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THE MATHEMATICAL EXAMPLE OF GNOMONS IN ARISTOTLE, *PHYSICS* 3.4, 203a10–16*

ABSTRACT

This article examines a complex passage of Aristotle's Physics *in which a Pythagorean doctrine is explained by means of a mathematical example involving gnomons. The traditional interpretation of this passage (proposed by Milhaud and Burnet) has recently been challenged by Ugaglia and Acerbi, who have proposed a new one. The aim of this article is to analyse difficulties in their account and to advance a new interpretation. All attempts at interpreting the passage so far have assumed that 'gnomons' should indicate 'odd numbers'. In this article it is argued that the usage of 'gnomon' related to polygonal numbers, which is normally considered late, could be backdated to at least the fifth/fourth centuries* B.C.*; in particular, it explains the link between the philosophical* explanandum *and the mathematical* explanans *in Aristotle's passage.*

Keywords: gnomon; polygonal numbers; Aristotle's *Physics*; Pythagorean mathematics; even and odd; Philolaus

In the fourth chapter of *Physics* 3, Aristotle begins to address the topic of the unlimited. As customary, before proposing his own ideas, he provides a history of previous philosophical accounts. He begins by talking about the Pythagoreans and Plato at the same time. According to him, in both philosophies the unlimited is considered as something *per se*, that is, not as an accident of something else, but as a substance. However, there are differences between them. First, the Pythagoreans place the unlimited among sensible things and outside the heavens, while Plato—who does not place anything outside the heavens, since Forms are nowhere—places the unlimited at the level of both sensible things and Forms. Then Aristotle explains the second difference (*Ph.* 3.4, 203a10–16):

καὶ οἱ μὲν τὸ ἄπειρον εἶναι τὸ ἄρτιον (τοῦτο γὰρ ἐναπολαμβανόμενον καὶ ὑπὸ τοῦ περιττοῦ περαινόμενον παρέχειν τοῖς οὖσι τὴν ἀπειρίαν· σημεῖον δ' εἶναι τούτου τὸ συμβαῖνον ἐπὶ τῶν ἀριθμῶν· περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἓν καὶ χωρὶς ὁτὲ μὲν ἄλλο ἀεὶ γίγνεσθαι τὸ εἶδος, ὁτὲ δὲ ἕν), Πλάτων δὲ δύο τὰ ἄπειρα, τὸ μέγα καὶ τὸ μικρόν.[1]

and the former (*sc*. the Pythagoreans) [say] that the unlimited is the even (for, [they say] that it [*sc*. the even], enclosed and delimited by the odd, provides entities with unlimitedness; a sign of this is what happens in the case of numbers: for, when the gnomons are placed around the unit and separately, now the form becomes always different, now [always] one), while Plato [says] that the unlimited are two, the large and the small.

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[1] I quote the *Physics* according to W.D. Ross's edition, *Aristotle's* Physics (Oxford, 1936). All translations, except where otherwise specified, are mine.

The interpretation of this passage has been debated for centuries. The entire history of the problem has recently been rediscussed by Ugaglia and Acerbi,[2] who retrace previous attempts at interpretation, demonstrate their difficulties and propose a new one. While the *pars destruens* of the article is well argued, their new proposal has in turn some difficulties that make it hard to endorse. The purposes of these pages are mainly (1) to show these difficulties, and (2) to put forward a new attempt at interpretation. Before doing so, I summarize the previous attempts; the reader is referred to Ugaglia and Acerbi's article for further discussion and references.

I. INTRODUCING THE PROBLEM

Before outlining a history of the interpretations, it is useful to summarize the most difficult points of this text:

1) What are the gnomons?
2) What does περὶ τὸ ἓν καὶ χωρίς mean?
3) How many mathematical constructions are referred to? If they are two or more, what is the relationship between them and how are they related with the ὁτὲ μέν … ὁτὲ δέ … opposition?
4) What is the εἶδος?

The term 'gnomon' originally indicated a tool used to draw right angles, that is, a set-square. From this usage, it began to indicate by analogy the similar geometrical figure obtained by subtracting from a square a smaller inner square with a vertex coinciding, which vice versa is also what the latter lacks to become again the larger square. Therefore, in this sense the γνώμων is what can make a square *bigger* by surrounding it on two sides. Eventually, this last usage was extended to indicate anything that was added to a polygonal number to obtain a larger one of the same kind. The exact chronology of this last extension is a matter of debate.[3]

That this mathematical example is actually Pythagorean, and was not designed by Aristotle to explain the Pythagorean idea he is talking about, has always been considered unproblematic, rightly.[4] Aristotle himself highlights this by using the infinitive γίγνεσθαι, which, along with the preceding ones, is part of a group of accusative-and-infinitive clauses governed by an implied verb of speaking whose subject is the Pythagoreans. Therefore any attempt at interpretation must consider what we know about ancient Pythagoreanism, and to distinguish its theories from the appropriation (and subsequent distortion) of Pythagoreanism made almost immediately within Platonism.

[2] M. Ugaglia and F. Acerbi, 'Aristotle on placing gnomons round (*Ph*. 3.4, 203a10–15)', *CQ* 65 (2015), 587–608.

[3] On this topic, cf. T.L. Heath, *A History of Greek Mathematics*, vol. 1: *From Thales to Euclid* (Oxford, 1921), 78–9; recent bibliography summarized by F.M. Petrucci, *Teone di Smirne, Expositio rerum mathematicarum ad legendum Platonem utilium. Introduzione, traduzione, commento* (Sankt Augustin, 2012), 334 n. 135. The term is also discussed in Ugaglia and Acerbi (n. 2), 603 and n. 59; I will return to their account later.

[4] Both Diels–Kranz (fr. 58 B 28 DK, 1.459.15–19) and Laks–Most ([17] Pyths. anon. D12a, 4.290–1) include this passage among the Pythagorean doctrines not attributed by name. They do not provide an interpretation of the passage (Laks–Most note only 'a notoriously obscure passage' [291 n. 1]).

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II. ANCIENT INTERPRETATIONS

Among ancient commentaries on this passage, those by Alexander (in Simplicius), Themistius, Simplicius and Philoponus are preserved. Their main features have been discussed by Ugaglia and Acerbi, with a detailed analysis of their shortcomings and an attempt to trace their mutual relationships.[5] I will briefly recall here the main data and objections, taking them mostly from their article.

In his paraphrase,[6] Themistius says that for the Pythagoreans the even, as unlimited, is cause of division into equal parts, while the odd is cause of limit since, added to the even, it prevents the division into equals. Furthermore, he says that taking the unit and adding to it successive odd numbers one always obtains the same form, that is, square; by contrast, when even numbers are added to the unit following their order they always produce a new form, and the difference proceeds unlimitedly, since they produce a triangle, then a heptagon, then whatever appears.[7] It seems quite clear that for him only odd numbers are to be considered as gnomons: 'This is the reason why arithmeticians call odd numbers "gnomons", because the successive [odd] numbers, placed around the preceding ones, preserve the form of the square, as do the geometrical ones. In general, one learns what a gnomon is in geometry.'[8] The even numbers, by contrast, are meant to be added to the unit in a purely arithmetical way, and each polygonal number has to be considered not according to its actual plane disposition, but according to its arithmetical properties.[9] Thus, as Ugaglia and Acerbi note, Themistius understands καὶ χωρίς as opposed to the whole expression περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἕν: while the latter refers to putting gnomons around the unit, καὶ χωρίς should refer to adding even numbers to the unit, but not by putting them around it as gnomons. However, Themistius also uses χωρίς to say that odd numbers should be added each one separately:[10] therefore, he seems to understand the clause περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἓν καὶ χωρίς also as a unitary concept. This dual interpretation suffers from inconsistency, although these two views are not strictly incompatible.

