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A CHARACTERIZATION OF SOME GEOMETRIES OF CHAINS

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The geometries considered here are the Möbius plane $\mathbf{M}(\mathfrak{K})$ (W. Benz [**1**]), the Laguerre plane $\mathbf{L}(\mathfrak{K})$ (W. Benz and H. Mäurer [**7**]) and the Minkowski plane $\mathbf{A}(\mathfrak{K})$ (W. Benz [**5**], G. Kaerlein [**18**]) over a field $\mathfrak{K}$. All of them are geometries of an algebra with identity over a field.

The characterization of the projective plane over a field by the proposition of Pappus first gave a close relation between algebraic and geometric structures. B. L. v. d. Waedern and L. J. Smid [**28**] presented a further example by characterizing the Möbius and Laguerre plane with incidence axioms and the "complete" proposition of Miquel. Other different characterizations and representations of the above three geometries are studied among others by W. Benz [**1-5**], Y. Chen [**8-11**], G. Ewald [**14**; **15**], A. J. Hoffman [**16**; **17**], G. Kaerlein [**18**], H. Mäurer [**20**], U. Melchior [**21**], B. Petkuntschin [**23**], L. J. Smid [**25**], W. Süss [**26**], A. Uhl [**27**] etc.

The purpose of this paper is to define the Möbius, Laguerre, and Minkowski planes by a common basic system of axioms and to show that they are isomorphic to $\mathbf{M}(\mathfrak{K}), \mathbf{L}(\mathfrak{K}), \mathbf{A}(\mathfrak{K})$, if a "simple" proposition of Miquel is satisfied. In [**8**] the complete proposition of Miquel is derived from the simple one in the Möbius plane. In this paper we often refer to [**8**], and derive for the Laguerre and Minkowski planes from the simple proposition of Miquel those properties (but not the complete proposition of Miquel) which are sufficient to algebrize the two geometries. We shall explain how the sufficient conditions for the algebrization of Laguerre and Minkowski planes are simpler than those for that of Möbius planes. An essential property, which we do not have in the Möbius plane is that two points can touch each other (i.e. there is no chain passing through them). We must check throughout every proof not only that a certain point is distinct from the others, but also whether it touches any other one or not.

General notations. Let $\mathscr{P}$ be a set of elements called points and denoted by the numbers $0, 0', 1, 1', 2, 2' \ldots$ or small Latin letters $a, b, a', a_1 \ldots$ etc. A certain subset $\mathscr{C}$ of the power set of $\mathscr{P}$ is called the set of chains which are

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denoted by Greek letters $\alpha, \beta, \alpha_1, \alpha', \ldots$ etc. Two points a, b of $\mathscr{P}$ "touch" each other denoted $a — b$, if $a = b$ or if there is no chain of $\mathscr{C}$ passing through them. Thus this "touch" relation "—" is symmetric and reflexive. $1 \mathbin{+\!\!\!\!+} 2$ is the negation of $1 — 2$. $(0, 1, 2, \ldots)_0$ means that $0, 1, 2, \ldots$ are connectable, i.e. there is a chain passing through them. $(0, 1, 2, \ldots)_\phi$ is its negation. We write $\alpha[1, 2, \ldots]$ for $1, 2, \ldots \in \alpha$ and use also $1 — \alpha$ (reading 1 touches α or α touches 1) for $1 \in \alpha$. $\alpha — \beta$ means either $\alpha = \beta$ or α and β have exactly one point in common. In the following let $x_1, y_1, x_2, y_2, \ldots$ be either points or chains and $P, Q, R, \ldots$ be relations $=, \neq, —$ and $+$. We write $(x_1, x_2, \ldots, x_n)_R$ for $x_i R x_j$, $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$. $(x_1, x_2, \ldots, x_n) R (y_1, y_2, \ldots, y_m)$ means $x_i R y_j$, $1 \leqq i \leqq n$ and $1 \leqq j \leqq m$. $(x_1, x_2, \ldots, x_n)_{PQ\ldots} R \ldots (y_1, y_2, \ldots)_{ST\ldots} U \ldots (z, \ldots) \ldots$ means $(x_1, x_2, \ldots, x_n)_P$, $(x_1, x_2, \ldots, x_n)_Q, \ldots$, $(x_1, x_2, \ldots, x_n) R (y_1, y_2, \ldots)$, $\ldots, (y_1, y_2, \ldots)_S, \ldots$ $(y_1, y_2, \ldots) U (z, \ldots), \ldots$ etc. For example, we write $(1, 2)_{\neq} + \alpha — (3, \beta, \gamma)_-$ for $1 \neq 2$, $1 + \alpha$, $2 + \alpha$, $\alpha — 3$, $\alpha — \beta$, $\alpha — \gamma$, $3 — \beta$, $3 — \gamma$ and $\beta — \gamma$.

