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$\mathcal{Q}_p$ Spaces and Dirichlet Type Spaces

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Abstract. In this paper, we show that the Möbius invariant function space $\mathcal{Q}_p$ can be generated by variant Dirichlet type spaces $\mathcal{D}_{\mu,p}$ induced by finite positive Borel measures μ on the open unit disk. A criterion for the equality between the space $\mathcal{D}_{\mu,p}$ and the usual Dirichlet type space $\mathcal{D}_p$ is given. We obtain a sufficient condition to construct different $\mathcal{D}_{\mu,p}$ spaces and provide examples. We establish decomposition theorems for $\mathcal{D}_{\mu,p}$ spaces and prove that the non-Hilbert space $\mathcal{Q}_p$ is equal to the intersection of Hilbert spaces $\mathcal{D}_{\mu,p}$. As an application of the relation between $\mathcal{Q}_p$ and $\mathcal{D}_{\mu,p}$ spaces, we also obtain that there exist different $\mathcal{D}_{\mu,p}$ spaces; this is a trick to prove the existence without constructing examples.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. The Möbius group $\text{Aut}(\mathbb{D})$ consists of all one-to-one analytic functions that map $\mathbb{D}$ onto itself. It is well known that each $\phi \in \text{Aut}(\mathbb{D})$ has the form

$$\phi(z) = e^{i\theta}\sigma_a(z), \quad \sigma_a(z) = \frac{a-z}{1-\overline{a}z},$$

where θ is real and $a \in \mathbb{D}$. Let X be a linear space of analytic functions on $\mathbb{D}$ which is complete in a norm or seminorm $\|\cdot\|_X$. The space X is called *Möbius invariant* if for each function f in X and each element ϕ in $\text{Aut}(\mathbb{D})$, the composition function $f \circ \phi$ also belongs to X and satisfies that $\|f\circ\phi\|_X = \|f\|_X$. L. Rubel and R. Timoney [20] have shown that the maximal Möbius invariant function space is the Bloch space $\mathcal{B}$, which consists of the functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The important space $BMOA$, the set of analytic functions on $\mathbb{D}$ with boundary values of bounded mean oscillation (see [8, 12]), is also Möbius invariant. We refer to J. Arazy, S. Fisher, and J. Peetre [4] for a general exposition on Möbius invariant function spaces.

In 1995, R. Aulaskari, J. Xiao, and R. Zhao [6] introduced the Möbius invariant $\mathcal{Q}_p$ spaces, which have attracted a lot of attention in recent years. For $0 \le p < \infty$, a

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function $f \in H(\mathbb{D})$ belongs to the space $\mathcal{Q}_p$ if

$$\|f\|_{\mathcal{Q}_p}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \big(1 - |\sigma_a(z)|^2\big)^p dA(z) < \infty,$$

where $dA(z) = dxdy$ for $z = x + iy$. Clearly, $\mathcal{Q}_1 = BMOA$. The space $\mathcal{Q}_0$ is equal to the Dirichlet space $\mathcal{D}$. By [5], we see that $\mathcal{Q}_p = \mathcal{B}$ for all $1 < p < \infty$. The theory of $\mathcal{Q}_p$ spaces has been developing very well and can be considered satisfactory. There are also several ways to generalize $\mathcal{Q}_p$ spaces (*cf.* [11, 26, 28]). See J. Xiao's monographs [24, 25] for rich results of $\mathcal{Q}_p$ spaces.

There exists a method to obtain all Möbius invariant function spaces. Let $(X, \|\cdot\|_X)$ be a Banach space of analytic functions in $\mathbb{D}$ containing all constant functions. Following A. Aleman and A. Simbotin [2], we denote by $M(X)$ the Möbius invariant function space generated by X. Namely, $M(X)$ is the class of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{M(X)} = \sup_{\phi\in \mathrm{Aut}(\mathbb{D})} \|f \circ \phi - f(\phi(0))\|_X < \infty.$$

This construction gives rise to all Möbius invariant Banach spaces on the open unit disk. To understand the Möbius invariant function spaces $BMOA$ and $\mathcal{B}$ well, we recall the classical Hardy spaces and Bergman spaces. For $0 < p < \infty$, H^p denotes the classical Hardy space of functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The Bergman space A^p consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

It is well known (*cf.* [7, 8]) that $BMOA = M(H^p)$ and $\mathcal{B} = M(A^p)$ for all $1 < p < \infty$. Note that if $p \neq q$, then $H^p \neq H^q$ and $A^p \neq A^q$. In other words, both $BMOA$ and $\mathcal{B}$ can be generated by different analytic function spaces. Up to now, it is only known that the space $\mathcal{Q}_p$, $0 < p < 1$, can be generated by the usual Dirichlet type space $\mathcal{D}_p$, which is the class of functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{D}_p}^2 = \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^p dA(z) < \infty.$$

It is natural to ask whether $\mathcal{Q}_p$, $0 < p < 1$, can be generated by different analytic function spaces. A positive answer will be given in this paper. We will denote by $\mathbb{F}$ the set of finite positive Borel measures on $\mathbb{D}$. Let $0 < p < \infty$ and let $\mu \in \mathbb{F}$. We introduce the Dirichlet type space $\mathcal{D}_{\mu,p}$ consisting of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_{\mu,p}}^2 = \int_{\mathbb{D}} |f'(z)|^2 U_{\mu,p}(z) dA(z) < \infty,$$

where

$$U_{\mu,p}(z) = \int_{\mathbb{D}} (1 - |\sigma_z(w)|^2)^p d\mu(w).$$

We will prove that $\mathcal{D}_{\mu,p} \subseteq \mathcal{D}_p$ for any $\mu \in \mathbb{F}$. Combining this with a similar proof in the book [10, Theorem 1.6.3], we see that $\mathcal{D}_{\mu,p}$ is a Hilbert space with respect to the norm $|f(0)|^2 + \|f\|_{\mathcal{D}_{\mu,p}}$. We will show that the space $\mathcal{Q}_p$ can be generated by variant Dirichlet type spaces $\mathcal{D}_{\mu,p}$.