Simplicius' interpretation is long and complex.[11] He says that the unlimited division does not concern numbers, but magnitudes, and questions the idea that it must be in

[5] Ugaglia and Acerbi (n. 2), 589–95; at 594 n. 28, they claim that no truly different or innovative contributions can be found in medieval, Renaissance and Arabic commentaries.

[6] Them. *in Phys.* 80.8–25 Schenkl; English translation in R.B. Todd (transl.), *Themistius: On Aristotle Physics 1–3* (London, 2012), 90–1.

[7] It seems that Themistius treated the series of squares differently from the series of polygonal numbers generated through the addition of consecutive even numbers: in the first case he gives priority to the common square form over the different polygons that each square number can represent (e.g. 9 is at the same time a square and a 9-sided polygon), while in the second case he emphasizes the diversity of forms, even when the numbers could be reduced to forms already encountered (e.g. 21 is a 21-sided polygonal, but also a triangular). Only if we assume such begging of the question does the expression ἀεί τι καινὸν εἶδος ποιοῦσι (80.22–3) make sense.

[8] Cf. 80.17–20. Todd (n. 6), 90 translates Themistius' ὥσπερ οἱ γραμμικοί 'just like geometrical points', but this can hardly be right: the correct meaning can be clarified by the parallel passage in Simplicius, who writes ὥσπερ καὶ οἱ ἐν γεωμετρίᾳ γνώμονες (457.3 Diels; cf. n. 12 below).

[9] The representation of numbers by means of geometrical forms is called σχηματογραφία (on this term, cf. Ugaglia and Acerbi [n. 2], 588 n. 4). Taking the unit and then proceeding to put the odd numbers around it, one actually obtains a series of squares. However, after having taken the unit, having put two points around it and having obtained an actual triangular shape, it is impossible to obtain an actual heptagon by putting four dots around this triangle. Hence, according to this interpretation, one must understand the addition of even numbers in a purely arithmetical way.

[10] λαβόντες γὰρ μονάδα τοὺς ἐφεξῆς αὐτῇ περισσοὺς ἐπισυντιθέασι χωρὶς ἕκαστον (80.14–15).

[11] Simpl. *in Phys.* 455.15–458.16 Diels; English translation in J.O. Urmson (transl.), *Simplicius: On Aristotle's Physics 3* (New York, 2002), 74–7.

equal parts to be related to the even, since also with unequal divisions the resulting parts are two. In his account of the mathematical example, the prefixes περι- and προσ- alternate, the former with geometrical, the latter with arithmetical meaning. As in Themistius, the gnomons indicate odd numbers.[12] In one case, odd numbers are put around the unit, always producing a square; in the second, even numbers are added to the unit, producing something that is not a square. Polygonal numbers other than squares are not brought into play in his explanation, as Themistius did. However, he seems particularly keen to emphasize that even numbers transform squares into something else. For instance, he says, adding an even like 6 to a square such as 4 produces a heteromecic (= rectangular) figure. Yet since this is not a case of continuous addition of even numbers to the unit, this must be interpreted as a fictitious example, which goes beyond Aristotle's text and is used to show the general fact that even numbers prevent conservation.

Simplicius mentions then a 'correct addition to the explanation'[13] given by Alexander: he thought that the phrase περιτιθεμένων γὰρ τῶν γνωμόνων obviously denoted the σχηματογραφία κατὰ τοὺς περιττοὺς ἀριθμούς, while the expression καὶ χωρίς referred to the arithmetical addition carried out χωρὶς περιθέσεως σχηματικῆς. Thus, χωρίς means *separately from gnomon-like addition*, hence *arithmetically*.[14] Then, Simplicius proposes a twofold interpretation, so that *both* the geometrical and the arithmetical additions apply to *both* kinds of number. It is evident that the odd numbers produce squares both when placed around the unit *as gnomons* and when added to it *arithmetically*; Simplicius, though, says that also even numbers alter the form *both* when placed around the unit like gnomons and when added to it arithmetically. Therefore, he proposes four constructions, two geometrical and two arithmetical. However, it is not entirely clear how even numbers are to be placed around the unit *like gnomons*,[15] since no closed figure can be created in this way.[16]

The same alternation of prefixes περι- and προσ- can be found in Philoponus' account.[17] However, his interpretation is quite different. He starts by saying that when the gnomons (= the odd numbers) are added to themselves starting from the unit, they always produce squares, while, when even numbers are added to the unit

[12] Cf. 457.1–8: 'And *the Pythagoreans* called the odd numbers "gnomons" because, placed around squares, they preserve the same shape, in the same way as gnomons in geometry; in fact, they call gnomons the two complements together with one of the parallelograms on the same diagonal, and this gnomon, being added to the other parallelogram on the same diagonal, makes the whole similar to the one to which it was added. Thus, also the odd numbers are called gnomons, because, when added to those that are already square, they always preserve their square shape.' I shall return to this passage later.

[13] Cf. 457.12–13 καλῶς δὲ καὶ οὕτως ἐπέβαλε τῇ ἐξηγήσει ὁ Ἀλέξανδρος, ὅτι κτλ.

[14] Alexander's interpretation is considered correct by M. Timpanaro Cardini, 'Una dottrina pitagorica nella testimonianza aristotelica', *Physis* 3 (1961), 105–12, at 109-12; M. Timpanaro Cardini, *Pitagorici. Testimonianze e frammenti* (Florence, 1964), 3.172–9.

[15] Cf. 458.5–7: 'In fact, although even [numbers] are not called gnomons in the proper sense (μὴ κυρίως γνώμονες) since they do not preserve the same form, nevertheless, if they are placed around like gnomons (ὡς γνώμονες), they make dissimilarity manifest even in figurative representation.'

[16] This interpretation was regarded as correct in E. Zeller, *Die Philosophie der Griechen in ihrer geschichtlichen Entwicklung* (Leipzig, 1892^5), 1.351–2 n. 2. In the Italian edition of this work (E. Zeller and R. Mondolfo, *La filosofia dei Greci nel suo sviluppo storico*. *Parte I: I presocratici, vol. II: Ionici e pitagorici* [Florence, 1967^2], 445–6 n. 3), this observation is followed by a lengthy note (at 446–8) in which Mondolfo highlights the weaknesses of Simplicius' interpretation and aligns himself with that of Milhaud–Burnet (cf. below).

[17] Phlp. *in Phys.*, 391.20–394.30 Vitelli. An English translation in M.J. Edwards (transl.), *Philoponus: On Aristotle Physics 3* (London, 1994), 60–3.

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or to any number, even or odd, they always produce different forms. However, this is not his full account of the mathematical example, which he explains later. In his opinion, Aristotle mentions two cases, one in which the gnomons are added to the unit and then to one another, producing squares, and another one in which the gnomons are added separately from the unit and from themselves (καὶ χωρίς), that is, to even numbers. The sum of odd numbers alone retains the same form (= square), while the addition of odd numbers to even numbers always produces different forms. The addition of even numbers alone would produce no form; therefore, the interweaving of even and odd numbers is necessary to produce something, but this something is always different. Thus, as Philoponus highlights, it is in this sense that even numbers produce unlimitedness when 'enclosed and delimited by the odd', since they must always intertwine with the odd numbers in order to generate something, and it is from this that unlimitedness arises. The odd numbers, on the other hand, need not intertwine with anything other than themselves. Hence Philoponus seems to be the only ancient commentator that tried to address the relationship between the mathematical *explanans* and this part of the *explanandum*. Still, the idea that περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἓν καὶ χωρίς refers to placing the gnomons first by themselves alone, then separately from themselves (that is, intertwining with the even numbers) seems a little arduous. This is probably the reason why Philoponus prefaces his interpretation with a remark about the obscurity of the Aristotelian text due to its conciseness.