Definition 1. $\mathbf{C} = (\mathscr{P}, \mathscr{C}, —)$ is called a *touch-plane of chains* if the following axioms (C0), (C1) and (C2) are satisfied

(C0) (1) (*The first axiom of abundance*) There are at least three points $0, 1, 2$ satisfying $(0, 1, 2)_+$; and (2) (*The second axiom of abundance*) There is no point touching all chains of $\mathscr{C}$, if $\mathscr{C}$ is not empty.

(C1) (*The axiom of uniqueness*) If $(0, 1, 2)_+$, then there is exactly one chain α with $\alpha — (0, 1, 2)$.

(C2) (*The touch-axiom*) If $1 — \alpha + 2 + 1$, then there exists exactly one chain β satisfying $\beta — (1, 2, \alpha)$.

It is called a *Möbius touch-plane* $\mathbf{C}_{\text{Möb}}$ if it satisfies

(Möb) (*The axiom of Möbius geometry*) Two distinct points never touch each other.

It is called a *Laguerre touch-plane* denoted $\mathbf{C}_{\text{Lag}}$ if it satisfies

(Lag) (*The axiom of Laguerre geometry*) If $1 + \alpha$, then there is exactly one point 2 with $2 — (1, \alpha)$.

It is called a *Minkowski touch-plane* denoted $\mathbf{C}_{\text{Min}}$ if it satisfies

(Min) (*The axiom of Minkowski geometry*) To two non-touching points or one chain and a point not on it there are exactly two points touching both ofthem and we require these two points do not touch each other.

By a *MLM touch-plane* of chains we mean any one of the above three touch-planes.

PROPOSITION 0. *In* $\mathbf{C}_{\text{Lag}}$ *the touch relation on points is an equivalence relation. In* $\mathbf{C}_{\text{Min}}$ *the touch relation induces two equivalence relations which will be called "up touching" (denoted* $\dot{-}$*) and "down touching" (denoted* $\underset{\cdot}{-}$*).*

Proof. It is clear that through any two non-touching points of a touch plane there is one chain. If in $\mathbf{C}_{\text{Lag}}$ $0 — 1 — 2 + 0$ then there would be no α with $\alpha—(0, 1)$, but one α with $(0, 2) — (1, \alpha)$, contradicts (Lag). In $\mathbf{C}_{\text{Min}}$ an equivalence class (called *touching class*) will be determined by two touching elements. Let $0 — 1$. If $2 — (0, 1)$ and $3 — (0, 1)$, then by (Min) $2 — 3$. If $0 — 1 + 2 — 0$, $x — (0, 1)$ and $y — (0, 2)$ then $x = y = 0$ or $x \neq y$. Thus two distinct classes have at most one common element. If $0 — 1 + 2 — 0$, and $x — 0$, then $x — 1$ or $x — 2$, otherwise there would be a chain α and $0 — (1, 2, x) — \alpha$, contradicts (Min). Thus one element can belong to at most (but also at least) two touching classes.

If there is only $1 — 2$, then $1 \dotminus 2$ or $1 \div 2$. We write $1 \dotminus 2$ together with $3 \dotminus 4$ (or $1 \div 2$ and $3 \div 4$) to denote that the touching class of 1, 2 and that of 3, 4 are of the same touching relation and we write $1 \dotminus 2$ together with $3 \div 4$ (or $1 \div 2$ and $3 \dotminus 4$) for the case that the touching class of 1, 2 and that of 3, 4 are of different touching relation.

The following theorem is now quite obvious:

THEOREM 0. *A Mobius touch-plane is a Möbius plane in narrow sense* (W. Benz [**1**]) *and vice versa; A Laguerre touch-plane is a Laguerre plane in narrow sense* (W. Benz and H. Mäurer [**7**]) *and vice versa and a Minkowski touch-plane is a pseudo-Euclidean plane* (in the sense of Kaerlein [**18**]) *and vice versa.*

THEOREM 1. *The above axioms of an MLM touch-plane are independent.*

Proof. We write $[(\bar{X})_{\text{Möb}}]$, $[(\bar{X})_{\text{Lag}}]$ or $[(\bar{X})_{\text{Min}}]$ respectively for a model which satisfies all the axioms (except (X)) of $\mathbf{C}_{\text{Möb}}$, $\mathbf{C}_{\text{Lag}}$ or $\mathbf{C}_{\text{Min}}$ respectively and the negation of axiom (X). $[(\bar{X})]$ gives a model satisfying the negation of (X) and all the other axioms.