The paper is organized as follows. In Section 2, we prove that $\mathcal{D}_{\mu,p} \subseteq \mathcal{D}_p$ for any $\mu \in \mathbb{F}$. We characterize the measures $\mu \in \mathbb{F}$ for which the equality $\mathcal{D}_{\mu,p} = \mathcal{D}_p$ holds. We also obtain a sufficient condition to construct different $\mathcal{D}_{\mu,p}$ spaces. Some examples of different $\mathcal{D}_{\mu,p}$ spaces are given. In Section 3, we prove decomposition theorems for $\mathcal{D}_{\mu,p}$ spaces. In Section 4, we give connections between $\mathcal{D}_{\mu,p}$ and $\mathcal{Q}_p$ spaces that are new even on $BMOA$ and the Bloch space. We show that $\mathcal{Q}_p = M(\mathcal{D}_{\mu,p})$, $0 < p < \infty$, for any $\mu \in \mathbb{F}$. Consequently, the space $\mathcal{Q}_p$ can be generated by different analytic function spaces. We also prove that $\mathcal{Q}_p = \bigcap_{\mu \in \mathbb{F}} \mathcal{D}_{\mu,p}$. In other words, the non-Hilbert space $\mathcal{Q}_p$, $0 < p < \infty$, is equal to the intersection of a family of Hilbert spaces. Applying the relation between $\mathcal{Q}_p$ and $\mathcal{D}_{\mu,p}$ spaces, we also obtain that there exist different $\mathcal{D}_{\mu,p}$ spaces. It is our hope that the theory of $\mathcal{Q}_p$ spaces can be developed further in terms of the investigation of $\mathcal{D}_{\mu,p}$ spaces.

Throughout this paper, we will write $a \lesssim b$ if there exists a constant C such that $a \le Cb$. Also, the symbol $a \approx b$ means that $a \lesssim b \lesssim a$.

2 Properties of Dirichlet Type Spaces $\mathcal{D}_{\mu,p}$

In this section, we consider the relation between $\mathcal{D}_{\mu,p}$ and $\mathcal{D}_p$ spaces and provide a method to construct different $\mathcal{D}_{\mu,p}$ spaces. Some examples of $\mathcal{D}_{\mu,p}$ spaces are also given.

Theorem 2.1 *Let $\mu \in \mathbb{F}$ and $0 < p < \infty$. Then the space $\mathcal{D}_{\mu,p}$ is always a subset of $\mathcal{D}_p$. Furthermore, $\mathcal{D}_{\mu,p} = \mathcal{D}_p$ if and only if*

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |w|^2}{|1 - \bar{z}w|^2} \right)^p d\mu(w) < \infty. \tag{2.1}$$

Proof Fix $0 < r < 1$ and let $\mu_r = \mu \chi_{r\mathbb{D}}$. Here, χ is the characteristic function and

$$r\mathbb{D} = \{z \in \mathbb{C} : |z| \le r\}.$$

Note that

$$U_{\mu_r,p}(z) = (1 - |z|^2)^p \int_{r\mathbb{D}} \frac{(1 - |w|^2)^p}{|1 - \bar{z}w|^{2p}} d\mu(w) \tag{2.2}$$

and

$$\frac{(1 - r^2)^p \mu(r\mathbb{D})}{2^{2p}} \le \int_{r\mathbb{D}} \frac{(1 - |w|^2)^p}{|1 - \bar{z}w|^{2p}} d\mu(w) \le \frac{2^p \mu(r\mathbb{D})}{(1 - r)^p}.$$

Consequently, $g \in \mathcal{D}_p$ if and only if $g \in \mathcal{D}_{\mu_r,p}$. Clearly, for any $f \in \mathcal{D}_{\mu,p}$, one gets that

$$\int_{\mathbb{D}} |f'(z)|^2 U_{\mu_r,p}(z) dA(z) \le \int_{\mathbb{D}} |f'(z)|^2 U_{\mu,p}(z) dA(z).$$

Thus, $\mathcal{D}_{\mu,p}$ is always a subset of $\mathcal{D}_p$.

Let (2.1) hold. It follows from equality (2.2) that $\mathcal{D}_p \subseteq \mathcal{D}_{\mu,p}$. Hence, $\mathcal{D}_{\mu,p} = \mathcal{D}_p$. On the other hand, let $\mathcal{D}_{\mu,p} = \mathcal{D}_p$. The closed graph theorem yields that the identity map from one of these spaces into the other is continuous. Thus, there exists a positive constant C such that

$$|f(0)| + \|f\|_{\mathcal{D}_{\mu,p}} \le C \left(|f(0)| + \|f\|_{\mathcal{D}_p} \right) \tag{2.3}$$

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for all $f \in \mathcal{D}_p$. For $a \in \mathbb{D}$, set

$$f_a(z) = (1-|a|^2)^{1+\frac{p}{2}} \int_0^z \frac{d\zeta}{(1-\overline{a}\zeta)^{2+p}}, \quad z \in \mathbb{D}.$$

By a similar calculation in [16, p. 684], $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{D}_p} < \infty$ for $0 < p < \infty$. Combining this with (2.3) gives that $\sup_{a\in\mathbb{D}} \|f_a\|^2_{\mathcal{D}_{\mu,p}} < \infty$. Namely,

$$\text{(2.4)} \quad \sup_{a\in\mathbb{D}} (1-|a|^2)^{2+p} \int_{\mathbb{D}} (1-|w|^2)^p \Big(\int_{\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\overline{z}w|^{2p}|1-\overline{a}z|^{4+2p}} dA(z) \Big) d\mu(w) < \infty.$$

Let $E(a) = \{z \in \mathbb{D} : |\sigma_a(z)| < 1/2\}$ be a pseudo-hyperbolic disk centered at a. It is well known that

$$1-|a| \approx |1-\overline{z}a| \approx 1-|z|$$

for all $z \in E(a)$, and the area of $E(a)$ is comparable with $(1-|a|)^2$. Furthermore, by [29, Lemma 4.30],

$$|1-\overline{w}z| \approx |1-\overline{w}a|$$

for all $z \in E(a)$ and $w \in \mathbb{D}$. Consequently,

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\overline{z}w|^{2p}|1-\overline{a}z|^{4+2p}} dA(z) &\geq \int_{E(a)} \frac{(1-|z|^2)^p}{|1-\overline{z}w|^{2p}|1-\overline{a}z|^{4+2p}} dA(z) \\ &\approx \frac{1}{|1-\overline{a}w|^{2p}(1-|a|)^{2+p}}.\end{aligned}$$

This, together with (2.4) shows that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|w|^2}{|1-\overline{a}w|^2} \Big)^p d\mu(w) < \infty.$$

