ON ISOMORPHISMS OF MEROMORPHIC FUNCTION FIELDS

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1. Introduction. In this work we deal with algebraic properties of some fields of functions meromorphic in the complex plane with a view to determining the possible isomorphisms between two such fields. Interest in problems of this type began with a paper by Bers (**2**), in which it was shown that the algebraic structure of the ring of functions analytic on a plane region determines the conformal structure of the region to within conformal or anti-conformal equivalence, and this result was later extended to arbitrary non-compact Riemann surfaces by Nakai (**7**). The corresponding problem where the ring of analytic functions is replaced by the field of meromorphic functions had long been outstanding. Contributions to this problem had been made by Royden (**9**), Heins (**5**), and Alling (**1**), but the solution was given only recently in a paper of Iss'sa (**6**). In particular, it is shown in (**6**) that the conformal structure of the complex plane C is determined by the algebraic structure of the field K of functions meromorphic on C.

In this paper we ask if the conformal structure of C is determined by the algebraic structure of a given subfield of K. The corresponding question for a given subring of the ring E of entire functions is generally easily treated and answered in the affirmative if the ring in question is at all reasonable (e.g., if it occurs in the classical theory of entire functions). For subfields of K, however, the problem is more difficult, and we have only partial results, for the methods of (**6**) do not readily generalize. This is because Iss'sa uses the fact that for $f \in E$ and $g \in K$ we have $g \circ f \in K$, whereas the composition $g \circ f$ need not belong to the subfield in question.

We present here a slightly modified version of Iss'sa's proof for the field K of all functions meromorphic on C. The proof given here will apply to certain subfields of K, and we are able to show that these subfields determine the conformal structure of the plane by their algebraic structure.

2. Algebraic preliminaries. In this section we consider some aspects of valuation theory which are required throughout the paper. Here we state results without proof and refer the reader to Zariski and Samuel (**12**) for further details. Throughout this section, unless otherwise specified, K will denote an arbitrary field and K^* the set $K - \{0\}$ considered as an abelian group under the operation of field multiplication.

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Definition 2.1. A *valuation* of K is a homomorphism v of K^* into the additive group N of integers (i.e., $v(xy) = v(x) + v(y)$ for all $x, y \in K^*$) such that

$$v(x - y) \geqq \min\{v(x), v(y)\}, \qquad x, y \in K^*, \ x \neq y.$$

The valuation is called non-trivial if $v(x) \neq 0$ for some $x \in K^*$. In this case the range of v is a subgroup of N, hence is isomorphic to N, and we shall always assume that a non-trivial valuation of K^* maps K^* *onto* N.

Definition 2.2. A *valuation ring* of K is a subring R of K such that for each $x \in K^*$, either $x \in R$ or $x^{-1} \in R$.

THEOREM 2.3. *Let R be a valuation ring of K. Then the set $\{x \in R: x^{-1} \notin R\}$ of all non-units of R is an ideal of R, hence is the unique maximal ideal of R.*

THEOREM 2.4. *Let v be a valuation of K. Then*
(i) $v(1) = 0$,
(ii) $v(x) = v(-x)$, $x \in K^*$,
(iii) $v(1/x) = -v(x)$, $x \in K^*$,
(iv) *If $x, y \in K^*$ with $v(x) \neq v(y)$, then $v(x - y) = \min\{v(x), v(y)\}$.*

If v is a valuation of K, we define $v(0) = \infty$, where ∞ is an element such that $\infty > n$ and $\infty + n = \infty$ for all $n \in N$. With this convention, statement (iv) of the above theorem is valid for all $x, y \in K$.

THEOREM 2.5. *Let v be a valuation of K. Then $R_v = \{x \in K: v(x) \geqq 0\}$ is a valuation ring of K with maximal ideal $M_v = \{x \in K: v(x) > 0\}$ and R_v is noetherian. Conversely, if R is a valuation ring of K which is noetherian, then there exists a unique valuation v of K such that $R = R_v$. We denote it by v_R.*

Example 2.6. Let K be a field of functions meromorphic on the complex plane C, K containing the polynomials. Let $a \in C$. Then, given $f \in K$, f admits a Laurent expansion at a,

$$f(z) = \sum_{-\infty}^{+\infty} A_k(z - a)^k,$$

where for some integer k_0 we have $A_k = 0$ for $k < k_0$. Define a function v_a on K^* by

$$v_a(f) = \inf\{k: A_k \neq 0\}.$$

Then v_a is a valuation of K. Its valuation ring is

$$R_a = \{f \in K: v_a(f) \geqq 0\} = \{f \in K: f(a) \neq \infty\},$$

and the corresponding maximal ideal is

$$M_a = \{f \in K: v_a(f) > 0\} = \{f \in K: f(a) = 0\}.$$

3. Isomorphisms of function fields. In this section we consider the classification of field isomorphisms between fields of meromorphic functions.