In addition to these ancient commentators, Ugaglia and Acerbi also quote a passage transmitted by John Stobaeus[18] that can be connected to this mathematical problem. He explains that, when the odd is mixed with the even, it always succeeds in dominating (what is born of both is always odd); when it is added to itself it generates the even, while the even never generates the odd. Then he notes that placing successive odd-numbered gnomons around the unit always produces squares, while placing even numbers (or even-numbered gnomons?) in the same way produces heteromecic and unequal numbers, and none is 'equal an equal number of times' (that is, square). This text is interesting due to its ambiguity. The author does not speak about different kinds of polygonal numbers, but about *rectangular numbers*. Moreover, the phrase τῇ μονάδι τῶν ἐφεξῆς περισσῶν γνωμόνων περιτιθεμένων … τῶν δὲ ἀρτίων ὁμοίως περιτιθεμένων, without being completely clear, leaves open the possibility of even gnomons, which was to be exploited, centuries later, by Milhaud and Burnet.

III. MODERN INTERPRETATIONS

The classical interpretation of this passage, most frequently found in modern commentaries on the *Physics*[19] and in works on the Pythagoreans,[20] is the so-called

[18] Stob. *Ecl.* I, prooem. 10, 1.22.6–23 Wachsmuth (= fr. 58 B 28 DK, 1.459.19–22). The passage, attributed in the manuscripts to Moderatus of Gades, was recognized as deriving from Plutarch by H. Diels, *Doxographi Graeci* (Berlin 1879), 96–7.

[19] Cf. e.g. Ross (n. 1), 542–5; P. Pellegrin (ed.), *Aristote: Physique* (Paris, 2002^2), 173 n. 1. Bostock, in R. Waterfield (transl.) and D. Bostock (intr., comm.), *Aristotle: Physics* (Oxford, 1996), 247–8 seems to endorse this interpretation, but with significant hesitations. G. Heinemann, in *Aristoteles: Physikvorlesung*. *Teilband 1: Bücher I–IV* (Hamburg, 2021), 280–1, makes use of this as the 'üblichen Interpretation', merely mentioning the 'abweichende Interpretation' of Ugaglia and Acerbi.

[20] Cf. e.g. I. Thomas, *Greek Mathematics*, vol. I: *From Thales to Euclid* (London and Cambridge, MA, 1957), 95 n. b; W. Burkert, *Lore and Science in Ancient Pythagoreanism* (Cambridge, MA,

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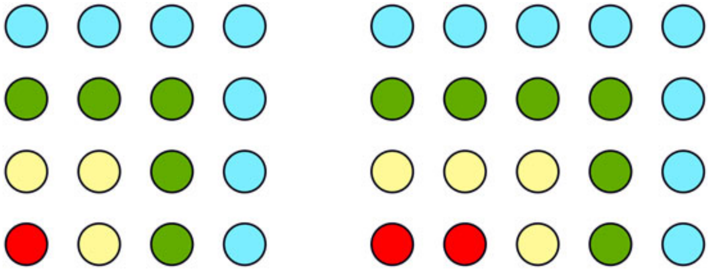

FIG. 1. The two constructions in the Milhaud–Burnet interpretation.

Milhaud–Burnet interpretation,[21] according to which Aristotle refers to two different constructions (Fig. 1).

In both cases there is a figurative representation, a σχηματογραφία. In one case, odd gnomons are placed around the unit; in the other, even gnomons are placed around the dyad. In fact, according to this interpretation, 'separately' means 'separately from the unit', which in this context should suggest 'around the dyad'. In this framework, the opposition ὁτὲ μέν … ὁτὲ δέ … is taken to refer to these two constructions, but in a chiastic way: when the odd gnomons are put around the unit the form is always the same, while when the even gnomons are put around the dyad (separately from the unit) the form always changes, since heteromecic figures with non-proportional sides (2:1, 3:2, 4:3, etc.) are produced. This chiasmus has not been considered problematic, as it is implied also by all ancient interpretations. I will recall here some of the objections to this classical interpretation raised by Ugaglia and Acerbi:[22]

1) If this were the sense of the passage, the text would be excessively implicit: one would have to understand 'separately' as 'around the dyad' and to think both of even-numbered gnomons (which is by no means obvious) and of their opposition to odd-numbered gnomons.
2) To make one of the two σχηματογραφίαι begin with the unit and the other with the dyad is more typical of Pythagorean Platonism than of ancient Pythagoreanism: while the opposition Unit/Dyad will be fundamental in later Platonism, in ancient Pythagoreanism the Unit is directly opposed to the Multiple.[23]
3) It is not clear how this mathematical example is a sign (that is, an *explanans*) of the previous statement, since in the second construction the even is in no way enclosed and delimited by the odd.
4) In the second case, the form does not actually always change, since all the resulting numbers are heteromecic, and in the Pythagorean list of principles the heteromecic is opposed to the square; therefore, it should represent a single numerical species.

1972), 33 n. 27; B. Centrone, *Introduzione ai Pitagorici* (Rome and Bari, 1999²), 129; L. Zhmud (transl. K. Windle and R. Ireland), *Pythagoras and the Early Pythagoreans* (Oxford, 2012), 282. Other general works are referred to in Ugaglia and Acerbi (n. 2), 595 n. 31 and 32.

[21] Cf. G. Milhaud, *Les philosophes-géomètres de la Grèce. Platon et ses prédécesseurs* (Paris, 1900), 113–18, especially 115–17, and J. Burnet, *Greek Philosophy. Part I. Thales to Plato* (London, 1914), 52–3 and n. 2. Milhaud's interpretation is explicitly based on a scholium (cf. 116 n. 1), whose account of this passage is similar to that of John Stobaeus: cf. Ugaglia and Acerbi (n. 2), 592 and n. 18, 595 n. 32.

[22] Cf. Ugaglia and Acerbi (n. 2), 597–8.

[23] This oddity had already been identified by Burkert (n. 20), 33 n. 27, who nevertheless endorsed the classical interpretation and attempted a lengthy justification of it.

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The first three objections seem entirely convincing. The fourth is also convincing, in so far as it refers to the list of principles (whose role in ancient Pythagoreanism, though, is much debated); sometimes in ancient mathematics, however, heteromecic numbers seem to be considered similar only if they have proportional sides[24] (for example 6=3*2 and 24=6*4); this does not happen with pairs of numbers of the type n*(n+1) and m*(m+1), like those produced in the second construction. Therefore, it could be true that, at least in this sense, the form of the rectangles always changes.[25]

An attempt to solve the problem of the dyad was made by Taylor, who explicitly took up Themistius' account.[26] He understood the opposition as 'putting *the gnomons* around the unit' vs 'putting something else around the unit'. This 'something else', opposed to odd numbers, is obviously even numbers. Using odd numbers one obtains squares, while using even numbers one obtains different regular polygons. In his interpretation, καὶ χωρίς simply means 'in the other case' or even *e contrario*. However, the question remains as to how the even numbers are to be placed around the unit: should we understand it only as an arithmetical addition? Moreover, Taylor does not address what is meant by 'enclosed and delimited by the odd'. Finally, as demonstrated above, it is not true that the regular polygons always change.[27]

Another interpretation was proposed by Vinel.[28] According to him, only one construction is implied. He highlights the emphasis on 'placing around' (περιτιθεμένων) and thinks that it is therefore necessary to understand the gnomons as placed around the unit in pairs and from opposite sides, to surround it completely. Thus, καὶ χωρίς simply indicates the act of placing the gnomons separately, that is, one pair after the other. This interpretation of χωρίς was already in Themistius' account, but without the idea of the pairs of gnomons. However, the most striking feature of this interpretation is the fact that the correlation ὁτὲ μέν … ὁτὲ δέ … is taken to indicate that in one sense the size of the square always changes, in another the numerical form (= the square) is always the same. Both size and numerical form would be indicated by the same term, εἶδος. However, according to Ugaglia and Acerbi:[29]

1) It is difficult to argue that for ancient mathematics sequences of squares only increasing in magnitude can be understood as having different forms.
2) The idea that περιτίθημι can only mean to surround completely is not true, as shown by many counterexamples.[30]

[24] Cf. the twenty-second definition of the seventh book of Euclid's *Elements* (105.3–4 Heiberg-Stamatis) and Theo Sm. 36.12–37.6 Hiller.