Thus, condition (2.1) holds. The proof is complete. ∎

Remark (i) For $0 < p < \infty$, it is well known that

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p dA(z) \approx \int_{\mathbb{D}} |f'(z)|^2 \Big(\log \frac{1}{|z|}\Big)^p dA(z)$$

for all $f \in \mathcal{D}_p$. Replacing f by $f \circ \sigma_w$, $w \in \mathbb{D}$, in the above formula, making the change of variables and using the Fubini theorem, one gets that $f \in \mathcal{D}_{\mu,p}$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^2 \Big(\int_{\mathbb{D}} \Big(\log \Big| \frac{1-\overline{w}z}{z-w} \Big| \Big)^p d\mu(w) \Big) dA(z) < \infty.$$

Thus, the space $\mathcal{D}_{\mu,1}$ is a Dirichlet type space with superharmonic weight studied by A. Aleman [1]. A result similar to Theorem 2.1 with $p = 1$ was obtained by A. Aleman [1], but the proof of Theorem 2.1 given here is different. We refer to [9] for the recent theory of $\mathcal{D}_{\mu,1}$. It is worth mentioning that, except Theorem 2.1, our results on $\mathcal{D}_{\mu,p}$ and $\mathcal{Q}_p$ spaces in this paper are new for all the range of p considered in the paper.

(ii) Let δ_a be a unit point mass measure at $a \in \mathbb{D}$. For $0 < p < 1$, $U_{\delta_a,p}(z) = (1-|\sigma_a(z)|^2)^p$ is a positive superharmonic function with zero boundary values on the unit disk. From the Riesz decomposition theorem for superharmonic functions,

$$U_{\delta_a,p}(z) = \int_{\mathbb{D}} \log \Big| \frac{1-\overline{w}z}{z-w} \Big| d\nu_a(w),$$

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where $d\nu_a(w) = -\Delta U_{\delta_a,p}(w)dA(w)$. However, $\int_{\mathbb{D}} -\Delta U_{\delta_a,p}(w)dA(w) = \infty$, so $\nu_a \notin \mathbb{F}$. In fact, for $0 < p \le 1$, $U_{\mu,p}$ is a superharmonic function. For $p > 1$, $U_{\mu,p}$ is not a superharmonic function and the space $\mathcal{D}_{\mu,p}$ is not of the Dirichlet type spaces studied in [9].

In light of the study of inclusion relation between a class of Möbius invariant spaces $\mathcal{Q}_K$ (see [11, Theorem 2.6]), we give a method to find different $\mathcal{D}_{\mu,p}$ spaces as follows.

Theorem 2.2 *Let $\mu, \nu \in \mathbb{F}$ and $0 < p < \infty$. If*

$$\lim_{|z|\to 1} \frac{U_{\mu,p}(z)}{U_{\nu,p}(z)} = 0 \quad \textit{and} \tag{2.5}$$

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|w|^2}{|1-\bar{z}w|^2} \Big)^p d\nu(w) = \infty, \tag{2.6}$$

then $\mathcal{D}_{\nu,p} \subsetneqq \mathcal{D}_{\mu,p}$.

Proof By (2.5), we see that $\mathcal{D}_{\nu,p} \subseteq \mathcal{D}_{\mu,p}$. Suppose that $\mathcal{D}_{\nu,p} = \mathcal{D}_{\mu,p}$. Denote by $\mathcal{D}^0_{\nu,p}$ the Banach space of functions $g \in \mathcal{D}_{\nu,p}$ with $g(0) = 0$. Then $\mathcal{D}^0_{\nu,p} = \mathcal{D}^0_{\mu,p}$. The closed graph theorem gives that there exists a positive constant C such that

$$\|f\|^2_{\mathcal{D}_{\nu,p}} \le C\|f\|^2_{\mathcal{D}_{\mu,p}} \tag{2.7}$$

for all $f \in \mathcal{D}^0_{\nu,p}$. From condition (2.5), there exists a constant $t \in (0,1)$ satisfying

$$U_{\mu,p}(z) \le \frac{U_{\nu,p}(z)}{2C}$$

for $t < |z| < 1$. This, together with (2.7), shows that

$$\begin{aligned} &\int_{\mathbb{D}} |f'(z)|^2 U_{\nu,p}(z)dA(z) \\ &\le C\Big(\int_{t<|z|<1} |f'(z)|^2 U_{\mu,p}(z)dA(z) + \int_{|z|\le t} |f'(z)|^2 U_{\mu,p}(z)dA(z) \Big) \\ &\le \frac{1}{2} \int_{\mathbb{D}} |f'(z)|^2 U_{\nu,p}(z)dA(z) + C\int_{|z|\le t} |f'(z)|^2 U_{\mu,p}(z)dA(z). \end{aligned}$$

Hence,

$$\int_{\mathbb{D}} |f'(z)|^2 U_{\nu,p}(z)dA(z) \le 2C\int_{|z|\le t} |f'(z)|^2 U_{\mu,p}(z)dA(z), \quad f \in \mathcal{D}^0_{\nu,p}. \tag{2.8}$$

Let $h \in \mathcal{D}_p$ with $h(0) = 0$. Set $h_r(z) = h(rz)$, $0 < r < 1$. A direct computation gives that $\|h_r\|^2_{\mathcal{D}_p} \le \|h\|^2_{\mathcal{D}_p}$. Clearly, $h_r \in \mathcal{D}^0_{\nu,p}$. Inequality (2.8) yields that

$$\int_{\mathbb{D}} r^2|h'(rz)|^2 U_{\nu,p}(z)dA(z) \le 2^{p+1}C\frac{\mu(\mathbb{D})}{(1-t)^p}\|h_r\|^2_{\mathcal{D}_p} \le 2^{p+1}C\frac{\mu(\mathbb{D})}{(1-t)^p}\|h\|^2_{\mathcal{D}_p}.$$

Using Fatou's Lemma, we get that

$$\|h\|^2_{\mathcal{D}_{\nu,p}} \le 2^{p+1}C\frac{\mu(\mathbb{D})}{(1-t)^p}\|h\|^2_{\mathcal{D}_p}$$

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for any $h \in \mathcal{D}_p$ with $h(0) = 0$. Therefore, $\mathcal{D}_p \subseteq \mathcal{D}_{\nu,p}$. Applying Theorem 2.1, we see that $\mathcal{D}_p = \mathcal{D}_{\nu,p}$ and

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|w|^2}{|1-\overline{z}w|^2} \Big)^p d\nu(w) < \infty,$$

which contradicts (2.6). Thus, $\mathcal{D}_{\nu,p} \subsetneqq \mathcal{D}_{\mu,p}$. We finish the proof. ∎

The following estimates will be useful in the paper and can be found in [13, p. 9] and [29, p. 55], respectively.