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Our discussions will be restricted to functions meromorphic on the complex plane, although some of the arguments apply to more general situations.

Definition 3.1. A *function field* is a field K of functions meromorphic on the complex plane such that:

(1) K contains the rational functions. (In particular, K contains the constants, which we identify with the complex field C, and K separates points of C.)

(2) If E denotes the ring of all entire functions, then K is the field of quotients of $K \cap E$ (i.e., if $f \in K$, there exist *entire* functions $g, h \in K$ such that $f = g/h$).

Definition 3.2. Let K be a function field. A *v-ring* of K is a subring R of K such that

(1) R is a noetherian valuation ring of K, and

(2) for each $f \in R$ there exists $\alpha \in C$ such that $(f - \alpha)$ is a non-unit of R. (Note that $R \neq K$ by condition (2).)

Example 3.3. Let K be a function field and $a \in C$. Then

$$R_a = \{f \in K: f(a) \neq \infty\}$$

is a v-ring of K. For we have $R_a = \{f \in K: v_a(f) \geqq 0\}$, v_a being the valuation of Example 2.6, and for each $f \in R_a$, we have that $f - f(a)$ is a non-unit of R_a.

LEMMA 3.4. *Let K be a function field and R a v-ring of K. Then R contains C, and for each $f \in R$ there is a unique $\alpha \in C$ such that $(f - \alpha) \in M(R)$.*

Proof. If $\alpha \in C$, $\alpha \neq 0$, then for each $n \geqq 1$ there exists $\beta \in C$ such that $\alpha = \beta^n$, whence $v_R(\alpha) = nv_R(\beta)$. That is, $v_R(\alpha)$ is an integer which is divisible by all integers $n \geqq 1$; thus, $v_R(\alpha) = 0$ and $\alpha \in R$. Further, suppose that $f \in R$ and $\alpha, \beta \in C$ so that $(f - \alpha), (f - \beta) \in M(R)$. Then

$$\delta = (\beta - \alpha) = (f - \alpha) - (f - \beta) \in M(R),$$

or $\delta \in C$ and $v_R(\delta) > 0$. By the above we have that $\delta = 0$, or $\alpha = \beta$.

LEMMA 3.5. *Let K_1 and K_2 be function fields and $\theta: K_1 \to K_2$ a field isomorphism of K_1 onto K_2. Then θ maps C onto C.*

Proof. Evidently, $\theta r = r$ for all real rationals r. Also, $(\theta i)^2 = -1$, thus either $\theta i = i$ or $\theta i = -i$. Thus, if C_r denotes the subfield of C of all complex numbers with rational coordinates, either $\theta\alpha = \alpha$ for all $\alpha \in C_r$, or $\theta\alpha = \bar{\alpha}$ for all $\alpha \in C_r$. Now take $f \in C$. Then $f - \alpha$ has an nth root in K_1 for all $n \geqq 1$ and all $\alpha \in C_r$; thus, $\theta f - \theta\alpha$ has an nth root in K_2 for all $n \geqq 1$ and all $\alpha \in C_r$. However, θ maps C_r onto C_r; thus, $\theta f - \beta$ has an nth root for all $n \geqq 1$ and all $\beta \in C_r$. This implies that θf is constant, thus θ maps C into C. We conclude that θ maps C onto C by considering the inverse isomorphism $\theta^{-1}: K_2 \to K_1$.

In general, an isomorphism of C onto itself need not be continuous. However, if discontinuous, it is necessarily very badly behaved, a fact we shall take advantage of below.

COROLLARY 3.6. *Let K_1 and K_2 be function fields and $\theta: K_1 \to K_2$ a field isomorphism of K_1 onto K_2. If R_1 is a v-ring of K_1, then θR_1 is a v-ring of K_2.*

Proof. Evidently, (1) of Definition 3.2 is invariant under isomorphisms, and the previous lemma shows that (2) remains valid as well.

In order to classify isomorphisms between function fields, more information is required concerning the v-rings of the fields.