$[\overline{(C0)(1)}_{\text{Möb}}]$: $\mathscr{P}$ has two points 1, 2 and $\mathscr{C}$ possesses three chains α, β, γ with $\alpha — 1 — \beta — 2 — \gamma$.

$[\overline{(C0)(1)}_{\text{Lag}}]$: $\mathscr{P}$ has four points 1, 2, 3, 4 and $\mathscr{C}$ possesses four chains $\alpha, \beta, \gamma, \delta$ with $1 — 3$, $2 — 4$ and $1 — \alpha — 2 — \beta — 3 — \gamma — 4 — \delta — 1$.

$[\overline{(C0)(1)}_{\text{Min}}]$: $\mathscr{P}$ has four points 1, 2, 3, 4 and $\mathscr{C}$ possesses two chains α, β with $(1, 2)_+ — (3, 4)_+$, $\alpha — (1, 2)$ and $\beta — (3, 4)$.

$[\overline{(C0)(2)}]$: There exists in $\mathscr{P}$ three points 1, 2, 3 and in $\mathscr{C}$ one chain with $\alpha — (1, 2, 3)$.

$[\overline{(C1)}_{\text{Möb}}]$: The classical three dimensional Möbius geometry.

$[\overline{(C1)}_{\text{Lag}}]$: The classical three dimensional Laguerre geometry, i.e. points are planes in $\mathbf{E}^3(\mathfrak{R})$ with an orientation, chains are spheres in $\mathbf{E}^3(\mathfrak{R})$ with an orientation. (where "sphere" means, of course, the set of planes tangent to a sphere with the same orientation). Any maximal set of parallel (with the same orientation) planes is also a chain.

$[\overline{(C1)}_{\text{Min}}]$: $\mathscr{P}$ has points a_{ij}, $i, j \in \{1, 2, 3\}$ and $\mathscr{C}$ is empty with $a_{ij} — a_{kl}$ for $i = k$ or $j = l$.

$[\overline{(C2)}_{\text{Möb}}]$: $\mathscr{P}$ contains points 1, 2, 3, 4 and $\mathscr{C}$ the chains $\alpha, \beta, \gamma, \delta$ with $\alpha — (1, 2, 3)$, $\beta — (1, 2, 4)$, $\gamma — (1, 3, 4)$ and $\delta — (2, 3, 4)$.

$[\overline{(C2)}_{\text{Lag}}]$: There exist in $\mathscr{P}$ points 1, 2, ... 8 and in $\mathscr{C}$ chains $\alpha, \beta, \ldots$ with $1 — 5$, $2 — 6$, $3 — 7$, $4 — 8$, $\alpha — (1, 2, 3, 8)$, $\beta — (1, 2, 4, 7)$, $\gamma — (1, 3, 4, 6)$,

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δ — (2, 3, 4, 5), ϵ — (4, 5, 6, 7), σ — (3, 5, 6, 8), ξ — (2, 5, 7, 8) and ζ — (1, 6, 7, 8).

[$\overline{(C2)}_{Min}$]: There exist in $\mathscr{P}$ points 1, 2, . . . 8 and in $\mathscr{C}$ chains $\alpha, \beta, \gamma, \delta$ with 1 — 6 — 3 — 8 — 1, 2 — 5 — 4 — 7 — 2, α — (1, 2, 3, 4), β — (5, 6, 7, 8), γ — (1, 3, 5, 7), and δ — (2, 4, 6, 8).

The last axiom of any *MLM* touch-plane is independent† of the other since the other *MLM* touch-planes exist. We omit the simple proof of the following combinatoric facts (P. Dembowski [**13**]) G. Kaerlein [**18**]).

THEOREM 2. *The conditions of each column in the following table are equivalent to one another:* (n is a positive integer $>$ 1 and is called the *order* of the plane).

	$\mathbf{C}_{\text{Möb}}$	$\mathbf{C}_{\text{Lag}}$	$\mathbf{C}_{\text{Min}}$
Points in total	$n^2 + 1$	$n(n + 1)$	$(n + 1)^2$
Chains in total	$n(n^2 + 1)$	n^3	$(n + 1)n(n - 1)$
Points per chain	$n + 1$	$n + 1$	$n + 1$
Chains per point	$n(n + 1)$	n^2	$n(n - 1)$
Chains passing two distinct points	$n + 1$	n	$n - 1$
Mutually touching chains through a common point	n	n	n
Touching chains per chain (the given one is excluded)	$n^2 - 1$	$n^2 - 1$	$n^2 - 1$
Disjoint chains per chain (the given one is excluded)	$\frac{n(n - 1)(n - 2)}{2}$	$\frac{n(n - 1)^2}{2}$	$\frac{n^2(n - 1)}{2}$
Touching classes	$n^2 + 1$	$n + 1$	$n + 1$ "$\dot{-}$" $n + 1$ "$\underset{\cdot}{-}$"
Points per touching class	1	n	$n + 1$