Lemma A (i) *Let $z \in \mathbb{D}$ and let β be any real number. Then*

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^{1+\beta}} \approx \begin{cases} 1 & \text{if } \beta < 0, \\ \log \frac{1}{1-|z|^2} & \text{if } \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta} & \text{if } \beta > 0, \end{cases}$$

as $|z| \to 1^-$.

(ii) *Suppose $z \in \mathbb{D}$, c is real and $t > -1$. Then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\overline{z}w|^{2+t+c}} dA(w) \approx \begin{cases} 1 & \text{if } c < 0, \\ \log \frac{1}{1-|z|^2} & \text{if } c = 0, \\ \frac{1}{(1-|z|^2)^c} & \text{if } c > 0, \end{cases}$$

as $|z| \to 1^-$.

Applying Theorems 2.1 and 2.2, we construct different Dirichlet type spaces $\mathcal{D}_{\mu,p}$. Consequently, the investigation of $\mathcal{D}_{\mu,p}$ spaces is reasonable. Note that the spaces $\mathcal{D}_{\mu,p}$, $\mu \in \mathbb{F}$, $0 < p < \infty$, contain polynomials. Thus, they are not trivial.

Example 1 For $0 < p < \infty$, let

$$d\mu(w) = \frac{1}{|1-w|^{2-p+\epsilon}} dA(w), \quad 0 < \epsilon < p.$$

Then $\mu \in \mathbb{F}$ and $\mathcal{D}_{\mu,p} \subsetneqq \mathcal{D}_p$. In fact, Lemma A(ii), we see that $\mu \in \mathbb{F}$. For $z \in \mathbb{D}$, we write that

$$D(z) = \{w \in \mathbb{D} : |z-w| < \tfrac{1}{2}(1-|z|)\}.$$

Then $D(z)$ is a subset of $E(z)$ as defined in the proof of Theorem 2.1. We deduce that

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|w|^2}{|1-\overline{z}w|^2} \Big)^p \frac{1}{|1-w|^{2-p+\epsilon}} dA(w) \gtrsim \sup_{0<r<1} (1-r)^{-p} \int_{D(r)} \frac{1}{|1-w|^{2-p+\epsilon}} dA(w).$$

If $w \in D(r)$, then

$$\frac{1}{2}(1-r) \le |1-w| \le \frac{3}{2}(1-r).$$

Thus,

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|w|^2}{|1-\overline{z}w|^2} \Big)^p \frac{1}{|1-w|^{2-p+\epsilon}} dA(w) \gtrsim \sup_{0<r<1} (1-r)^{-\epsilon} = \infty.$$

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This together with Theorem 2.1 implies that $\mathcal{D}_{\mu,p} \subsetneqq \mathcal{D}_p$.

The next examples are only valid for $p > 1$. In Section 4, using the theory of $\mathcal{Q}_p$ spaces, we will point out that for all $0 < p < \infty$, there exist Dirichlet type spaces $\mathcal{D}_{\mu_1,p}$ and $\mathcal{D}_{\mu_2,p}$, $\mu_1, \mu_2 \in \mathbb{F}$ such that $\mathcal{D}_{\mu_i,p} \subsetneqq \mathcal{D}_p$, $i = 1, 2$, and $\mathcal{D}_{\mu_1,p} \nsubseteq \mathcal{D}_{\mu_2,p}$.

Example 2 For $p > 1$, let

$$d\mu_1(w) = (1 - |w|^2)^{q_1} dA(w) \quad \text{and} \quad d\mu_2(w) = (1 - |w|^2)^{q_2} dA(w),$$

where $-1 < q_1 < q_2 < p - 2$. Then $\mu_1, \mu_2 \in \mathbb{F}$. Furthermore, $\mathcal{D}_{\mu_1,p} \subsetneqq \mathcal{D}_{\mu_2,p} \subsetneqq \mathcal{D}_p$. In fact, applying Lemma A(ii) yields that

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \Big(\frac{1-|w|^2}{|1-\bar{z}w|^2} \Big)^p d\mu_i(w) = \infty, \quad i = 1, 2,$$

$$\lim_{|z|\to 1} \frac{U_{\mu_2,p}(z)}{U_{\mu_1,p}(z)} \approx \lim_{|z|\to 1} (1-|z|)^{q_2-q_1} = 0.$$

By Theorems 2.1 and 2.2, we know that $\mathcal{D}_{\mu_1,p} \subsetneqq \mathcal{D}_{\mu_2,p} \subsetneqq \mathcal{D}_p$.

3 Decomposition Theorems for $\mathcal{D}_{\mu,p}$ Spaces

The theory of decomposition has appeared in many research areas and it is also important for the study of analytic function spaces. For every function in a given analytic function space, it is interesting to write the function as a linear combination of functions that are elementary in some sense. Decomposition theorems for the Bloch space $\mathcal{B}$, $BMOA$ and $\mathcal{Q}_p$, $0 < p < 1$, were established in [18, 19, 22] respectively. The purpose of this section is to obtain decomposition theorems for $\mathcal{D}_{\mu,p}$ spaces. We also compare decomposition theorems on different analytic function spaces.

For any $z, w \in \mathbb{D}$, the Bergman metric between z and w is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\sigma_z(w)|}{1 - |\sigma_z(w)|}.$$

Fix $r > 0$. Denote by

$$D(z, r) = \{w \in \mathbb{D} : \beta(z, w) < r\}$$

the hyperbolic disk. A sequence $\{z_k\}_{k=1}^{\infty}$ in $\mathbb{D} \setminus \{0\}$ is called an r-lattice if

$$\mathbb{D} = \bigcup_{k=1}^{\infty} D(z_k, r)$$

and $\beta(z_i, z_j) \geq r/2$ for $i \neq j$. The last condition is usually expressed by saying that $\{z_k\}_{k=1}^{\infty}$ is $\frac{r}{2}$-separated. We refer to Zhu's book [29] for these notations.

The following theorem is the main result of the section. One can compare it with decomposition theorems of $\mathcal{Q}_p$ spaces given in [22].