Definition 3.7. A function field K is said to be *pointlike* if for each v-ring R of K there exists $a \in C$ such that $R = R_a = \{f \in K: f(a) \neq \infty\}$.

THEOREM 3.8. *Let K_1 and K_2 be function fields and $\theta: K_1 \to K_2$ a field isomorphism of K_1 onto K_2. Suppose that K_1 is pointlike. Then there is a unique one-to-one map ϕ from C into C such that for all $f \in K_1$ and all $b \in C$ we have*

$$(*) \qquad (\theta f)(b) = \theta[f(\phi(b))],$$

where $\theta\infty = \infty$.

Proof. Take $b \in C$; thus, $R_2 = \{g \in K_2: g(b) \neq \infty\}$ is a v-ring of K_2 with maximal ideal $M_2 = \{g \in K_2: g(b) = 0\}$. Then $R_1 = \theta^{-1}R_2$ is a v-ring of K_1; thus, there exists a unique point $a \in C$ such that $R_1 = \{f \in K_1: f(a) \neq \infty\}$. (The point a is unique since K_1 separates points of C.) Thus, taking $a = \phi(b)$ yields a well-defined map ϕ of C into C, and ϕ is one-to-one since K_2 separates points of C. Take $g \in R_2$; thus, $g - g(b) \in M_2$, and let $f = \theta^{-1}g$. Then $\theta^{-1}[g - g(b)] = f - \theta^{-1}[g(b)]$ is a non-unit of R_1, hence vanishes at a. Now $\theta^{-1}[g(b)]$ is a constant function, and therefore

$$f(\phi(b)) = f(a) = \theta^{-1}[g(b)] = \theta^{-1}[(\theta f)(b)],$$

and an application of θ to this equation yields (*) for $\theta f \in R_2$. If $g = \theta f \notin R_2$, then (*) follows on consideration of the function $1/g$, which belongs to R_2; thus, (*) holds for all $f \in K_1$.

We now require further information on the function fields involved. In the following we shall *always* suppose that a field isomorphism θ between two function fields satisfies the condition $\theta i = i$. The case $\theta i = -i$ can be treated by similar methods.

Definition 3.9. A function field K is said to be *saturated* if, given any infinite discrete subset $\{a_k\}_{k=1}^{\infty}$ of C, there exists $f \in K, f \not\equiv 0$, such that $f(a_k) = 0$ for infinitely many $k \geqq 1$.

For saturated subfields we have a considerably stronger result than that given previously.

THEOREM 3.10. *Let K_1 and K_2 be function fields and $\theta: K_1 \to K_2$ a field isomorphism of K_1 onto K_2. Suppose that K_1 is pointlike and saturated. Then there exists a unique conformal map ϕ of C onto C such that $\theta f = f \circ \phi$ for all $f \in K_1$.*

Proof. Let ϕ be the unique one-to-one map of Definition 3.9, therefore

$$(**) \qquad (\theta f)(b) = \theta[f(\phi(b))], \qquad f \in K_1,\ b \in C.$$

We claim that ϕ takes relatively compact sets of C into relatively compact sets of C. For otherwise there exists a convergent sequence $\{b_k\}_{k=1}^{\infty}$ of C such that $\{\phi(b_k)\}_{k=1}^{\infty}$ is an infinite discrete subset of C. Take $f \in K_1$, $f \not\equiv 0$, so that $f(\phi(b_k)) = 0$ for infinitely many $k \geqq 1$. We may suppose that $f(\phi(b_k)) = 0$ for all $k \geqq 1$, therefore

$$(\theta f)(b_k) = \theta[f(\phi(b_k))] = \theta(0) = 0, \qquad k \geqq 1.$$

Thus, $\theta f \in K_2$ is a non-constant meromorphic function whose zeros have a finite cluster point, a contradiction.

We now show that $\theta | C$ is continuous. To see this, take f to be the identity map on C and define $D = \{b \in C: |b| < 1\}$. Then $\phi(D)$ is a relatively compact subset of C and (**) gives

$$\sup\{|\theta^{-1}[g(b)]|: b \in D\} = \sup\{|\phi(b)|: b \in D\} < +\infty,$$

where $g = \theta f$. Now, this implies that g is analytic on D and that θ^{-1}, considered as an isomorphism of the complex field, is bounded on $g(D)$. However, $g(D)$ is an open set, since D is open and g is a non-constant meromorphic function. Thus, $\theta^{-1}|C$ is an isomorphism of the complex field onto itself which is bounded on an open set, and it is well known that this implies that $\theta^{-1}|C$ is continuous. From Lemma 3.5 we then have that $\theta^{-1}\alpha = \alpha$ for all $\alpha \in C$, as desired, and (**) now becomes

$$\theta f = f \circ \phi, \qquad f \in K_1.$$

Taking f to be the identity map on C, we see that $\phi = \theta f$ is meromorphic, hence entire, and therefore ϕ is a one-to-one analytic map of C onto C. Thus ϕ is necessarily of the form $\phi(z) = \alpha + \beta z$ for some $\alpha, \beta \in C$.