The touch plane $\mathbf{C} = (\mathscr{P}, \mathscr{C}, —)$ induces affine planes $A = (\mathscr{P}', \mathscr{L}, \epsilon)$ as follows: Let 0 be any point of $\mathscr{P}$ and $\mathscr{P}' = \mathscr{P} \backslash \{0\}$. A line of $\mathscr{L}$ is either a chain of $\mathscr{C}$ passing through 0 or a touching class. A point is incident with a line, if the former is contained in the latter. If the touching classes are not considered as lines, then the induced structure is called an affine plane with neighbour elements (W. Benz [**6**], W. Klingenberg [**19**]).

Definition 2. Four points a, b, c, d of a touch plane are *dependent* denoted $(a, b, c, d)_\Delta$, if a point or a chain touches any three of them, then it touches also the last one and if $a \nparallel (c, d)$ or $b \nparallel (c, d)$.

†The axiom (Min) contains two statements (1) To $0 \nparallel \alpha$ there are exactly 1 and 2 with $(0, \alpha)$ — (1, 2). (2) To $0 \nparallel 1$ there are precisely 2,3 with (a) (0, 1) — (2, 3) and furthermore (b) $2 \nparallel 3$. The author did not succeed in deriving (b) from the other conditions. It seems also not very easy to find a model satisfying all axioms except (1). A model satisfying all axioms except (2) is as follows: $\mathscr{P}$ has points 0, 1, 2, 3, 4 and $\mathscr{C}$ possesses α, β with (1, 2) — (3, 4), α — (0, 1, 2) and β — (0, 3, 4). A model satisfying (C0), (C1), (C2), (2) (a) of (Min) and the negation of (1) of (Min): There are points 0, 1, 2, 3, 4 in $\mathscr{P}$ and α in $\mathscr{C}$ with α — $(0, 1, 2)_+$ — $(3, 4)_-$.

LEMMA 0. *$(a, b, c, d)_\Delta$ if and only if*

(1) *$(a, b, c, d)_0$ for the Möbius case;*

(2) *$(a, b, c, d)_0$ or $a — b$ implies $c — d$ for the Laguerre case;*

(3) *$(a, b, c, d)_0$ or $a \dot{-} b$ implies $c \underset{\cdot}{-} d$ (similarly $a \underset{\cdot}{-} b$ implies $c \dot{-} d$) for the Minkowski case* (proof omitted).

Definition 3. *The proposition of Miquel* (M): If 0, 1, 2, 3, 4, 5, 6, 7 are eight distinct points and $(0, 2, 1, 3)_\Delta$, $(0, 4, 1, 7)_\Delta$, $(0, 6, 3, 7)_\Delta$, $(1, 5, 2, 4)_\Delta$, $(2, 6, 3, 5)_\Delta$ then $(4, 6, 5, 7)_\Delta$ (Figures 1 (Four points on one face of dice are dependent. Two points of one edge do not touch each other if they are not identical.), 2, 3, 4).