Theorem 3.1 *Let $\mu \in \mathbb{F}$, $0 < p < 2$ and $b \geq p + 1$. There exists an $r_0 > 0$, such that for any r-lattice $\{z_k\}_{k=1}^{\infty}$ in $\mathbb{D}$ with $0 < r < r_0$, the following are true.*

(i) *If* $f \in \mathcal{D}_{\mu,p}$, *then there exists a sequence* $\{\lambda_k\} \in \ell^2$ *such that*

$$f(z) = f(0) + \sum_{k=1}^{\infty} \frac{\lambda_k}{\sqrt{U_{\mu,p}(z_k)}} \Big(\frac{1-|z_k|^2}{1-\overline{z_k} z} \Big)^b \tag{3.1}$$

and

$$\sum_{k=1}^{\infty} |\lambda_k|^2 \le C \|f\|^2_{\mathcal{D}_{\mu,p}}. \tag{3.2}$$

(ii) *For any* $\{\lambda_k\} \in \ell^2$, *the function* f *defined by* (3.1) *is in* $\mathcal{D}_{\mu,p}$ *and*

$$\|f\|^2_{\mathcal{D}_{\mu,p}} \le C \sum_{k=1}^{\infty} |\lambda_k|^2.$$

Remark The proof of Theorem 3.1 given here is invalid for $p \ge 2$, because we need to use Lemma C.

Before proving Theorem 3.1, we give some auxiliary results. The following lemma can be found in [29, p. 72].

Lemma B *Suppose* $0 < r < 1$ *and* $\{z_k\}_{k=1}^{\infty}$ *is an* r*-lattice. For each* k *there exists a measurable set* D_k *with the following properties:*

(i) $D(z_k, r/4) \subseteq D_k \subseteq D(z_k, r)$ *for all* $k \ge 1$.
(ii) $D_i \cap D_j = \varnothing$ *if* $i \ne j$.
(iii) $\mathbb{D} = \cup_{k=1}^{\infty} D_k$.

The following sharp inequality can be found in [15, Lemma 2.5] (see also [27, Lemma 1]).

Lemma C *Suppose that* $s > -1$, r, $t > 0$, *and* $r + t - s > 2$. *If* $t < s + 2 < r$, *then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\overline{w}z|^r |1-\overline{w}\zeta|^t} dA(w) \le C \frac{(1-|z|^2)^{2+s-r}}{|1-\overline{\zeta}z|^t},$$

for all $z, \zeta \in \mathbb{D}$.

For $\nu \in \mathbb{F}$, let $L^2(\mathbb{D}, d\nu)$ be the space of all measurable functions g on $\mathbb{D}$ with

$$\|g\|^2_{L^2(\mathbb{D},d\nu)} = \int_{\mathbb{D}} |g(z)|^2 d\nu(z) < \infty.$$

To prove Theorem 3.1, we need to consider a certain operator on $L^2(\mathbb{D}, U_{\mu,p} dA)$ as follows.

Lemma 3.2 *Let* $\mu \in \mathbb{F}$, $0 < p < 2$ *and* $b > \max\{2p-1, \frac{p+1}{2}\}$. *Then the operator*

$$Tg(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+1}} |g(w)| dA(w), \quad g \in L^2(\mathbb{D}, U_{\mu,p} dA),$$

is bounded on $L^2(\mathbb{D}, U_{\mu,p} dA)$.

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Proof We prove the result by Schur's test. Define a linear operator T_H on $L^2(\mathbb{D}, dA)$ as follows:

$$T_H f(z) = \int_{\mathbb{D}} H(z,w) f(w) dA(w), \quad f \in L^2(\mathbb{D}, dA),$$

where

$$H(z,w) = \frac{(1-|w|^2)^{b-1}}{|1-\overline{z}w|^{b+1}} \sqrt{\frac{U_{\mu,p}(z)}{U_{\mu,p}(w)}}.$$

Fix a number β with $\max\{p-b+1, 0\} < \beta < \min\{p+1, 2-p, b\}$ and take the test function

$$h(z) = \frac{\sqrt{U_{\mu,p}(z)}}{(1-|z|^2)^{\beta}}.$$

Note that $\beta \in (0, b)$. Using Lemma A we get that

$$\int_{\mathbb{D}} H(z,w) h(w) dA(w) = \sqrt{U_{\mu,p}(z)} \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-1-\beta}}{|1-\overline{z}w|^{b+1}} dA(w) \lesssim h(z). \tag{3.3}$$

Note that $b > 0 > -1$ and $1-p-b < p-b+1 < \beta < \min\{p+1, 2-p\}$. Applying the Fubini theorem and Lemma C, we deduce that

$$\begin{aligned}
&\int_{\mathbb{D}} H(z,w) h(z) dA(z) \\
&= \frac{(1-|w|^2)^{b-1}}{\sqrt{U_{\mu,p}(w)}} \int_{\mathbb{D}} (1-|\zeta|^2)^p d\mu(\zeta) \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-\beta}}{|1-\overline{z}w|^{b+1}|1-\overline{z}\zeta|^{2p}} dA(z) \\
&\lesssim \frac{(1-|w|^2)^{p-\beta}}{\sqrt{U_{\mu,p}(w)}} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^p}{|1-\overline{w}\zeta|^{2p}} d\mu(\zeta) \approx h(w).
\end{aligned}$$

Bear in mind (3.3) and the above inequality. Using the Schur theorem (*cf.* [29, Theorem 3.6]), we get that T_H is a bounded operator on $L^2(\mathbb{D}, dA)$.

For any $g \in L^2(\mathbb{D}, U_{\mu,p} dA)$, let

$$f(z) = |g(z)| \sqrt{U_{\mu,p}(z)}.$$

Then

$$\int_{\mathbb{D}} |Tg(z)|^2 U_{\mu,p}(z) dA(z) = \int_{\mathbb{D}} |T_H f(z)|^2 dA(z) \lesssim \int_{\mathbb{D}} |f(z)|^2 dA(z),$$

which gives the desired result. The proof is complete. ∎

As mentioned in Section 2, we let $\mathcal{D}^0_{\mu,p}$ be the Banach space of functions $g \in \mathcal{D}_{\mu,p}$ with $g(0) = 0$. Suppose $0 < r < 1$, $p > 0$, $b \geq p+1$, and $\{z_k\}_{k=1}^{\infty}$ is an r-lattice. Define a linear operator $S_{r,b}$ on $\mathcal{D}^0_{\mu,p}$ by

$$S_{r,b} f(z) = \frac{1}{\pi} \sum_{k=1}^{\infty} f'(z_k) |D_k| \frac{(1-|z_k|^2)^{b-1}}{\overline{z_k}(1-\overline{z_k}z)^b}, \quad f \in \mathcal{D}^0_{\mu,p}, \tag{3.4}$$

where D_k is defined as in Lemma B and $|D_k|$ is the area of D_k.