4. Pointlike function fields. As we have seen in the previous section, the isomorphisms between function fields may be classified by determining the v-rings of the fields in question. We begin with the following simple theorem.

THEOREM 4.1. *Let K be a function field and R a v-ring of K. If R contains $K \cap E$, then there exists $a \in C$ such that $R = R_a = \{f \in K: f(a) \neq \infty\}$.*

Proof. Since $K \cap E \subset R$, $v_R(f) \geqq 0$ for all $f \in K \cap E$. In particular, $v_R(z) \geqq 0$, and by Lemma 3.4 there exists a unique $a \in C$ such that $v_R(z - a) > 0$. Now if $f \in K \cap E$ and $f(a) = 0$, then we have that

$f = g \cdot (z - a)$, where $g \in K \cap E$, whence

$$v_R(f) = v_R(g) + v_R(z - a) \geqq v_R(z - a) > 0.$$

Thus, by Lemma 3.5, $v_R[f - f(a)] > 0$ for all $f \in K \cap E$, and $f \in K \cap E$ is a non-unit of R if and only if $f(a) = 0$. Since K is the field of quotients of $K \cap E$, we conclude that $R \supset R_a$. Conversely, if $f \in K - R_a$, then we may write $f = g/h$ with $g, h \in K \cap E$ and $g(a) \neq 0$, $h(a) = 0$. Then g is a unit of R_a, hence of R, and $v_R(f) = v_R(g) - v_R(h) = -v_R(h) < 0$, or $f \notin R$, whence $R \subset R_a$.

In order to show that a function field K is pointlike, it suffices to show that each v-ring of K contains $K \cap E$. This has been done by Iss'sa (**6**) for the field K of all functions meromorphic on C. However, in this result, use is made of the fact that if $g \in K$ and $f \in E$, then $g \circ f \in K$. This is not available for general subfields of K, where the problem becomes much more difficult, and we are able to exhibit pointlike function fields only in some special cases. In what follows we present a modified version of Iss'sa's proof which may be applied to other fields of meromorphic functions.

LEMMA 4.2. *Let m be a positive integer. Then there exists an integer $p > 1$ such that for all integers q (positive, negative, or zero), the quantity*

$$S_n = S_n(q) = q + m \sum_{k=0}^{n-1} p^k, \qquad n \geqq 1,$$

is divisible by p^n for only finitely many integers $n \geqq 1$.

Proof. First, note that if p^{n+1} divides $S_{n+1}(q)$, then p^n divides $S_n(q)$. Thus, either p^n divides $S_n(q)$ for *all* $n \geqq 1$ or for only finitely many $n \geqq 1$. Secondly, note that for $p > 1$, $S_n(q)p^{-n} \to m/(p - 1)$ as $n \to \infty$. Hence, if we initially choose $p > 1$ so that $m/(p - 1)$ is not an integer, then $S_n(q)p^{-n}$ cannot be an integer for all $n \geqq 1$, regardless of the value of q.

THEOREM 4.3. *Let K be the field of all functions meromorphic on C. Then K is pointlike.*

Proof. Evidently, K is a function field, and thus it suffices to show that each v-ring of K contains the ring E of entire functions. Therefore, let R be a v-ring of K and suppose that $f \in E - R \neq \emptyset$. Then $m = v_R(1/f) > 0$. We let p be any integer such that $m/(p - 1)$ is not an integer; thus, Lemma 4.2 applies. Construct sequences $\{R_k\}_{k=0}^{\infty}$ and $\{a_k\}_{k=0}^{\infty}$ as follows: take $R_0 > 0$ arbitrary and $a_0 = M(R_0; f)$, where $M(r; f)$ denotes the maximum modulus of f on $\{|z| = r\}$. Given $R_0, \ldots, R_n$ and $a_0, \ldots, a_n$, we choose R_{n+1} so that $R_{n+1} \geqq 2R_n$ and we choose $a_{n+1} > a_n$ so that $a_{n+1} \geqq M(R_{n+1}; f)$.