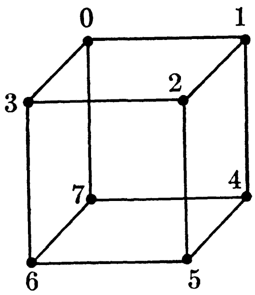

FIGURE 1

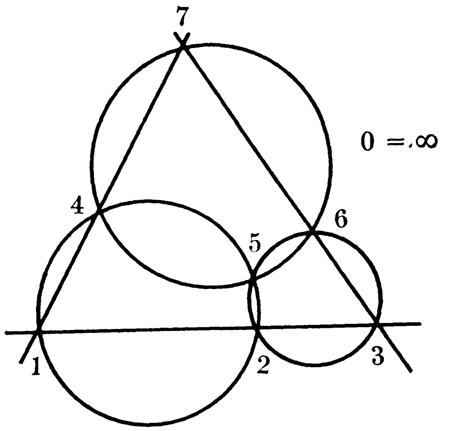

FIGURE 2

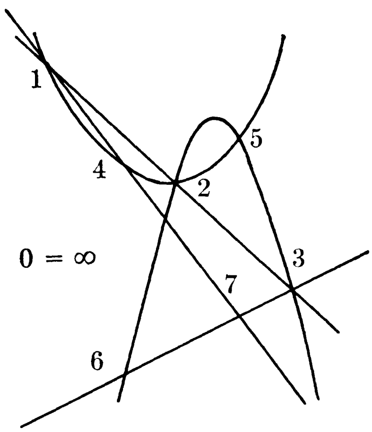

FIGURE 3

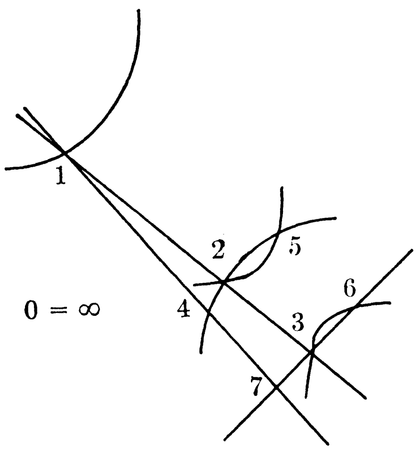

FIGURE 4

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One can easily show that if $\mathfrak{A}$ is a quadratic ring extension of a field $\mathfrak{K}$ where the unity of $\mathfrak{A}$ is equal to that of $\mathfrak{K}$, then $\mathfrak{A}$ is either $\mathfrak{K}(i)$, quadratic field extension of $\mathfrak{K}$ by adjoint with i, or $\mathfrak{K}(j)$ with $j^2 = 0$ or $\mathfrak{K}(k)$ with $k^2 = k$. Let $l = \{i, j, k\}$ and write $\mathfrak{A} = \mathfrak{K}(l)$. The elements of $\mathfrak{K}(l)$ will be denoted by Latin letters with subscript $a_1, a_2, a_i, a_j, \ldots$ We know that there is no non-trivial ideal in $\mathfrak{K}(i)$, but one non-trivial maximal ideal $\langle j \rangle$ in $\mathfrak{K}(j)$ and two distinct non-trivial maximal ideals $\langle k \rangle$ and $\langle 1 - k \rangle$ in $\mathfrak{K}(k)$.

Definition 4. A *touch plane* $\mathbf{C}(\mathfrak{A})$ over $\mathfrak{A}$ has as point set the set of points of the projective line over $\mathfrak{A}$. (i.e. every point a is an ordered pair

$$[a_1, a_2] \in \{\mathfrak{A} \times \mathfrak{A}\}\backslash\{\langle l_1 \rangle \times \langle l_1 \rangle \cup \langle l_2 \rangle \times \langle l_2 \rangle\}.$$

$[a_1, a_2] = [b_1, b_2]$ if and only if $a_1 = rb_1$ $a_2 = rb_2$ with $r \in \mathfrak{A}\backslash\{\langle l_1 \rangle \cup \langle l_2 \rangle\}$ where $l_1 = l_2 = 0$ for $\mathfrak{A} = \mathfrak{K}(i)$, $l_1 = l_2 = j$ for $\mathfrak{A} = \mathfrak{K}(j)$ and $l_1 = k$, $l_2 = 1 - k$ for $\mathfrak{A} = \mathfrak{K}(k)$; two points $[a_1, a_2]$, $[b_1, b_2]$ touch each other if

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \in \langle l_1 \rangle \cup \langle l_2 \rangle$$

and as chain set the set of images of the points of the projective line $p^1(\mathfrak{K})$ over $\mathfrak{K}$ under a projective transformation of PGL $(2, \mathfrak{A})$ (i.e. every chain is a set of points $[a_1, a_2]^\alpha$, where $[a_1, a_2] \in \{\mathfrak{K} \times \mathfrak{K}\backslash[0, 0]\}$ and $\alpha \in$ PGL $(2, \mathfrak{A})$.)

The following is known (W. Benz [**2**; **5**; **7**]): $\mathbf{C}(\mathfrak{A})$ *is* $\mathbf{C}_{\text{Möb}}$, $\mathbf{C}_{\text{Lag}}$, $\mathbf{C}_{\text{Min}}$ *if* $\mathfrak{A}$ *is* $\mathfrak{K}(i)$, $\mathfrak{K}(j)$ *and* $\mathfrak{K}(k)$ *respectively*.

We also know (W. Benz [**2**; **5**; **7**]) that there is an automorphism group of $\mathbf{C}(\mathfrak{A})$ which is sharply transitive with respect to three mutually non-touching points. This almost implies (details in [**10**]):

PROPOSITION 1. (M) *is valid in* $\mathbf{C}(\mathfrak{A})$.