[25] According to Timpanaro Cardini (n. 14), 108, all these rectangles should be considered similar precisely because they are all epimoric, i.e. the numerical ratio between their sides takes the form (n+1):n. Still, as Euclid and Theon's passages quoted in the previous note show, in Greek arithmetic there is at least one sense in which rectangles with epimoric sides are not considered similar.

[26] Cf. A.E. Taylor, 'Two Pythagorean philosophemes', *CR* 40 (1926), 149–51.

[27] Cf. above, n. 7. Other difficulties are noted by Ugaglia and Acerbi (n. 2), 598.

[28] Cf. N. Vinel, *L'*In Nicomachi arithmeticam *de Jamblique. Introduction, édition critique, traduction et commentaire* (Diss., Université Blaise Pascal, Clermont-Ferrand II, 2008), 1.xl–lviii (known to me from Ugaglia and Acerbi [n. 2]). The interpretation of this passage is not included in N. Vinel (ed.), *Jamblique. In Nicomachi arithmeticam* (Pisa and Rome, 2014).

[29] Cf. Ugaglia and Acerbi (n. 2), 598–9 and n. 46. Some of these objections, along with a first sketch of what would become their new interpretation, had already been stated in M. Ugaglia (ed.), *Aristotele. Fisica. Libro III* (Rome, 2012), 131–2.

[30] Cf. Ugaglia and Acerbi (n. 2), 603 and n. 57.

A further objection that Ugaglia and Acerbi do not raise is that in this way the result concerning the size and the one concerning the geometrical form occur simultaneously, so that one is forced to understand the opposition ὁτὲ μέν… ὁτὲ δέ … as 'in one sense … in another …', rather than as 'now … now …', as would be normal. I will return to this issue in the next section, since it could be problematic for their interpretation, too.

IV. UGAGLIA AND ACERBI'S INTERPRETATION

According to Ugaglia and Acerbi, Aristotle is referring to a single construction: the odd numbers are placed around the unit (without surrounding it from opposite sides) to form the series of squares. The term εἶδος has two different meanings in the passage: 'geometrical form' and 'form or species of number' (that is, parity, the property of a number to be even or odd). Ugaglia and Acerbi convincingly argue that this latter meaning is most definitely Pythagorean (I will return to this). In their interpretation, the expression περὶ τὸ ἓν καὶ χωρίς simply means 'around the unit, but *apart from it*': as the unit is neither even nor odd, it would not be true to say that the εἶδος (= *parity*) changed in the passage from the unit to the first square (= 4). Therefore, one should look at the two different behaviours of the εἴδη only once the first gnomon is placed. According to them, the unit being neither even nor odd is a feature of Pythagorean arithmetic and therefore the remark περὶ τὸ ἓν καὶ χωρίς would make sense in this context.[31] In this framework, the correlation ὁτὲ μέν … ὁτὲ δέ … is not chronological; rather, it indicates two different points of view. Hence, the text should be read in this sense: 'when the gnomons are placed around the unit, but apart from it, in one sense the form always changes (= the parity of the square numbers is always different), in another sense the form is always the same (= the geometrical form of the number is always square)'.

Two main objections can be raised to this interpretation.

1) The term εἶδος assumes two completely different meanings at such a short distance and despite being not repeated (ὁτὲ μὲν ἄλλο ἀεὶ γίγνεσθαι τὸ εἶδος, ὁτὲ δὲ ἕν); hence, instead of being implied precisely because of its semantical equivalence to the explicit occurrence, the second implicit εἶδος should mean something very different from it. Ugaglia and Acerbi are aware of this 'linguistic trick'[32] and try to justify it by saying that 'Aristotle is fond of this argumentative strategy, and one might surmise that he presents the example both to implicitly show its inconsistency … and in admiration of its clever conception.'[33] In another work, Acerbi argues that such semantic ambiguity and tenuous connection between the *explanandum* and the *explanans* are not surprising in Aristotle.[34] I am not sure these arguments are conclusive.
2) In this interpretation, the correlation ὁτὲ μέν … ὁτὲ δέ … has no temporal value. They analyse the occurrences of this opposition, trying to show that the two correlated outcomes do not always occur in an irregular pattern.[35] Their demonstration

[31] If the interpretation of the unit as even–odd (ἀρτιοπέρισσον) were true, since 'even-odd' is an εἶδος of the number just like 'even' and 'odd' this remark would not be needed; however, this is a complex issue and I will not elaborate on it.

[32] As they themselves call it; cf. Ugaglia and Acerbi (n. 2), 607 n. 80.

[33] Ugaglia and Acerbi (n. 2), 607 n. 80.

[34] Cf. F. Acerbi (ed.), *Diofanto. De polygonis numeris* (Pisa and Rome, 2011), 39.

[35] Cf. Ugaglia and Acerbi (n. 2), 606–7, from which the following quotations are taken.

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of this point is convincing. However, in the examples they quote the correlation always describes events that do not happen simultaneously. This, however, is precisely what would happen to the two εἴδη according to them.

In fact, they start by exhibiting an example in which 'two facets of the same subject or of the same phenomenon are being singled out, irrespective of their happening according to a constant pattern', that is, *De an.* 2.7, 418b31–419a1 ἡ γὰρ αὐτὴ φύσις ὁτὲ μὲν σκότος ὁτὲ δὲ φῶς ἐστιν; however, the same nature (*sc.* the transparent) is not said to be simultaneously darkness and light from two different points of view. Then, to show that 'in some cases the two facets are the regular outcomes of a procedure', they quote (1) *Metaph.* 2.5, 1002a34–b2: ὅταν γὰρ ἅπτηται ἢ διαιρῆται τὰ σώματα, ἅμα ὁτὲ μὲν μία ἁπτομένων ὁτὲ δὲ δύο διαιρουμένων γίγνονται; however, the bodies are not said to touch and be divided, becoming by that very fact one and two, simultaneously; (2) *Cael.* 1.10, 279b12–16, where Aristotle reports an opinion on the cosmos, according to which it ἐναλλὰξ ὁτὲ μὲν οὕτως ὁτὲ δὲ ἄλλως ἔχειν, without meaning that the cosmos can be at the same time οὕτως and ἄλλως; (3) *Ph.* 4.9, 217a26–31, where the same thing becomes air from water and water from air, moving from greatness to smallness and to smallness to greatness, but without the process and its contrary happening simultaneously. In an addendum to their article,[36] they also refer to (4) *Ph.* 5.2, 225b31–3, a passage complicated by a textual problem (ὑγίειαν vs ἄγνοιαν); regardless of the chosen reading, though, Aristotle cannot refer to simultaneous alterations neither towards knowledge and health, nor, *a fortiori*, towards knowledge and ignorance. In sum, these parallel passages do not prove that the correlation ὁτὲ μέν … ὁτὲ δέ … can indicate two different facets of the same phenomenon happening simultaneously from different points of view, like the change of parity and the contemporary preservation of the square-form in their interpretation.

V. A NEW INTERPRETATION

Since the previous interpretations have been convincingly refuted by Ugaglia and Acerbi, but their new proposal does not seem completely persuasive, a new interpretation of this mathematical example is called for.

My suggestion is that the 'gnomons' indicated here do not necessarily have to be those connected with squares, and that they therefore do not need to indicate odd numbers. As we saw at the beginning, odd numbers were originally called gnomons because the form they assume when put around a square is similar to that of a set-square. Then, the usage of this term was extended in arithmetic to denote any number added to a polygonal number to form the next polygonal number with the same number of sides. In fact, all polygonal numbers are obtained from the unit by successive additions of 'gnomons', that is, of increasing quantities with constant difference. This is, for example, the construction of the first pentagonal numbers (the gnomons added each time are highlighted in different colours): Fig. 2.