For $k \geqq 1$, define $f_k = f - a_k$ and $Z_k = \{a \in C: f_k(a) = 0\}$; thus, the Z_k are disjoint discrete subsets of C. Also, since $R_k \to +\infty$ and

$$Z_k \cap \{|z| < R_k\} = \emptyset,$$

the set $Z = \bigcup_{k \geqq 0} Z_k$ is a discrete subset of C.

Given $z \in Z_k$, let $n_k(z)$ denote the order of the zero of f_k at z, and define an integer-valued function δ on Z by

$$\delta(z) = p^k n_k(z), \qquad z \in Z_k,\ k \geqq 0.$$

Then, by the Weierstrass theorem, there exists an entire function F so that $Z = \{a \in C\colon F(a) = 0\}$ and such that $\delta(z)$ is the order of the zero of F at z for each $z \in Z$.

We may now obtain the desired contradiction. First note that for all $n \geqq 1$ the function

$$g_n = \prod_{k=0}^{n-1} [f_k]^{p^k}$$

divides F in the ring of entire functions for all $n \geqq 1$. Moreover, if $F_n = F/g_n$, then the order of each zero of F_n is divisible by p^n. Thus, by the simple connectivity of C, we conclude that F_n has a root of order p^n in K for all $n \geqq 1$, and, consequently, p^n divides $v_R(g_n)$ for all $n \geqq 1$.

However, $v_R(f) < 0$ and $v_R(\alpha) \geqq 0$ for all constants α (by Lemma 3.5). Applying (iv) of Theorem 2.4 we have that

$$v_R(f_k) = v_R(f - a_k) = v_R(f) = -m, \qquad k \geqq 0.$$

Consequently, by Theorem 2.4,

$$v_R(F_n) = v_R(F) - v_R(g_n) = v_R(F) - \sum_{k=0}^{n-1} p^k v_R(f_k) = v_R(F) + m\sum_{k=0}^{n-1} p^k.$$

In the notation of Lemma 4.2, $v_R(F_n) = S_n[v_R(F)]$, and therefore, by our choice of p, p^n cannot divide $v_R(F_n)$ for all $n \geqq 1$. This is a contradiction and we conclude that each v-ring of K contains E.

The method of this proof, which we shall apply to other function fields, is to first suppose that there is a function $f \in K \cap E$ with $m = v_R(1/f) > 0$. We then choose $p > 1$ so that Lemma 4.2 applies and construct functions $F \in K \cap E$ and $f_k \in K \cap E$, $k \geqq 0$, such that

(i) $v_R(f) = v_R(f_k)$ for all $k \geqq 0$, and

(ii) if we define

$$g_n = \prod_{k=0}^{n-1} [f_k]^{p^k} \quad \text{for } n \geqq 1,$$

then the function $F_n = F/g_n$ has a p^nth root in K for each $n \geqq 1$.

From the existence of these functions, the desired contradiction may be obtained exactly as in Theorem 4.3, and thus we may conclude that $v_R(f) \geqq 0$ for all $f \in K \cap E$ and that K is pointlike. In the remainder of this paper we shall show that this is possible for certain function fields defined in terms of growth restrictions on the Nevanlinna characteristic function, which we denote by $T(r;f)$, or on the maximum modulus, denoted by $M(r;f)$. We first consider functions of finite order.

Given an entire function f, we define $Z(f) = \{a \in C: f(a) = 0\}$, and if $a \in Z(f)$ then $m(a;f)$ will denote the order of the zero of f at a. We also define, for f not identically zero, the following:

$$n(r;f) = \sum_{\substack{a \in Z(f);\\ |a| \leqq r}} m(a;f), \qquad r \geqq 0.$$

We shall say that the entire function f is of order $\rho > 0$ if $\log M(r;f) = O(r^\rho)$ as $r \to +\infty$. The collection of all such functions is a ring under pointwise operations, and if f is entire of order $\rho > 0$, then $n(r;f) = O(r^\rho)$ as $r \to +\infty$. Given $\rho > 0$, the ring of entire functions of order *strictly less than* ρ is denoted by $E(\rho)$.