Definition 5. We call a *MLM* touch-plane $\mathbf{C}$ *Miquelian* if the following *special proposition of Miquel* with respect to point 0 denoted (SM0) holds: In $\mathbf{C}$ there is a point 0 such that $(4, 5, 6, 7)_\Delta$ follows from $(0, 1, 2, 3, 4, 5, 6, 7)_\ne$, $(0, 1, 2, 3)_o$, $(0, 1, 4, 7)_o$, $(0, 3, 6, 7)_o$, $(1, 2, 4, 5)_o$ and $(2, 3, 5, 6)_o$.

For Möbius plane (SM0) is simply the usual proposition of Miquel (i.e. (1SMM0) in [**8**]). For Laguerre plane it is a little stronger than the usual proposition of Miquel (W. Benz [**7**]). We have already remarked in [**8**], that the derivation of axiom VIIa from VII in V. d. Waerden and Smid [**28**] is incomplete. For Minkowski plane it is stronger than (M) of G. Kaerlein [**18**]. But Kaerlein needs (M) and $(\overline{\text{M}})$ for the characterization, where $(\overline{\text{M}})$ is equivalent to the statement that (2) and (3) imply (1) in lemma 7 of this paper. Besides, he does not specify any point 0 as we do here. It may be interesting and worthwhile to mention the following. In summer 1968 I succeeded in deriving the complete proposition of Miquel from the simple one for the Möbius plane. Naturally one would try to get a similar result for the Laguerre and Minkowski plane. I did that not only for the problem itself but also because of my belief that there is an axiom system with one language for the plane geometry of Möbius, Laguerre, Minkowski and perhaps some more geometries and they can be algebrized by a proposition of Miquel or by means of "cross ratio" or by a certain properties of the automorphism group or by means of a polarity (in case of the characteristic of field unequal to two). This study may lead us to find out a direct equivalence between the different properties being used for algebrization as in the projective plane one shows the equivalence between the proposition of Pappus and the projective group being transitive with respect to four points in general position without applying the coordinates. By solving the first indicated problem I found that it is easy to prove the proposition of Pappus in the induced affine plane if the following proposition of Miquel in **C** holds. There is a point 0 such that $(4, 5, 6, 7)_o$ follows from $(0, 1, 2, 3, 4, 5, 6, 7)_{\neq}$, $(0, 1, 2, 3)_o$, $(0, 1, 4, 7)_o$, $(0, 3, 6, 7)_o$, $(1, 2, 4, 5)_o$, $(2, 3, 5, 6)_o$ and $(4, 5, 6, 7)_+$. But this does not seem to be sufficient for showing that every chain is a conic. For the latter one can make different requirements. Insisting that there should be eight direct points in the assumption I finally found that (SM0) is at least sufficient. The proof I got in 1969 is more complicated than that given here. Independent of my try and almost at the same time G. Kaerlein looked for characterizations different from that of W. Benz [**5**]. He has also derived the proposition of Pappus from a simple proposition of Miquel. By means of $(\overline{\mathrm{M}})$ where only six distinct points appear in the assumption it follows immediately that the nonlinear chains are conics. While in this paper we derive $(\overline{\mathrm{M}})$ and emphasize how $(1, 2, 3, 4)_\Delta$ takes the place of $(1, 2, 3, 4)_o$, when **C** is not necessarily a Möbius plane.

Our general assumption in the following is a Miquelian *MLM* touch-plane. 0 should be used exclusively for the distinguish point 0 of (SM0). We remark that $\alpha[0, 1, \ldots]$, $\beta[2, 3, \ldots] \ldots$ are often used without mentioning of $(0, 1, \ldots)_+$, $(2, 3, \ldots)_+, \ldots$ and the existence of $\alpha, \beta \ldots$ at first. For example the statement if $\alpha[0, 1, \ldots] - \beta[2, 3, \ldots]$, then $\gamma[4, 5, \ldots] + \delta[6, 7, \ldots]$ should mean: If $(0, 1, \ldots)_+$, $(2, 3, \ldots)_+$ and there are α, β with $(0, 1, \ldots) - \alpha - \beta - (2, 3, \ldots)$ then $(4, 5, \ldots)_+$, $(6, 7, \ldots)_+$ and there are γ, δ with $(4, 5, \ldots) - \gamma + \delta - (6, 7, \ldots)_+$. An immediate consequence of (SM0) is

LEMMA 1. (a) *If* $(0, 1, 2, 3, 4, 5, 6)_+$, $(0, 1, 2, 3)_o$, $(0, 4, 5, 6)_o$, $(1, 2, 3, 5)_o$, $(2, 3, 5, 6)_o$, *then* $\alpha[0, 1, 4] - \beta[0, 3, 6]$ (Figure 5 (If two elements of an edge are identical, then the two chains containing the points of the two dice faces having this edge in common must touch each other)).