This use of the term γνώμων is certainly attested at least since Nicomachus of Gerasa and Theon of Smyrna in the first/second century A.D. Nicomachus employs the term in chapters VI–XII of the second book of his *Introductio arithmetica*, dealing with

[36] M. Ugaglia and F. Acerbi, 'Aristotle on placing gnomons round (*Ph.* 3.4, 203a10–15): an addendum', *CQ* 65 (2015), 608.

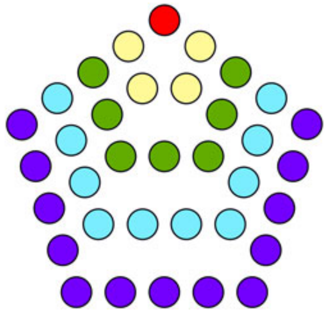

FIG 2. The first pentagonal numbers.

polygonal numbers, but does not define it.[37] Theon, instead, writes (37.11–13 Hiller): 'All successive numbers that generate triangular or square or polygonal numbers are called gnomons.'[38]

On this basis, scholars generally think that the semantic extension of the term 'gnomon' to all polygonal numbers dates back to a fairly late period. They say that it certainly developed after Euclid, since he uses this term only for squares and parallelograms;[39] cf. *Elem.* II, def. 2, 67.5–7 Heiberg–Stamatis: 'Of any surface in the shape of a parallelogram, let any one of the parallelograms constructed around its diagonal together with the two complements be called "gnomon".'[40]

This explains why all commentators understand 'gnomon' in its narrow usage. Themistius explains why the arithmeticians use 'gnomons' to indicate odd numbers, even though he brings into play other kinds of polygonal numbers when talking about the even. According to Simplicius and Philoponus, the gnomons are related to squares or at most to parallelograms (as in Euclid). Taylor says that they indicate 'the successive series of odd numbers which have to be "put round" to produce the series of squares.'[41] Ugaglia and Acerbi say that a γνώμων is, 'in the numerical case, an array of identical signs that can be placed round a number laid out as a species in order to produce another number laid out as the same species […]. In general, and by definition, a γνώμων is a numerical/geometrical shape preserving the species/form of the object to which it is applied, but we have no reasons to suppose that, to the early Pythagoreans, a γνώμων was something different from an odd number.'[42]

[37] Nicomachus sometimes provides a history of the notions he presents (as in the case of the discovery of the various proportions in *Ar.* II XXII, 122.11–123.26 Hoche), but does not mention the evolution of this term; this may indicate that he finds no ambiguity in it. Yet, in *Ar.* I IX 4, 20.17–20 Hoche, a passage that has nothing to do with polygonal numbers, Nicomachus uses the term in its *narrow* meaning, that is, to refer to odd numbers and to gnomons of squares. Thus the two usages were already clearly distinct.

[38] Thus Theon, like Nicomachus, does not problematize the evolution of this term, which seems to imply that he considered its usages already standard. On Theon's arithmetical sources, probably of Academic derivation via Moderatus, see Petrucci (n. 3), 40–1.

[39] Cf. e.g. Heath (n. 3), 79 and Timpanaro Cardini (n. 14), 112, who explicitly attribute the extension to all polygonal numbers to Heron of Alexandria.

[40] The meaning is that, given a figure such as the one in Fig. 3, in the parallelogram ABCD, cut by the diagonal BD, the sum of one of the two parallelograms with the same diagonal as ABCD, i.e. either EBFI or HIGD, and the two 'complements', i.e. AEIH and IFCG, is called a 'gnomon'.

[41] Taylor (n. 26), 150.

[42] Ugaglia and Acerbi (n. 2), 603.

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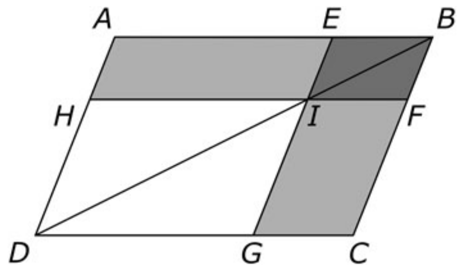

FIG 3. The gnomons in parallelograms.

It may be true that we have no cogent reasons to think that; however, it is also true that there are no particular reasons to think that it cannot be so. First, the semantic broadening of the term does not happen at the expense of the narrow meaning, which continues to coexist with the extended one. For example, Nicomachus uses both,[43] and Themistius, Simplicius and Philoponus claim that the term 'gnomon' in our passage must be understood as related only to quadrangular figures because 'gnomon' means odd, although at their time the term surely indicated also gnomons added to other polygonal numbers. This is especially evident with Philoponus, who wrote a long commentary on Nicomachus' *Introductio arithmetica*, where this extended, arithmetical usage can be found many times.[44] Hence it cannot be said that, just because only the narrow meaning is attested before Nicomachus and Theon, the extended one had not yet developed: to deny the existence of the extended meaning on the basis of the lack of attestations is an argument from silence. Furthermore, as we have already seen, Nicomachus and Theon do not describe the extended meaning as something new, but employ it as already known and widespread.

In fact, the first mathematical theory of polygonal numbers that has come down to us is due to Hypsicles (second century B.C.), at least two and a half centuries earlier than them; moreover, it is also difficult to determine to what extent the material he provides is derived from previous sources.[45] Moreover, the history of arithmetical studies on polygonal numbers goes back even further, to the fourth century B.C., since Speusippus wrote a work *On Pythagorean Numbers* (Περὶ Πυθαγορικῶν ἀριθμῶν), which is said to be based on Philolaus,[46] and the Suda attributes to Philip of Opus a work *On Polygonal Numbers* (Περὶ πολυγόνων ἀριθμῶν).[47] Clearly, it is difficult to develop a complete theory of polygonal numbers without the concept of 'gnomons', since they are necessary to construct a progressive series of polygonal numbers of the same kind; difficult, too, to think that these 'additions with constant difference' did not have a name, or that they had a different one which was later dropped and replaced by

[43] Cf. above, n. 37.

[44] On the complex genesis of this commentary, cf. F. Acerbi, 'The textual tradition of Nicomachus' *Introductio arithmetica* and of the Commentaries thereon: a thematic cross-section', *Estudios bizantinos* 8 (2020), 83–148, at 93–5.

[45] On this issue, cf. Acerbi (n. 34), 39–41. Hypsicles is also the author of the fourteenth book of the *Elements*, handed down under the name of Euclid; cf. B. Vitrac, 'Euclide', in R. Goulet (ed.), *Dictionnaire des philosophes antiques* (Paris, 2000), 3.252–72, in particular 266.

[46] Fr. 28 Tarán = 122 Isnardi Parente, preserved in [Iambl.] *Theologumena arithmeticae* (82.10–85.23 De Falco-Klein); I will return to it later.

[47] *Suda* φ 418 (4.733.33 Adler).

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'gnomon'. Hence, it does not seem far-fetched—though obviously not ascertainable—to suppose that the extended usage of 'gnomon' goes back at least to these fourth-century authors of works on polygonal numbers.

