u(y, x)$.

Let P and Q be the topologies on X determined by u and v respectively. (X, P, Q) is quasi-Hausdorff but not pairwise Hausdorff.

3.5. PROPOSITION. *Let (X, P, Q) be a quasi-Hausdorff space. Let U_1 and U_2 be quasi-open sets such that $U_1 \cap U_2 = \emptyset$. Then $\bar{U}_1 \cap U_2 = \emptyset$.*

PROOF. Let U_1 and U_2 be quasi-open sets such that $U_1 \cap U_2 = \emptyset$. We shall show that $\bar{U}_1 \cap U_2 = \emptyset$. Suppose $\bar{U}_1 \cap U_2 \neq \emptyset$. Let $y \in \bar{U}_1 \cap U_2$. $y \in \bar{U}_1$ implies every P-neighborhood and Q-neighborhood of y meets U_1. But U_2 is a quasi-open set containing y and so there exists either a P-neighborhood or a Q-neighborhood W of y such that $W \subset U_2$. But by hypothesis U_2 does not meet U_1, so $W \cap U_1 = \emptyset$. This contradiction establishes that $\bar{U}_1 \cap U_2 = \emptyset$.

3.6. DEFINITION. Let $(X_i, P_i, Q_i)_{i \in I}$ be a family of bitopological spaces. On the product set $X = \prod_{i \in I} X_i$, we define a *bitopological structure* (P, Q) by taking P as the product topology generated by the $P_i s$ and Q as the product topology generated by the $Q_i s$.

3.7. PROPOSITION. *The natural projection from a product bitopological space (X, P, Q) on the component spaces $(X_i, P_i, Q_i)_{i \in I}$ is quasi-continuous.*

PROOF. Let $\prod_i$ denote the projection from X onto X_i. Let U be any quasi-open set of X_i. Then $U = V \cup W$ where V is P_i-open and W is Q_i-open. $\prod_i^{-1}(U) = \prod_i^{-1}(V) \cup \prod_i^{-1}(W)$. $\prod_i^{-1}(V)$ is P-open and $\prod_i^{-1}(W)$ is Q-open. Therefore $\prod_i^{-1}(U)$ is quasi-open in X. Hence $\prod_i$ is quasi-continuous.

3.8. PROPOSITION. *The product of any arbitrary number of semi-compact bitopological spaces is semi-compact.*

PROOF. Let $(X_i, P_i, Q_i)_{i \in I}$ be a family of semi-compact spaces. Let τ_i be the upper bound topology of P_i and Q_i. Let $(X, \tau) = \prod_{i \in I} (X_i, \tau_i)$ be the product space, and let $(X, P, Q) = \prod_{i \in I} (X_i, P_i, Q_i)$ be the product bitopological space. Then (X, τ) is compact since by definition each (X_i, τ_i) is compact. Since τ is finer than the upper bound topology of P and Q, (X, P, Q) is semi-compact.

3.9. PROPOSITION. Let (A, P_1, Q_1), (B, P_2, Q_2) and (C, P_3, Q_3) be quasi-Hausdorff spaces. Let $f: B \to C$ and $g: A \to C$ be quasi-continuous maps. Then $D = \{(a, b) : g(a) = f(b)\}$ is a semi-closed subset of $A \times B$.

PROOF. Let $(A \times B, P, Q)$ be the product space of (A, P_1, Q_1) and (B, P_2, Q_2). Let $(a_1, b_1) \in (A \times B) - D$. Therefore $g(a_1) \neq f(b_1)$ in C. Because C is quasi-Hausdorff, there exists disjoint quasi-open sets U_1 and U_2 containing

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$g(a_1)$ and $f(b_1)$. So, $a_1 \in g^{-1}(U_1)$ and $b_1 \in f^{-1}(U_2)$. Since g and f are quasi-continuous, $g^{-1}(U_1)$ and $f^{-1}(U_2)$ are quasi-open sets in A and B respectively. Since a quasi-open set is also semi-open, $g^{-1}(U_1) \times f^{-1}(U_2)$ is a semi-open subset of $A \times B$.

We shall now show that $D \cap \{g^{-1}(U_1) \times f^{-1}(U_2)\} = \emptyset$. Suppose these two sets are not disjoint, then let $(a_2, b_2) \in D \cap \{g^{-1}(U_1) \times f^{-1}(U_2)\}$. $(a_2, b_2) \in D$ implies $g(a_2) = f(b_2)$. $(a_2, b_2) \in \{g^{-1}(U_1) \times f^{-1}(U_2)\}$ implies $g(a_2) \in U_1$ and $f(b_2) \in U_2$. So, $g(a_2) = f(b_2) \in U_1 \cap U_2$ contradicting the fact that U_1 and U_2 are disjoint. Therefore $\{g^{-1}(U_1) \times f^{-1}(U_2)\}$ is disjoint from D.