THEOREM 4.4. *Let K be a function field and suppose that $\rho > 0$ such that K contains $E(\rho)$. Then each v-ring of K contains $E(\rho)$.*

Proof. Let R be a v-ring of K and suppose that $E(\rho) - R \neq \emptyset$. Let $f \in E(\rho) - R$; thus, $m = v_R(1/f) > 0$ and there exists an integer $p > 1$ so that Lemma 4.2 applies. Since f has order less than ρ, there exists a *non-integral* constant σ such that $0 < \sigma < \rho$ and such that f has order less than σ. Therefore, $f - a$ has order less than σ for all constants $a \in C$, whence

$$n(r; f - a) = o(r^\sigma)$$

as $r \to \infty$ for all $a \in C$.

We construct sequences $\{a_k\}_{k \geqq 0}$, $\{R_k\}_{k \geqq 0}$, and $\{f_k\}_{k \geqq 0}$ as follows: first take $a_0 = 0$ and $f_0 = f$, and choose $R_0 > 0$ so that

$$n(r;f_0) \leqq r^\sigma, \qquad r \geqq R_0.$$

Given $\{a_0, \ldots, a_k\}$, $\{R_0, \ldots, R_k\}$, and $\{f_0, \ldots, f_k\}$ we define

$$a_{k+1} = 2 \max\{a_k, M(R_k;f)\}, \qquad f_{k+1} = f - a_{k+1},$$

and then choose $R_{k+1} \geqq 2R_k$ so that

$$n(r;f_{k+1}) \leqq (2p)^{-(k+1)} r^\sigma, \qquad r \geqq R_{k+1}.$$

Now let $Z_k = Z(f_k)$, so that the Z_k are disjoint discrete subsets of C. Also, since R_k increases to $+\infty$ with k and since $Z_{k+1} \cap \{|z| < R_k\} = \emptyset$ for all $k \geqq 1$, the set $Z = \bigcup_{k \geqq 0} Z_k$ is a discrete subset of C. We now define an integer-valued function δ on Z by

$$\delta(a) = p^k m(a;f_k), \qquad a \in Z_k,\ k \geqq 0,$$

and we further consider

$$n(r) = \sum_{\substack{a \in Z;\\ |a| \leqq r}} \delta(a) = \sum_{k=0}^{\infty} p^k n(r;f_k),$$

where $n(r;f_{k+1}) = 0$ for $r \leqq R_k$.

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Let $r \geqq R_0$ and let $N \geqq 0$ be such that $R_N \leqq r < R_{N+1}$. Then

$$n(r) = \sum_{k=0}^{N} p^k n(r; f_k) \leqq \sum_{k=0}^{N} p^k \left(\frac{1}{2p}\right)^k r^\sigma \leqq 2r^\sigma.$$

That is, $n(r) \leqq 2r^\sigma$ for all $r \geqq R_0$, whence $n(r) = O(r^\sigma)$ as $r \to +\infty$. Now σ is not an integer, and thus it follows from a classical theorem of Lindelöf that there exists an entire function F of order σ such that $Z(F) = Z$ and $m(a; F) = \delta(a)$ for all $a \in Z$ (see Boas (**3**) and Rubel and Taylor (**11**) for the Lindelöf theorem). In particular, $F \in K$ since $F \in E(\rho)$.

Now, since $v_R(f) < 0$, (iv) of Lemma 2.4 implies that $v_R(f_k) = v_R(f)$, $k \geqq 0$. We are now in the situation referred to in the remarks following the proof of Theorem 4.3, for it is evident that if a function $g \in E(\rho)$ satisfies the equation $g = h^n$ for some $n \geqq 1$ and some entire function h, then $h \in E(\rho)$. The desired contradiction now follows as in Theorem 4.3, and we conclude that $E(\rho) - R = \emptyset$.

This last theorem leads immediately to other examples of pointlike function fields.

COROLLARY 4.5. *Let $\rho > 0$ and let K be the field of quotients of the ring $E(\rho)$. Then K is a pointlike function field.*

COROLLARY 4.6. *Let K be the field of quotients of the ring of all entire functions of finite order. Then K is a pointlike function field.*

Theorem 4.4 can be slightly improved upon, although, to do this, it is necessary to distinguish between functions of integral and non-integral order. As the methods used are almost identical to those used in the proof of Theorem 4.4, we shall omit the proof and simply state the result.