In Euclid the term 'gnomon' is defined in Book 2 and then occurs in Books 2, 6, 10 and 13, that is, only in *geometrical* books. It has no occurrences in the purely arithmetical Books 7–9. This is enough to explain why he only employs the geometrical meaning of the term, and not the arithmetical one. Moreover, as already noted by Heath,[48] the fact that the definition of gnomons in the *Elements* also includes parallelograms shows that Euclid had already accepted an extended usage of the term: since the gnomons of the parallelograms have no right angle, their similarity with set-squares, which was the cause of their name, disappears. One should note that the mathematical abstraction allowing the term 'gnomon' to be applied to parallelograms is exactly the same that allows its extension to polygonal numbers. Lastly, Euclid does indeed deal with figurate numbers in Books 7–9; however, he limits his analysis to figurate numbers representing products (that is, among plane numbers, squares and rectangles; among solid ones, cubes and parallelepipeds), omitting regular polygons and other polyhedra altogether. In this way, Euclid ensures that his figured numbers actually represent areas and volumes. By contrast, Nicomachus and Theon no longer associate figured numbers with areas and volumes, since they also deal with various other kinds of polygonal and polyhedral numbers. Therefore, since Euclid does not address polygonal numbers, although they were already known, the fact that he does not employ the extended usage of the term 'gnomon' referring to polygonal numbers comes as no surprise. Thus Euclid's silence cannot be used to prove that the extended meaning did not exist at his time. Quite the contrary: his analogically extended usage of the term with reference to parallelograms even in a strictly geometrical context already goes beyond the narrow usage.

In the Aristotelian corpus, the term γνώμων has only one other relevant occurrence,[49] *Cat.* 14, 15a29–31: 'But there are some things which, when increased, do not alter; for example, the square, after a gnomon is placed [around it], has indeed increased, but has not become anything different.'[50] Clearly, this example does not allow us to say that Aristotle did not know the extended meaning of the term. What can be said, at most, is that the example can show that gnomons have a privileged relationship to squares; however, this remains true even when the extended usage is certainly attested.

Within this framework, we can examine the fifty-seventh and fifty-eighth of Heron's *Definitions*.[51] In def. 57 we find the question 'What is a gnomon in the parallelogram?'. The provided answer is a definition very similar to the Euclidean one, that is, purely geometrical. However, def. 58 deals with what a gnomon is in the common sense (κοινῶς), and the answer explicitly includes both numbers and figures. The fact that the two meanings are explicitly and clearly distinct allows for the preservation of both together.

[48] Cf. Heath (n. 3), 78–9.

[49] Excluding two cases in the *Problemata*, whose authenticity is doubtful, the only other occurrence is found in *Hist. An.* 6, 577a18–21, where it denotes a type of mule's teeth.

[50] Philoponus explicitly refers to this passage in *in Phys*. 392.23–5.

[51] The attribution of this text is notoriously difficult. Ugaglia and Acerbi (n. 2), 603 n. 59, talk about 'pseudo-Heronian *Definitiones*'. As Acerbi writes, 'ad Erone è […] attribuita una raccolta di *Definizioni*, nel loro assetto attuale sicuramente una compilazione bizantina ma contenenti un nucleo eroniano di cui ci sono poche ragioni di dubitare, sebbene la sua identificazione sia compito improbo' (*Il silenzio delle sirene. La matematica greca antica* [Rome, 2010], 62).

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Up to now, I have shown that it is not impossible that the Pythagoreans referred to by Aristotle already knew the extended usage of the term 'gnomon'. In the following pages, I show some texts that—although much debated—could perhaps positively demonstrate that this usage was known at least to the Pythagoreans of the fifth and fourth century B.C.[52]

It is generally agreed that the main source from which Aristotle drew information about Pythagorean philosophy is Philolaus, and that whenever Aristotle attributes something to 'the Pythagoreans' he is probably referring to a *particular* Pythagorean, that is, Philolaus himself.[53] According to the *Theologumena*, Speusippus' work on figured numbers, too, was based on Philolaus. It may be useful to read at least the beginning of this testimony (fr. 28.1-9 Tarán = 122.1–11 Isnardi Parente):

Also Speusippus, the son of Potone, Plato's sister, scholarch of the Academy before Xenocrates, having composed an accurate booklet based on the teachings of the Pythagoreans, that he had always carefully studied, *and especially on the writings of Philolaus*, entitled it *On Pythagorean Numbers*; from the beginning to the middle he discusses with the greatest precision those among them that are linear and *polygonal* and all kinds of those among numbers that are plane and at the same time solid, etc.

This text has been the subject of bitter debate. Among the many issues raised by it, the one most relevant to the present discussion concerns Speusippus' sources, and in particular the precise relationship between his work, the written συγγράμματα attributed to Philolaus and the presumably oral ἀκροάσεις of some unspecified Pythagoreans.[54] We do not know when the connection between Speusippus' work and Philolaus was made, and the very existence of an arithmetical book by Philolaus has been questioned.[55] However, if Speusippus' account on polygonal numbers derives from Philolaus and/or other Pythagorean sources—irrespective of whether an actual book by Philolaus ever existed[56]—, then it is likely that at least Philolaus and/or these other Pythagoreans employed such numbers, and it is also possible that they used the

[52] This possibility is also compatible with accounts of the history of Pythagoreanism such as that of R. Netz, 'The problem of Pythagorean mathematics', in C.A. Huffman (ed.), *A History of Pythagoreanism* (Cambridge, 2014), 167–84, according to which Archytas was the first Pythagorean to be also a mathematician, while Philolaus was not a mathematician but simply a philosopher who 'did however pay much more attention than his predecessors to those fields that would emerge, ultimately, as "mathematics"' (172).

[53] Cf. Centrone (n. 20), 117–30 and O. Primavesi, 'Aristotle on the "so-called Pythagoreans": from lore to principles', in C.A. Huffman (ed.), *A History of Pythagoreanism* (Cambridge, 2014), 227–49.

[54] These and many other issues are discussed in detail by L. Tarán, *Speusippus of Athens. A Critical Study with a Collection of the Related Texts and Commentary* (Leiden, 1981), 257–98 (in particular 259–65) and M. Isnardi Parente, *Speusippo. Frammenti. Edizione, traduzione e commento* (Naples, 1980), 368–77.

[55] Burkert (n. 20), 246 and n. 40. Cf. C.A. Huffman, *Philolaus of Croton, Pythagorean and Presocratic* (Cambridge, 1993), 361–3, at 363: 'it is tempting to assign all references to Philolaus' number theory to this spurious work, which then is seen by the later tradition as the source of Speusippus' book.' On the other hand, the authenticity of Philolaus' work and its connection to Speusippus' one has been recently endorsed by Zhmud (n. 20), 409–10.

[56] Cf. J. Dillon, 'Pythagoreanism in the Academic Tradition', in C.A. Huffman (ed.), *A History of Pythagoreanism* (Cambridge, 2014), 250–73, at 251–2. Dillon writes (n. 2): 'Speusippus, like Nicomachus after him, regards triangular and pentagonal numbers as plane numbers […]. In this, Speusippus is probably nearer to the mathematics of such figures as Archytas and Philolaus, but we cannot be sure.'

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term 'gnomon' in this sense, since it would have been difficult to develop a theory of polygonal numbers without the concept of 'gnomon'.[57]

Another interesting testimony can be found in *Metaph*. 14, 1092b9–13. Aristotle discusses the theory according to which numbers are causes of substances and being, and explains that it is not clear in which sense it could be so. The first possible meaning is that numbers are causes as ὅροι:[58]

> like the points of the magnitudes, and in the manner in which Eurytus established which was the number of which thing, for example, this here of man and this here of horse, imitating with pebbles the forms of the living beings in the manner of those who bring the numbers into the shapes of triangle and square.

I will not deal here with the complex issue of the interpretation of this passage and of Eurytus' practice.[59] It will be sufficient to note that Eurytus, most likely a disciple of Philolaus,[60] represented in some way non-geometrical figures, and that, in doing so, he was simply expanding the already widespread practice of representing geometrical figures using a certain number of pebbles. The phrasing εἰς τὰ σχήματα τρίγωνον καὶ τετράγωνον can be understood as putting forward a pair of examples, implying that also other polygonal figures were representable, just like man and horse are just examples of living beings.