Thus for every point $(a_1, b_1) \in (A \times B) - D$ there exists a semi-open set disjoint from D. Therefore $(A \times B) - D$ is semi-open and hence D is semi-closed.

4. Extremally disconnected bitopological spaces

4.1. Definition. (X, P, Q) is said to be *extremally disconnected* if for every semi-open set its quasi-closure is quasi-open.

4.2. Proposition. *A semi-compact subset of a quasi-Hausdorff bitopological space is semi-closed.*

This is the analogue for bitopological spaces of the classical result that a compact subset of a Hausdorff space is closed. The proof runs on the same pattern. We have only to recall that every quasi-open set is semi-open.

However this proposition is not enough for the proof of our main theorem. So we prove the following

4.3. Proposition. *A semi-compact subset of a quasi-Hausdorff bitopological space which is also extremally disconnected is quasi-closed.*

Remark. The hypothesis of extremally disconnectedness is not necessary for the conclusion as is shown by the space (X, P, Q) of 3.3 (Remark). A semi-compact subset here is a closed bounded interval $[a, b]$ which is quasi-closed in X. But we do not know how to prove the proposition without this extra hypothesis.

Proof of the Proposition. Let A be a semi-compact subset of a quasi-Hausdorff space (X, P, Q) which is also extremally disconnected. We shall show that $X - A$ is quasi-open.

Let $s \in X - A$. By proposition 3.5, there exist for each $a \in A$ disjoint quasi-open sets U_a and V_a containing a and s respectively such that $U_a \cap \overline{V}_a = \emptyset$. The collection $\{U_a\}_{a \in A}$ is a semi-open covering of A. By compactness of A, there exists a finite subset $A_1 \subset A$ such that $\bigcup_{a \in A_1} U_a$ covers A. Thus $A \subset \bigcup_{a \in A_1} U_a$. Let $W_s = \bigcap_{a \in A_1} \overline{V}_a$. Then $\overline{\bigcap_{a \in A_1} V_a} \subset \bigcap_{a \in A_1} \overline{V}_a = W_s$. But $\overline{\bigcap_{a \in A_1} V_a}$ is quasi-open (because (X, P, Q) is extremally disconnected). So W_s contains a quasi-open set containing s. Also $A \cap W_s \subset (\bigcup_{a \in A_1} U_a) \cap W_s = \emptyset$. Thus for each $s \in X - A$, there exists a quasi-open set $\subset W_s$ such that $W_s \cap A = \emptyset$. This proves $X - A$ is quasi-open. Hence A is quasi-closed.

5. Generalization of Gleason's lemmas

5.1. LEMMA. *Let $\rho : (E, P, Q) \to (A_1, P_1, Q_1)$ be quasi-continuous and onto such that $\rho(E_0) \neq A$ for any proper semi-closed subset $E_0 \subset E$. Then for any quasi-open set $G \subset E$, $\rho(G) \subset \overline{(A-\rho(E-G))}$.*

The proof runs on the same lines as that of lemma 2.1 of [1].

5.2. LEMMA. *In an extremally disconnected quasi-Hausdorff bitopological space, if U_1 and U_2 are two disjoint quasi-open sets then $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.*

The proof follows from proposition 3.5 and definition 4.1.

5.3. LEMMA. *Let (E, P, Q) be a semi-compact space and (A, P_1, Q_1) be a quasi-Hausdorff and extremally disconnected space. Let $\rho : E \to A$ be quasi-continuous, one-one and onto. Then ρ^{-1} is quasi-continuous.*

PROOF. Let F be a quasi-closed subset of E. Then F is semi-compact. Since ρ is quasi continuous, $\rho(F)$ is semi-compact. So, by proposition 4.3, $\rho(F)$ is quasi-closed in A. This proves ρ^{-1} is quasi-continuous.

5.4 LEMMA. *Let (E, P, Q) be semi-compact and quasi-Hausdorff, and (A, P_1, Q_1) be semi-compact, quasi-Hausdorff and extremally disconnected spaces. Let $\rho : E \to A$ be quasi-continuous and onto. Let $\rho(E_0) \neq A$ for any proper semi-closed subset $E_0 \subset E$. Then ρ is $1-1$ and ρ^{-1} is quasi-continuous.*

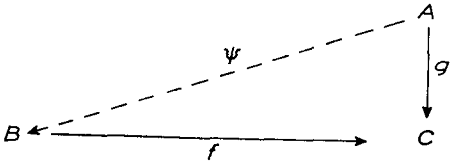
PROOF. In view of Lemma 5.3 we have only to prove that ρ is one-one. The proof of this follows on the same lines as that of Lemma 2.3 of [1].