(b) *If* $(0, 1, 2, 3, 4, 5, 6)_{\neq}$, $(0, 1, 2, 3)_o$, $(0, 3, 5, 6)_o$, $(1, 4, 5, 6)_o$, $(2, 3, 4, 5)_o$, *then* $\alpha[0, 1, 6] - \beta[1, 2, 4]$ (Figure 6).

(c) *If* $(0, 1, 2, 3, 4, 5, 6)_{\neq}$, $(0, 1, 2, 3)_o$, $(0, 1, 4, 6)_o$, $(0, 3, 5, 6)_o$, $(2, 4, 5, 6)_o$, *then* $\alpha[1, 2, 4] - \beta[2, 3, 5]$ (Figure 7).

LEMMA 2. *If* $(0, 1, 2, 3, 4, 5, 6)_{\neq}$, $(0, 1, 2, 3)_o$, $(0, 3, 5, 6)_o$, $(1, 4, 5, 6)_o$, $\alpha[1, 2, 4] - \beta[0, 1, 6]$, *then* $(2, 3, 4, 5)_o$ *or* $2 - 5$ *and* $3 - 4$ (Figure 6).

Proof. (a) Assume $3 + 4$. There are $\mu[1, 3, 4]$ and $\delta[3, 4, 5]$. If $\mu - \beta$ then

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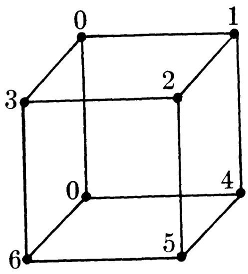

FIGURE 5

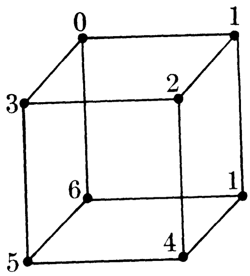

FIGURE 6

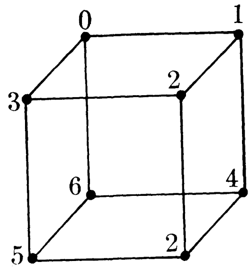

FIGURE 7

$\mu = \alpha$ i.e. $(0, 1, 2, 3, 4)_o$, thus $(0, 1, 2, 3, 4, 5, 6)_o$. In case of $\mu + \beta$ there is 7 with $(1, 7)_{\ne} — (\mu, \beta)$. Let the trivial case $(0, 1, 2, 3, 4, 5, 6)_o$ be excluded. $7 \ne (2, 3, 4, 5)$ due to $\beta + (2, 3, 4, 5)$. From $7 = 0$ or $7 = 6$ it follows $(0, 1, 2, 3, 4)_o$ or $(1, 3, 4, 6)_o$ respectively and thus finally $(0, 1, 2, 3, 4, 5, 6)_o$. Thus $7 \ne (0, 2, 3, 4, 5, 6)$. If $\delta + \xi[0, 3, 7]$ were true, then there would be a 8 with $(3, 8)_{\ne} — (\delta, \xi)$, $8 \ne 1$ otherwise $\mu = \delta$, $8 \ne 6$ otherwise $3 — \beta$. From $8 = 4$ or $8 = 5$ it follows $(0, 1, 3, 4, 7)_o$ or $(0, 3, 5, 6, 7)_o$ respectively. $8 \ne (0, 7)$ due to $\delta + (0, 7)$. Thus $8 \ne (0, 1, 3, 4, 5, 6, 7)$. Owing to (SM0) and $4 + 7$ we get $(1, 4, 7, 8)_o$ (Figure 8) and further $(0, 1, 2, 3, 4, 5, 6, 7, 8)_o$. If $\delta — \xi$, then $\delta + \gamma[0, 1, 2, 3]$, i.e. there is $(3, 9)_{\ne} — (\gamma, \delta)$. $9 \ne (0.1)$ for $\delta + (0, 1)$. $9 \ne (4, 5, 6)$ for $\gamma + (4, 5, 6)$. Thus by Lemma 1 (b) $\beta[0, 1, 6] — \rho[1, 4, 9]$ (Figure 9) and furthermore due to $\alpha — \beta$ $\alpha = \rho$, i.e. $(1, 2, 4, 9)_o$ and finally $(0, 1, 2, 3, 4, 9)_o$ if not $2 = 9 — \delta$.