Lemma 3.3 *Let $\mu \in \mathbb{F}$, $0 < p < 2$, and $b \geq p+1$. There exists a small enough positive constant r_0 such that if $0 < r < r_0$, then the operator $S_{r,b}$ defined by (3.4) is bounded and invertible on the Banach space $\mathcal{D}^0_{\mu,p}$.*

Proof Let $f \in \mathcal{D}^0_{\mu,p}$. Then Theorem 2.1 gives that $f \in \mathcal{D}_p$. Since $b \geq p+1$, we obtain that $f \in \mathcal{D}_{b-1}$. Applying the reproducing formula of Bergman spaces (*cf.* [29, Proposition 4.23]), we get

$$f'(z) = \frac{b}{\pi}\int_{\mathbb{D}} \frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+1}} f'(w)\,dA(w).$$

Combining this with Lemma B yields

$$f'(z) - (S_{r,b}f)'(z) = \frac{b}{\pi}\sum_{k=1}^{\infty}\int_{D_k} \frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+1}} f'(w)\,dA(w) - \frac{b}{\pi}\sum_{k=1}^{\infty} f'(z_k)|D_k|\frac{(1-|z_k|^2)^{b-1}}{(1-\overline{z_k}z)^{b+1}}.$$

Z. Wu and C. Xie [22, p. 395] proved that

$$|f'(z) - (S_{r,b}f)'(z)| \lesssim r\int_{\mathbb{D}} \frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+1}} |f'(w)|\,dA(w).$$

Note that $b \geq p+1 > \max\{2p-1, \frac{p+1}{2}\}$. Applying Lemma 3.2, we see that

$$\begin{aligned}\int_{\mathbb{D}} |f'(z) - (S_{r,b}f)'(z)|^2 U_{\mu,p}(z)\,dA(z) &\lesssim r^2\int_{\mathbb{D}} |Tf'(z)|^2 U_{\mu,p}(z)\,dA(z)\\ &\lesssim r^2\|f\|^2_{\mathcal{D}_{\mu,p}},\end{aligned}$$

which means that $I - S_{r,b}$ is a bounded operator on $\mathcal{D}^0_{\mu,p}$. Here I is the identity operator. Hence,

$$\|(I-S_{r,b})f\|_{\mathcal{D}_{\mu,p}} \lesssim r\|f\|_{\mathcal{D}_{\mu,p}}$$

for all $f \in \mathcal{D}^0_{\mu,p}$. Thus, $S_{r,b}$ is bounded on $\mathcal{D}^0_{\mu,p}$. If r is small enough, then the operator $I - S_{r,b}$ has norm less than one. By standard functional analysis, the operator $S_{r,b}$ is invertible on $\mathcal{D}^0_{\mu,p}$. The proof is complete. ∎

Proof of Theorem 3.1 (i) Let $f \in \mathcal{D}_{\mu,p}$. Then the function $g(z) = f(z) - f(0)$ belongs to $\mathcal{D}^0_{\mu,p}$. Using Lemma 3.3, we obtain that

$$\begin{aligned} g(z) = S_{r,b}S^{-1}_{r,b}g(z) &= \frac{1}{\pi}\sum_{k=1}^{\infty}(S^{-1}_{r,b}g)'(z_k)|D_k|\frac{(1-|z_k|^2)^{b-1}}{\overline{z_k}(1-\overline{z_k}z)^b}\\ &= \sum_{k=1}^{\infty}\frac{\lambda_k}{\sqrt{U_{\mu,p}(z_k)}}\Big(\frac{1-|z_k|^2}{1-\overline{z_k}z}\Big)^b,\end{aligned}$$

where

$$\lambda_k = \frac{(S^{-1}_{r,b}g)'(z_k)|D_k|}{\pi\overline{z_k}(1-|z_k|^2)}\sqrt{U_{\mu,p}(z_k)}.$$

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Bear in mind that $|D_k| \approx (1-|z_k|^2)^2$. Applying Lemma B and the subharmonicity of $|(S_{r,b}^{-1}g)'|^2$ (*cf.* [29, Proposition 4.13]), we get that

$$\sum_{k=1}^{\infty}|\lambda_k|^2 \approx \sum_{k=1}^{\infty}\frac{|(S_{r,b}^{-1}g)'(z_k)|^2|D_k|^2}{(1-|z_k|^2)^2}U_{\mu,p}(z_k)$$
$$\lesssim \sum_{k=1}^{\infty}\int_{D(z_k,r/4)}|(S_{r,b}^{-1}g)'(z)|^2U_{\mu,p}(z_k)dA(z).$$

By [29, Proposition 4.5] and [29, Lemma 4.30], we know that

$$1-|z| \approx 1-|z_k| \approx |1-\overline{z_k}z|, \quad |1-\overline{w}z| \approx |1-\overline{w}z_k|,$$

for all $z \in D(z_k, r/4)$ and $w \in \mathbb{D}$. Hence, $U_{\mu,p}(z_k) \approx U_{\mu,p}(z)$ for all $z \in D(z_k, r/4)$. Note that the operator $S_{r,b}^{-1}$ is also bounded on $\mathcal{D}_{\mu,p}^0$. Consequently,

$$\sum_{k=1}^{\infty}|\lambda_k|^2 \lesssim \sum_{k=1}^{\infty}\int_{D(z_k,r/4)}|(S_{r,b}^{-1}g)'(z)|^2U_{\mu,p}(z)dA(z)$$
$$\lesssim \int_{\mathbb{D}}|(S_{r,b}^{-1}g)'(z)|^2U_{\mu,p}(z)dA(z) \lesssim \|g\|_{\mathcal{D}_{\mu,p}}^2 \approx \|f\|_{\mathcal{D}_{\mu,p}}^2.$$