As we have seen, the investigation of polygonal numbers must date back at least to the 4th century B.C., that is, to Speusippus and Philip of Opus. Since gnomons are necessary to construct series of polygonal numbers, it is reasonable to think that the extended usage of this term, too, could date back at least to this period. All later occurrences of the narrow usage do not invalidate this hypothesis because the two usages coexist. Moreover, it seems plausible to trace the extended usage back specifically to fifth- and fourth-century Pythagoreans.[61] Under this hypothesis, I will now explore the possibility that the Aristotelian expression περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἓν καὶ χωρίς refers to the act of putting gnomons around the unit *to produce the various polygonal numbers*, and not just square numbers.

The fact that the ancient commentators did not consider this possibility is no argument against an attempt at doing so, since it cannot be assumed that they did not

[57] The only attestation of γνώμων in a text related to early Pythagoreans is in a fragment that John Stobaeus attributes to Philolaus (Stob. *Ecl*. I, prooem., coroll. 3, 1.17.4–12 Wachsmuth = Philolaus, fr. 44 B 11 DK, 1.411.14–412.3 = Laks–Most, [18] Pyths. rec. R48, 4.414–7). However, Huffman (n. 55), 349–50 has shown that the fragment is spurious through an analysis of its language and content. Ugaglia and Acerbi (n. 2), 603 n. 59, state that this text has a lexical affinity with our Aristotelian passage, which should further prove its inauthenticity. However, the philosophical content of the two texts seems too different for the fragment to be securely related to Aristotle's passage; the lexical affinity is limited to the presence of γνώμων, ἄπειρον and χωρίς, the latter not referring to an act involving gnomons. Hence the lack of context and the obscurity of the text make it unusable for our purposes, and the lexical affinities with Aristotle's passage may be only superficial (I owe to discussion with Concetta Luna the main points of what precedes).

[58] The meaning of the term in this context is discussed by Netz (n. 52), 174 n. 26.

[59] A similar account of Eurytus' practice can be found in Theophrastus, *Metaph*. 6a14–22. For a history of the interpretations of these passages, cf. Netz (n. 52), 173–8. While proposing a completely new interpretation of Eurytus' practice involving the abacus, Netz agrees that 'triangle' and 'square' must refer to 'numbers … made into figures' (at 176 n. 29).

[60] Cf. B. Centrone, 'Eurytos de Tarente', in Goulet (n. 45), 353.

[61] However, even if one does not accept this possibility, what really matters here is that the term could have been used in this sense at least in the Academy and in Aristotle: if so, we will attribute to these late Pythagoreans the arithmetical procedure without the usage of 'gnomon' to denominate it.

do that because of some reliable source explicitly showing them that the Pythagoreans did not use the term in this way. Instead, when commenting on this passage, they associated gnomons with odd numbers and squares probably under the influence of *Cat.* 14, 15a29–31 (mentioned above), which is in fact explicitly quoted by Philoponus while commenting on our passage.[62] Simplicius explicitly says that the Pythagoreans called the odd numbers 'gnomons',[63] but this does not invalidate our hypothesis, since (1) any usage of the narrow meaning does not automatically imply the absence of the extended one, and (2) this is more likely an inference he made than a historical information drawn from earlier sources.

When constructing polygonal numbers, the constant difference between consecutive gnomons coincides with the number of sides of the polygon decreased by two. Therefore, when forming polygonal numbers with an even number of sides (such as squares or hexagons), the gnomons have a constant even difference and thus, starting from the unit (which is not a gnomon), they retain parity and are always odd. Conversely, when forming polygonal numbers with an odd number of sides (such as triangles or pentagons), the gnomons have an odd difference too; hence, their resulting parity always changes, that is, the gnomons are alternately even and odd. Thus, in the case of an even number of sides, since the gnomons are all odd, the parity of the polygonal numbers obtained by adding these gnomons changes constantly (O, E, O, E, etc.); conversely, in the case of an odd number of sides, since the gnomons are alternately even and odd, the parity of the polygonal numbers changes every two (O, O, E, E, O, O, E, E, etc.). In no cases the parity of the polygonal numbers remains the same. Yet, if one compares the parities of the gnomons themselves, the two cases resemble precisely those described in Aristotle's passage, that is, in one case the parity is always different, in the other the parity is always the same. For example, in the construction of squares (1, 4, 9, 16, 25, 36, etc.) the gnomons are 3, 5, 7, 9, 11, etc., that is, all gnomons are odd (constant difference = 2, the number of the sides of the square minus two), while in the construction of pentagons (1, 5, 12, 22, 35, 51, etc.) the gnomons are 4, 7, 10, 13, 16, etc., that is, they are alternately even and odd (constant difference = 3, the number of the sides of the pentagon minus two).

Therefore, according to this interpretation:

1) περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἓν καὶ χωρίς means 'placing the gnomons around the unit and separately', that is, in separate constructions producing the various kinds of polygonal numbers: the series of triangles, squares, pentagons, etc. Once one starts placing gnomons around the unit to form, say, triangles, it is not possible to place the gnomons needed to form, say, pentagons around the same unit. This will have to be done through another construction, that is, separately and around a different unit.
2) The term εἶδος is no longer ambiguous. In both cases it means 'form of number', that is, parity. As already shown by Ugaglia and Acerbi, this is 'the only meaning of εἶδος attested in a fragment that can directly be ascribed with some certainty to the early Pythagoreans.'[64] They refer to a fragment by Philolaus (fr. 44 B 5 DK, 1.408.7–10 = Laks–Most, [12] Philol. D9, 4.160–1 = Stob. *Ecl.* I, 21.7c, 1.188.9–12 Wachsmuth):[65]

[62] Cf. above, n. 50.
[63] Cf. above, n. 12.
[64] Ugaglia and Acerbi (n. 2), 605. This exegetical possibility, even though within the framework of Milhaud and Burnet's interpretation, was first suggested by R. Besnier, 'Le rôle des nombres figurés dans la cosmologie pythagoricienne, d'après Aristote', *RPhilos* 183 (1993), 301–54, at 341.
[65] Cf. Huffman (n. 55), 177–93, discussing the connection between this fragment and our passage from the *Physics* at 180. However, Huffman adopts the Milhaud–Burnet interpretation (185).

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> So, the number has two proper forms (δύο … ἴδια εἴδεα), odd and even, and a third [form] from both mixed together, the even-odd. Of each of the two forms there are many shapes, of which each thing itself gives signs.[66]

Moreover, understanding the term εἶδος only in its arithmetical usage, and not in the geometrical one,[67] is consistent with the fact that in Aristotle's passage the mathematical *explanans* is introduced with the expression σημεῖον δ' εἶναι τούτου τὸ συμβαῖνον ἐπὶ τῶν ἀριθμῶν.[68]

3) The correlation ὁτὲ μέν … ὁτὲ δέ … refers to non-contemporaneous results of one process: when gnomons are placed around the unit and in separate constructions, to form the various kinds of polygonal numbers, *sometimes* the form (= parity) always changes, *sometimes* it is always the same. In some cases, the gnomons are alternately even and odd, in other cases only odd. ὁτὲ μέν … ὁτὲ δέ … is neither chiastic, as was considered by the ancient commentators and in Milhaud and Burnet's interpretation,[69] nor does it refer to any regular pattern, since there is no need to construct the squares after the triangles and the pentagons after the squares: one can construct the polygonal numbers in any order, and yet see that sometimes the parity of the gnomons changes, and sometimes not.[70]
4) The relation between the *explanandum* and the *explanans* seems now clear in all its parts. The mathematical example shows that the even can bestow unlimitedness–in this case, the continuous change of parity (continuous change is a form of unlimitedness)[71]–**only** when it is enclosed and delimited by the odd. The even could not produce anything on its own.[72] On the other hand, the odd on its own produces limitation and stability. Only when the odd and the even are together, the latter, enclosed and delimited by the odd, which alternates with it, generates unlimitedness, that is, continuous change. One might wonder why in this alternation the even is not also said to enclose and delimit the odd. However, the odd acts properly, as a producer of limitation, only when it is alone; when joined with the even, it no longer produces limitation, but helps the even produce unlimitedness (since the even could not do it alone). Hence, it would be meaningless to say that the odd does something when enclosed and delimited

[66] The meaning of the last sentence is particularly complex due to textual issues: cf. Huffman (n. 55), 192–3.