THEOREM 4.7. *Let K be a function field and suppose that $\rho > 0$ and that K contains the ring of all entire functions of order ρ. Let f be entire with $\log M(r; f) = o(r^\rho)$ as $r \to +\infty$. Then f belongs to every v-ring of K.*

Note that if the "o" of this theorem could be replaced by "O", then we could conclude that the field of quotients of the ring of all entire functions of order ρ is a pointlike function field. However, we have not been able to do this with these methods. Similarly, we cannot show that the field of quotients of the ring of all entire functions f of order ρ, minimal type, is pointlike. Roughly speaking, one can show that an entire function $f \in K \cap E$ belongs to each v-ring of K if there exist other functions in $K \cap E$ which grow much faster than f, and the difficulty is in showing that f belongs to each v-ring of K when the rate of growth of f is maximal in K.

We finally consider function fields which are defined by growth restrictions on the Nevanlinna characteristic function. The results of Nevanlinna theory will be taken for granted here, and the reader is referred to Hayman (**4**) or Nevanlinna (**8**) for a detailed discussion of these results.

Definition 4.8. A *growth function* is a non-decreasing continuous function λ defined on $[0, \infty)$ such that $\lambda(x) \geqq 1$ for all $x \geqq 0$ and $\lambda(x) \to \infty$ as $x \to \infty$. Given a growth function λ, we denote by $K(\lambda)$ the collection of all functions f meromorphic on C for which there exist constants $A = A(f) > 0$ and $B = B(f) > 0$ such that $T(r; f) \leqq A\lambda(Br)$ for all $r \geqq 0$.

Such classes of meromorphic functions have been considered previously by Rubel (**10**) and by Rubel and Taylor (**11**), who have shown that $K(\lambda)$ is a field. In the following, we shall consider only growth functions satisfying the following conditions:

(i) $x \to \log \lambda(e^x)$ is convex on $(-\infty, \infty)$,

(ii) $\lambda(r)/r^k \to +\infty$ for all $k \geqq 0$. Evidently, $K(\lambda)$ contains all entire functions of finite order if (ii) is valid, and it has been shown that under these conditions, $K(\lambda)$ is a function field (**11**). We shall also need the following two results; see (**11**) for details. In the following, λ will always denote a growth function satisfying the conditions above.

THEOREM 4.9. *For $k \geqq 0$ define the quantities*

$$\alpha_k = \inf\{\lambda(x)/x^k \colon x > 0\},$$

$$X_k = \sup\{x > 0 \colon \lambda(x)/x^k = \alpha_k\}.$$

Then $\{X_k\}_{k \geqq 0}$ is a strictly increasing sequence of positive numbers, $X_k \to +\infty$ as $k \to +\infty$, and $x \to \lambda(x)/x^k$ is non-decreasing on $[X_k, +\infty)$.

THEOREM 4.10. *Let $f \in K(\lambda)$ be entire. Then there exist constants $A > 0$ and $B > 0$ and a family $\{g_x \colon x \geqq 0\}$ of entire functions such that for all $x \geqq 0$,*

(i) $Z(g_x) = Z(f) \cap \{|z| \leqq x\}$,

(ii) *f/g_x is analytic on $\{|z| \leqq x\}$,*

(iii) *$f/g_x \to 1$ uniformly on compacta as $x \to +\infty$,*

(iv) *$T(r; h) \leqq A\lambda(Br)$ for all $r \geqq 0$, where h is any of the functions f, g_x, f/g_x, $x \geqq 0$. (In particular, $g_x \in K(\lambda) \cap E$ and $f/g_x \in K(\lambda) \cap E$ for all $x \geqq 0$.)*

The fields $K(\lambda)$ will now be shown to be pointlike, though the proof for this case is somewhat different from the above.

THEOREM 4.11. *The function field $K(\lambda)$ is pointlike.*

Proof. Let R be a v-ring of $K(\lambda)$ and suppose that $K(\lambda) \cap E - R \neq \emptyset$. Let $f \in K(\lambda) \cap E - R$, thus $m = v_R(1/f) > 0$ and there exists $p > 1$ so that Lemma 4.2 applies. Now by Corollary 4.6, R contains the entire functions of finite order and there exists $a \in C$ with $v_R(z - a) > 0$. Furthermore, if $f(a) = 0$ and k is the order of the zero of f at a, then $(z - a)^{-k}f$ belongs to $K(\lambda) \cap E - R$ and does not vanish at a, thus we may suppose that $f(a) \neq 0$.