(ii) Suppose $\{\lambda_k\} \in \ell^2$. We consider the function f defined by (3.1). For any $z \in \mathbb{D}$, one gets that

$$|f'(z)| \le b\sum_{k=1}^{\infty}\frac{|\lambda_k||z_k|}{\sqrt{U_{\mu,p}(z_k)}}\frac{(1-|z_k|^2)^b}{|1-\overline{z_k}z|^{b+1}}$$
$$\approx \sum_{k=1}^{\infty}\frac{|\lambda_k z_k|}{(1-|z_k|)\sqrt{U_{\mu,p}(z_k)}}\int_{D(z_k,r/4)}\frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+1}}dA(w)$$
$$\approx \int_{\mathbb{D}}\frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+1}}\Big(\sum_{k=1}^{\infty}\frac{|\lambda_k z_k|\chi_{D(z_k,r/4)(w)}}{(1-|z_k|)\sqrt{U_{\mu,p}(z_k)}}\Big)dA(w).$$

Set

$$g(w) = \sum_{k=1}^{\infty}\frac{|\lambda_k z_k|\chi_{D(z_k,r/4)(w)}}{(1-|z_k|)\sqrt{U_{\mu,p}(z_k)}}.$$

Then

$$\int_{\mathbb{D}}|g(w)|^2U_{\mu,p}(w)dA(w) \lesssim \int_{\mathbb{D}}\sum_{k=1}^{\infty}\frac{|\lambda_k|^2\chi_{D(z_k,r/4)(w)}}{(1-|z_k|)^2U_{\mu,p}(z_k)}U_{\mu,p}(w)dA(w)$$
$$\approx \sum_{k=1}^{\infty}\frac{|\lambda_k|^2}{(1-|z_k|)^2U_{\mu,p}(z_k)}\int_{D(z_k,r/4)}U_{\mu,p}(w)dA(w)$$
$$\approx \sum_{k=1}^{\infty}|\lambda_k|^2 < \infty.$$

Combining the above estimates and Lemma 3.2, we see that

$$\|f\|_{\mathcal{D}_{\mu,p}}^2 \lesssim \|Tg\|_{L^2(\mathbb{D},U_{\mu,p}dA)}^2 \lesssim \|g\|_{L^2(\mathbb{D},U_{\mu,p}dA)}^2 \lesssim \sum_{k=1}^{\infty}|\lambda_k|^2 < \infty.$$

The proof of Theorem 3.1 is complete. ∎

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Let ν be a positive Borel measure on the unit circle $\partial\mathbb{D}$. Motivated by the study of cyclic analytic two-isometries, S. Richter [17] introduced a certain Dirichlet type space $\mathcal{D}(\nu)$, which consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|^2_{\mathcal{D}(\nu)} = \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'(z)|^2 P_\nu(z)\,dA(z) < \infty,$$

where

$$P_\nu(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\frac{d\nu(t)}{2\pi}.$$

Recently, the decomposition theorems for $\mathcal{D}(\nu)$ spaces were established in [14] as follows.

Theorem D *Let ν be a positive Borel measure on $\partial\mathbb{D}$ and $b > 2$. Then there exists a d-separated sequence $\{z_j\}_{j=1}^\infty$ in $\mathbb{D}$ such that the following are true.*

(i) *If $f \in \mathcal{D}(\nu)$, then there exists a sequence $\{\lambda_j\}$ in $\mathbb{C}$ such that*

$$f(z) = f(0) + \sum_{j=1}^\infty \lambda_j (1-|z_j|^2)^b \Big(\frac{1}{(1-\overline{z_j}z)^b} - 1\Big) \tag{3.5}$$

and

$$\sum_{j=1}^\infty |\lambda_j|^2 P_\nu(z_j) \le C\|f\|^2_{\mathcal{D}(\nu)}.$$

(ii) *If a sequence $\{\lambda_j\} \subseteq \mathbb{C}$ satisfies that $\sum_{j=1}^\infty |\lambda_j|^2 P_\nu(z)\delta_{z_j}$ is a ν-Carleson measure, that is,*

$$\sum_{j=1}^\infty |\lambda_j|^2 P_\nu(z_j)|f(z_j)|^2 \lesssim \|f\|^2_{\mathcal{D}(\nu)}, \quad \text{for all } f \in \mathcal{D}(\nu), \tag{3.6}$$

then the series defined in (3.5) converges in $\mathcal{D}(\nu)$ and

$$\|f\|^2_{\mathcal{D}(\nu)} \le C \sum_{j=1}^\infty |\lambda_j|^2 P_\nu(z_j).$$

Remark We point out that condition (3.6) in Theorem D can be replaced by

$$\sum_{j=1}^\infty |\lambda_j|^2 P_\nu(z_j) < \infty.$$

Comparing decomposition theorems stated in the section with that on other analytic function spaces (*cf.* [18, 19, 22]), we can understand decomposition theorems on analytic function spaces as follows. Let $X \subseteq H(\mathbb{D})$ be a Banach space. Roughly speaking, there exists a sequence $\{z_j\}_{j=1}^\infty$ in $\mathbb{D}$ and a large enough number b such that the space X consists exactly of functions of the form

$$f(z) = \sum_{j=1}^\infty \lambda_j \Big(\frac{1-|z_j|^2}{1-\overline{z_j}z}\Big)^b,$$

where $\{\lambda_j\}$ satisfies certain condition depending only on the space X.

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4 $\mathcal{Q}_p$ Spaces and $\mathcal{D}_{\mu,p}$ Spaces

As mentioned in Section 1, $BMOA = M(H^p)$ and $\mathcal{B} = M(A^p)$ for $1 < p < \infty$. If $0 < p < 1$, it is only known that $\mathcal{Q}_p = M(\mathcal{D}_p)$. In this section, we show that, just like $BMOA$ and $\mathcal{B}$, the Möbius invariant function space $\mathcal{Q}_p$, $0 < p < 1$, can be generated by different analytic function spaces. In fact, $\mathcal{Q}_p = M(\mathcal{D}_{\mu,p})$ for any $\mu \in \mathbb{F}$. We also prove that the non-Hilbert space $\mathcal{Q}_p$ is equal to the intersection of Hilbert spaces $\mathcal{D}_{\mu,p}$. Applying the relation between $\mathcal{Q}_p$ and $\mathcal{D}_{\mu,p}$ spaces, we see that there exist different $\mathcal{D}_{\mu,p}$ spaces.