[67] As already shown, all other interpretations understand εἶδος only in a geometrical sense, while Ugaglia and Acerbi interpret the passage as employing both the arithmetical and the geometrical meaning at the same time.

[68] This point was made to me by Stefano Demichelis.

[69] Timpanaro Cardini (n. 14), 110–11 attempted to explain this alleged chiasmus by imagining that Aristotle had written the first (geometrical) series and then the second (arithmetical) one on a blackboard-like surface during a lecture, and that it was then 'naturale e spontaneo […] partire proprio dalla seconda, che aveva appena terminato di scrivere, per risalire alla prima, la cui perfezione, rappresentata dal quadrato, risultava, dal contrasto, più evidente'.

[70] This addresses an objection raised by Ugaglia (n. 29), 132, and then by Ugaglia and Acerbi (n. 2), 606, who write that 'independently of the interpretation adopted for the mathematical example, the use of this correlative does not seem fully justified'.

[71] As recognized by Ugaglia and Acerbi (n. 2), 604 and nn. 64–6, ἄλλο ἀεί refers to a recurrent pattern of alternation, and it is used properly even in the case of only two alternatives: 'it is enough that the output of any step be different from the output of the immediately preceding step; it is not required that *all* outputs be different' (604).

[72] Contrary to what was assumed in the Milhaud–Burnet interpretation, in which even gnomons were placed around the dyad, which is also even. The idea that nothing can be produced from the unlimited alone was clearly Pythagorean; cf. Timpanaro Cardini (n. 14), 108 and n. 8.

by the even. Moreover, since the odd is, in itself, limiting, whereas the even produces the lack of limit, it would be absurd to say that the even, which has no limiting power, encloses and limits the odd, which is the source of limitation. This interpretation of the *explanandum* echoes that of Philoponus (*in Phys.* 394.1–4 Vitelli):

> the even, being unlimited, when it is interwoven together with the odd, becomes a cause of unlimitedness even for the things that are generated from them [τοῖς ἐξ αὐτῶν γινομένοις, that is, from even and odd together], while the odd, since it is limiting and limit,[73] becomes a cause of definition and identity for what is generated from it [τοῖς ἐξ αὐτοῦ, that is, from the odd alone].

One last point remains. The verb ἐναπολαμβάνω, employed in the *explanandum*, is peculiar due to its double prefix. It has many occurrences, also in non-technical contexts. This is the only mathematical occurrence in the Aristotelian corpus. The use of this term by Iamblichus, though late, may indicate that he witnesses what the technical use of this term was in mathematical contexts. In fact, in his extensive writings on mathematics and philosophy of mathematics, he used this verb only on two occasions, both in his *Introduction to Nicomachus' Arithmetic*:[74]

1) Iamb. *in Nic.* 59.2–5 Pistelli–Klein = 128.12–14 Vinel:

> in the formation of plane figures the fourth [triangle] will begin to enclose (ἐναπολαμβάνειν) the first, the fifth the second, and so the others in succession, until in turn the seventh surrounds the first surrounding one, etc.

2) Iamb. *In Nic.* 62.10–16 Pistelli–Klein = 130.35–8 Vinel

> and also in the figurative representation of the polygonal numbers two sides will in every case remain the same while becoming longer each time, while the others beside them will be enclosed by the act of placing the gnomons around (ἐναποληφθήσονται τῇ τῶν γνωμόνων περιθέσει), changing continuously, one in the triangle, two in the square, three in the pentagon, and so on ad infinitum.

In both cases this verb refers to polygonal numbers 'enclosing' each other; in the second case, in particular, Iamblichus talks precisely about *gnomons* enclosing each other, since at every step of the construction the enclosed sides are part of the gnomon added in the previous step. This usage of ἐναπολαμβάνω by Iamblichus seems to me particularly useful to understand better the precise meaning of the verb in Aristotle's passage.

In fact, Iamblichus himself describes the property of gnomons which (if my interpretation is correct) is referred to in Aristotle's passage, that is, the alternative stability and change of their parity (*in Nic.* 60.21–61.5 Pistelli-Klein = 130.5–13 Vinel):

> If one were to set forth in successive rows the polygonal numbers beginning with the triangular ones, by putting also in front of them the following number,[75] it would appear in the diagram that

[73] πέρας depends on εἶναι as περατωτικόν and is not connected with ὁρισμοῦ (*contra* Edwards [n. 17], 62). The odd has a limiting power because it is limited. The term περατωτικός is typical of Neoplatonism. (I am grateful to Concetta Luna for these suggestions).

[74] These passages were already noted by Vinel and can be found in Ugaglia and Acerbi (n. 2), 601–2, along with an analysis of this verb. In C. Mugler, *Dictionnaire historique de la terminologie géométrique des Grecs* (Paris, 1958), 176–7, ἐναπολαμβάνειν is defined as an 'expression verbale désignant la délimitation d'une portion d'une figure par les contours extérieurs d'une autre figure' (Mugler does not quote these passages by Iamblichus).

[75] This passage has been explained by Vinel (n. 28), 238 n. 168: 'Jamblique suggère de mettre la série des nombres naturels au-dessus du tableau qu'on trouve dans Nicomaque *Ar.* II,12 (où l'unité est

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the triangles are even and odd two by two, the squares one by one, the pentagons, like the triangles, two by two, and so in general for those of the same row as them, that is, <two by two those of odd rows, one by one those of>[76] even rows. And in fact all polygonal numbers got gnomons according to a certain natural order, the triangular one odd and one even, the square only odd, the pentagonal again one and one, the hexagonal only odd, and so for all the following kinds.

VI. CONCLUSIONS

Aristotle reports a Pythagorean opinion—perhaps, but not necessarily, taken from Philolaus—according to which the even coincides with the unlimited and provides entities with unlimitedness only when it is enclosed and delimited by the odd. To substantiate the latter assertion, he refers to a mathematical example, almost certainly found in his Pythagorean source: when gnomons are placed around the unit and in separate constructions to form different series of polygonal numbers, sometimes (for example, with triangles, pentagons, heptagons) even gnomons are enclosed and delimited by odd ones, thus their form (= their parity) always changes; sometimes (for example, with squares, hexagons, octagons) only odd gnomons are to be found, thus their form is always the same.

This interpretation may have important consequences on the history of Pythagorean mathematics. If it is correct, then we must recognize that some Pythagoreans, maybe Philolaus himself, were accustomed to the construction of polygonal numbers, not only of squares and rectangles. This would greatly increase the scope and importance of mathematical practice within the philosophy of at least these late Pythagoreans. While constructing and studying polygonal numbers, they noticed that, when adding gnomons around separate units to form the various series of polygonal numbers, sometimes the gnomons were all odd, sometimes alternately even and odd, never all even. Hence, they employed this mathematical fact to exemplify the philosophical idea that the odd gives stability, while the even does not; rather, the even is so chaotic that it cannot produce anything by itself, while, when enclosed and delimited by the odd, it produces constant alteration, which is a form of unlimitedness.

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au début de chaque ligne). Sans le dire explicitement, il veut compléter les considérations de Nicomaque sur la composition des polygones à partir des polygones qui les précèdent. Cet ajout met en évidence que chaque nombre triangulaire est la somme du nombre naturel placé au-dessus et de ceux qui le précèdent.'

[76] The integration, proposed by Pistelli (who, however, left it in the apparatus) and put in the text by Vinel, is plausible and based on the parallel with 96.6–8 Pistelli–Klein = 166.16–17 Vinel.

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