Let $\{X_k\}_{k \geqq 0}$ be the sequence defined in Theorem 4.9, and let $A > 0$, $B > 0$, and $\{g_x \colon x \geqq 0\}$ be the quantities referred to in Theorem 4.10. Since each function g_x has only finitely many zeros, we have that $g_x = p_x h_x$ with p_x a polynomial and h_x a non-zero entire function with no zeros in C. Now, if q is a

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polynomial, evidently, $v_R(q) > 0$ if and only if $q(a) = 0$, whereas $p_x(a) \neq 0$ since f/g_x is entire and $f(a) \neq 0$. Also, since $h_x \neq 0$ on C, it follows from the simple connectivity of C and from elementary properties of the Nevanlinna characteristic, that h_x has roots of all orders *in* $K(\lambda)$. From this it follows (as in the proof of Lemma 3.4) that $v_R(h_x) = 0$ for all $x \geqq 0$, whence $v_R(g_x) = 0$ for all $x \geqq 0$.

Now $f/g_x \to 1$ uniformly on compacta as $x \to +\infty$; thus, there is a sequence $\{R_k\}_{k\geqq 0}$ increasing to $+\infty$ such that for all $k \geqq 0$

$$\sup\{|f(z)/g_{R_k}(z) - 1| : |z| \leqq X_k\} < (2p)^{-k}.$$

For $k \geqq 0$, define $f_k = f/g_{R_k}$. Then, evidently, $v_R(f_k) = v_R(f)$ and the infinite product

$$\prod_{k=0}^{\infty} [f_k(z)]^{p^k}$$

converges uniformly on compacta to an entire function F. In view of our previous remarks, the proof will be complete if we can show that $F \in K(\lambda)$.

To see this, we first note that the function $x \to \lambda(x)x^{-k}$ is non-decreasing on $[X_k, +\infty)$; thus, if $k \geqq 0$ and $x \geqq X_k$, then $p^k\lambda(x) \leqq \lambda(px)$. Let $r \geqq X_0$ and consider $T(r; F)$. We have that

$$T(r;F) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|F(re^{i\theta})|\, d\theta$$
$$\leqq \sum_{k=0}^{\infty} \frac{p^k}{2\pi}\int_0^{2\pi} \log^+|f_k(re^{i\theta})|\, d\theta.$$

Now since $\{X_k\}_{k\geqq 0}$ increases to $+\infty$, there exists an integer $n \geqq 1$ with $X_{n-1} \leqq r < X_n$, and $T(r; F) \leqq \alpha(r) + \beta(r)$, where

$$\alpha(r) = \sum_{k=0}^{n-1} \frac{p^k}{2\pi}\int_0^{2\pi} \log^+|f_k(re^{i\theta})|\, d\theta,$$

$$\beta(r) = \sum_{k=n}^{\infty} \frac{p^k}{2\pi}\int_0^{2\pi} \log^+|f_k(re^{i\theta})|\, d\theta.$$

First, $r \geqq X_{n-1}$ and we may suppose that $B \geqq 1$ and $p^{n-1}\lambda(Br) \leqq \lambda(Bpr)$. Thus

$$\alpha(r) \leqq \sum_{k=0}^{n-1} p^k \log^+ M(r; f_k) \leqq \sum_{k=0}^{n-1} p^k A\lambda(Br)$$
$$\leqq p^n A\lambda(Br) \leqq Ap\lambda(Bpr).$$

Secondly, since $|f_k(z) - 1| < (2p)^{-k}$ for $|z| \leqq X_k$, we have that

$$\log^+|f_k(re^{i\theta})| \leqq \log[1 + (2p)^{-k}] \leqq (2p)^{-k};$$

thus $\beta(r) \leqq 2$. Hence, $T(r; F) \leqq (Ap)\cdot\lambda(Bpr) + 2$ for all $r \geqq X_0$, whence $F \in K(\lambda)$, as desired. The contradiction now follows as in the proof of

Theorem 4.3. It suffices to note that if $n \geqq 1$ and $f \in K(\lambda)$ such that $f = g^n$ for some function g meromorphic on C, then $g \in K(\lambda)$. This follows from the properties of the characteristic $T(r;f)$; thus we conclude that $K(\lambda)$ is pointlike.

Note also that the function fields considered in Corollaries 4.5 and 4.6 and Theorem 4.11 are easily seen to be saturated (as defined in Definition 3.9), thus Theorem 3.10 on field isomorphisms applies to these fields.

Finally, some of these results can be extended to fields of functions meromorphic on other plane regions. However, we have been unable to apply the methods of this last section to situations where the region is multiply connected, for in showing that a function field is pointlike, the simple connectivity of C is seen to be essential in all of the above proofs.

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