To prove our main result in the section, we recall $\mathcal{Q}_{p,0}$ spaces. For $0 < p < \infty$, $\mathcal{Q}_{p,0}$ is the class of functions $f \in H(\mathbb{D})$ with

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2 \big(1 - |\sigma_a(z)|^2\big)^p dA(z) = 0.$$

By the characterization of lacunary series of $\mathcal{Q}_{p,0}$ and $\mathcal{Q}_p$ spaces in [6], the Dirichlet space $\mathcal{D}$ is strictly contained in $\mathcal{Q}_{p,0}$ for $0 < p < \infty$. K. Wirths and J. Xiao [21] proved that $\mathcal{Q}_{p,0}$ is the closure of polynomials in the norm of $\mathcal{Q}_p$, and $\mathcal{Q}_{p,0}$ is Möbius invariant space in the strict sense of Arazy, Fisher, and Peetre [4].

The following theorem is new even for the classical function spaces $BMOA$ and $\mathcal{B}$.

Theorem 4.1 *Let $\mu \in \mathbb{F}$ and $0 < p < \infty$. Then the following are true:*

(i) $\mathcal{Q}_p \subsetneqq \mathcal{D}_{\mu,p}$;
(ii) $\mathcal{Q}_p = M(\mathcal{D}_{\mu,p})$;
(iii) $\mathcal{Q}_p = \bigcap_{\mu\in\mathbb{F}} \mathcal{D}_{\mu,p}$.

Proof (i) For any $f \in \mathcal{Q}_p$, applying the Fubini theorem yields that

$$\int_{\mathbb{D}} |f'(z)|^2 U_{\mu,p}(z) dA(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_w(z)|^2)^p dA(z) d\mu(w) \le \mu(\mathbb{D}) \|f\|^2_{\mathcal{Q}_p}.$$

Hence, $\mathcal{Q}_p \subseteq \mathcal{D}_{\mu,p}$. Suppose that $\mathcal{Q}_p = \mathcal{D}_{\mu,p}$. From the closed graph theorem we obtain that the norms of $\mathcal{Q}_p$ and $\mathcal{D}_{\mu,p}$ are equivalent. Therefore, $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$ are Hilbert spaces. J. Arazy and S. Fisher [3] proved that the unique Hilbert space among Möbius invariant spaces in the strict sense of Arazy–Fisher–Peetre [4] is the Dirichlet space $\mathcal{D}$. Thus, $\mathcal{Q}_{p,0} = \mathcal{D}$ contradicting the fact that $\mathcal{D}$ is strictly included in $\mathcal{Q}_{p,0}$. Thus, $\mathcal{Q}_p \subsetneqq \mathcal{D}_{\mu,p}$.

(ii) By Theorem 2.1 and (i) of the theorem, we know that $\mathcal{Q}_p \subsetneqq \mathcal{D}_{\mu,p} \subseteq \mathcal{D}_p$. This implies that $M(\mathcal{Q}_p) \subseteq M(\mathcal{D}_{\mu,p}) \subseteq M(\mathcal{D}_p)$. Note that $M(\mathcal{Q}_p) = M(\mathcal{D}_p) = \mathcal{Q}_p$. Thus, $\mathcal{Q}_p = M(\mathcal{D}_{\mu,p})$.

(iii) Since $\mathcal{Q}_p \subsetneqq \mathcal{D}_{\mu,p}$ for any $\mu \in \mathbb{F}$, we obtain that $\mathcal{Q}_p \subseteq \bigcap_{\mu\in\mathbb{F}} \mathcal{D}_{\mu,p}$. Now let $f \in H(\mathbb{D})$ and $f \notin \mathcal{Q}_p$. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that

$$\beta_n = \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_{a_n}(z)|^2)^p dA(z) \ge 2^n$$

for any positive integer n. Set $t_n = 1/2^n$ and $\nu = \sum_{n=1}^{\infty} t_n \delta_{a_n}$. Then

$$\nu(\mathbb{D}) = \sum_{n=1}^{\infty} t_n < \infty \quad \text{and} \quad \|f\|^2_{\mathcal{D}_{\nu,p}} = \sum_{n=1}^{\infty} t_n \beta_n = \infty.$$

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This implies that $f \notin \mathcal{D}_{\nu,p}$. Thus $f \notin \bigcap_{\mu\in\mathbb{F}} \mathcal{D}_{\mu,p}$. The conclusion follows. ∎

In Section 2, we gave some examples of different $\mathcal{D}_{\mu,p}$ spaces only for $p > 1$. Applying (i) and (iii) of Theorem 4.1, we prove the existence of different $\mathcal{D}_{\mu,p}$ spaces for every $0 < p < \infty$, without constructing examples.

Corollary 4.2 *Let $0 < p < \infty$. There exist Dirichlet type spaces $\mathcal{D}_{\mu_1,p}$ and $\mathcal{D}_{\mu_2,p}$, μ_1, $\mu_2 \in \mathbb{F}$, such that $\mathcal{D}_{\mu_i,p} \subsetneqq \mathcal{D}_p$, $i = 1, 2$, and $\mathcal{D}_{\mu_1,p} \neq \mathcal{D}_{\mu_2,p}$.*

Proof By Theorem 2.1, $\mathcal{D}_{\mu,p} \subseteq \mathcal{D}_p$ for all $\mu \in \mathbb{F}$ and $0 < p < \infty$. Combining this with (i) and (iii) of Theorem 4.1, we see that there exists $\mu_1 \in \mathbb{F}$ such that $\mathcal{D}_{\mu_1,p} \subsetneqq \mathcal{D}_p$. Applying these facts again, we get the desired result. ∎

5 Final Remark

The theory of $\mathcal{Q}_p$ spaces is very well developed. But there are still unresolved problems. For example, the problem of composition operators on $\mathcal{Q}_p$ spaces for $0 < p < 1$. Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic self-map of the unit disk. The function φ induces a composition operator C_φ acting on $H(\mathbb{D})$ by the formula $C_\varphi f = f \circ \varphi$. As pointed out in [23, 24], it is still an open question to characterize the boundedness and compactness of the composition operator C_φ acting on $\mathcal{Q}_p$, $0 < p < 1$, in terms of the function properties of the symbol φ.

Based on Theorem 4.1, we hope that the theory of $\mathcal{Q}_p$ spaces can be developed further in terms of the content of $\mathcal{D}_{\mu,p}$ spaces.

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