5.5. LEMMA. *Let (A, P, Q) and (D, P_1, Q_1) be semi-compact, quasi-Hausdorff spaces. Let $\prod : D \to A$ be quasi-continuous and onto. Then D contains a semi-compact subset E such that $\prod(E) = A$ but $\prod(E_0) \neq A$ for any proper semi-closed subset $E_0 \subset E$.*

The proof runs along the same lines as that of lemma 2.4 of [1].

6. Proof of the main theorem

Let us first prove that every extremally disconnected semi-compact quasi-Hausdorff space (A, P, Q) is projective. Let (B, P_1, Q_1) and (C, P_2, Q_2) be semi-compact quasi-Hausdorff spaces and $f : B \to C$ and $g : A \to C$ be quasi-continuous maps and f is also onto.



We have to produce a quasi-continuous map $\psi : A \to B$ such that the above diagram commutes. Consider $A \times B$ and its subset $D = \{(a, b) : g(a) = f(b)\}$.

By proposition 3.8, D is semi-closed subset of $A\times B$. By proposition 3.7, $A\times B$ is semi-compact and hence D is semi-compact. Consider the projection $\prod_1 : A\times B \to A$. Since f is onto, $\{f(b) : b\in B\}$ exhausts C and therefore $\{g(a) : (a, b)\in D\}$ exhausts C which means $\prod_1$ maps D onto A. By Lemma 5.5, there exists a semi-compact subset $E\subset D$ such that $\prod_1(E)=A$ but $\prod_1(E_0)\neq A$ for any proper semi-closed subset $E_0 \subset E$. Let $\rho=\prod_1 E$. ρ satisfies all the conditions of Lemma 5.4 and so ρ^{-1} exists and is quasi-continuous. Let

$$\psi=\prod\nolimits_2\circ\rho^{-1} : A\xrightarrow{\rho^{-1}} E\subset A\times B\xrightarrow{\Pi_2} B.$$

ψ is quasi-continuous. We shall show that ψ is the required map. Let $a\in A$. Since $\rho^{-1}(a)\in D$,

$$f(\textstyle\prod_2(\rho^{-1}(a)))=g(\prod_1(\rho^{-1}(a)))=g(a).$$

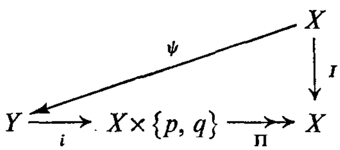
Therefore $g=f\circ\prod_2\circ\rho^{-1}=f\circ\psi$. This proves one part of the theorem.

Now let us prove that every projective space is extremally disconnected.

Let (X, P, Q) be a projective space. Let G be semi-open subset of X. We shall show that $\bar{G}$ is quasi-open. Consider

$$Y=((X-G)\times\{p\})\cup(\bar{G}\times\{q\})\subset X\times\{p, q\},$$

where $\{p, q\}$ has the bitopological structure $(\emptyset, \{p\}, \{p, q\})$ and $(\emptyset, \{q\}, \{p, q\})$.



Since X is projective there exists a quasi-continuous map $\psi : X\to Y$ such that the above diagram commutes, where I is the identity map from X into X. $\prod\circ i$ is one-one on $(G\times\{q\})$. Therefore $\psi(x)=(x, q)$ for all $x\in G$. Since ψ is quasi-continuous $\psi(x)\in\overline{(G\times\{q\})}$ for all $x\in\bar{G}$. But $\overline{(G\times\{q\})}\subset\bar{G}\times\{q\}$. Therefore $\psi^{-1}(\bar{G}\times\{q\}\supset\bar{G}$. Also if $x\notin\bar{G}$, $\psi(x)=(x, p)$. So $\psi^{-1}(\bar{G}\times\{q\})=\bar{G}$. But ψ is quasi-continuous and $(X-G\times\{p\})$ is quasi-closed in Y and so $\bar{G}\times\{q\}$ is quasi-open in Y. Therefore $\bar{G}$ is quasi-open in X.

Remark. In the second part of the proof of the theorem the quasi-Hausdorff and semi-compact property of the spaces was not used and so in the category of all bitopological spaces and quasi-continuous maps every projective space is extremally disconnected.

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