(b) If $2 + 5$, we exchange 0, 2, 3 with 6, 4, 5 respectively and construct 7, 8, 9 as in (a) to get $(2, 3, 4, 5)_o$.

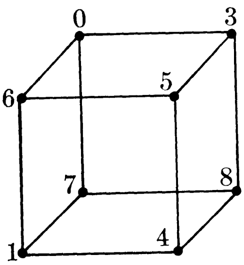

FIGURE 8

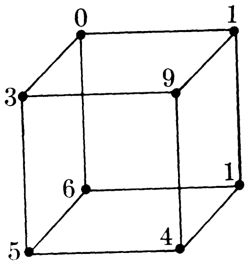

FIGURE 9

LEMMA 3. *If* $(0, 1, 2, 3, 4, 5, 6)_{\ne}$, $(0, 1, 2, 3)_o$, $(0, 4, 5, 6)_o$, $(2, 3, 5, 6)_o$ $\alpha[0, 1, 4] — \beta[0, 3, 6]$, *then* $(1, 2, 4, 5)_o$ *or* $1 — 5$ *and* $2 — 4$ (Figure 5).

Proof. Because of symmetry we need only to prove $(1, 2, 4, 5)_o$ under the assumption $2 + 4$. Due to $4 — \alpha — \beta$ and (C2) there is 7 with $(0, 7)_{\ne} — (\beta, \mu[0, 2, 4])$, $7 \ne (2, 4, 5)$ for $\beta + (2, 4, 5)$. From $7 = 3$, $7 = 5$ or $7 = 6$ it follows $(0, 1, 2, 3, 4, 5, 6)_o$. Thus $7 \ne (0, 2, 3, 4, 5, 6)$. If $\gamma[0, 1, 2, 3] — \delta[2, 4, 5]$, then according to Lemma 2 $(4, 5, 6, 7)_o$ (Figure 10) and furthermore

$(0, 1, 2, 3, 4, 5, 6, 7)_o$, therefore $\gamma \,\boldsymbol{+}\, \delta$. i.e. there is 8 with $(2, 8)_{\mp}$ — (γ, δ). $8 \mp (0, 2, 3, 4, 5, 6)$. By Lemma 1(a) $\alpha'[0, 4, 8]$ — $\beta[0, 3, 6]$ (Figure 11). From this it follows by (C2) $\alpha' = \alpha$ and furthermore by (C1) $8 = 1$.

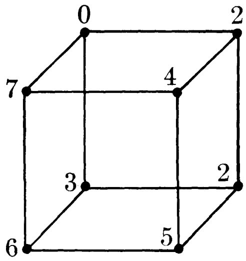

FIGURE 10

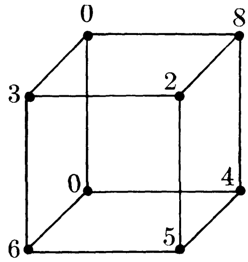

FIGURE 11

In [**12**] we prove that every Laguerre and Minkowski plane is categoric, if it exists and if it is of order less than eight. In the following we shall assume the order of **C** sufficient large, but not necessarily greater than seven.

LEMMA 4. *If* $(0, 1, 2, 3, 4, 5)_{\mp}$, $(0, 1, 2, 3)_o$, $(0, 3, 4, 5)_o$, $\gamma[0, 2, 4]$ — $\delta[1, 4, 5]$, *then* $\kappa[1, 2, 4]$ — $\sigma[1, 3, 5]$ (Figure 12).

Proof. (a) We prove that $\beta[0, 1, 4] \,\boldsymbol{+}\, \sigma[1, 3, 5]$. If β — σ then we would construct η with $(0, 1)$ — η — γ. $\gamma \,\boldsymbol{+}\, \epsilon[0, 3, 4, 5]$. It follows $(0, 6)_{+}$ — (η, ϵ). $(0, 1, 2, 3, 4, 5, 6)_o$ or $6 \mp (1, 2, 3, 4)$. If 6 ∓ 5 we construct ξ with $(0, 1)$ — ξ — $\nu[1, 5, 6]$ there is $(0, 7)_{\mp}$ — (ξ, δ). Obviously $7 \mp (1, 4, 5, 6)$. Let μ pass through 0, 4, 7. Due to 4 — γ — η we get $(0, 8)_{\mp}$ — (η, μ). $8 \mp (1, 4, 5, 6, 7)$. On account of $4 \,\boldsymbol{+}\, 6$ and Lemma 2 it follows $(4, 5, 6, 8)_o$ (Figure 13) a contradiction, namely $(0, 6, 8)_{\mp}$ — $(\eta, \epsilon)_{\mp}$. If $6 = 5$ we find by means of sufficient large order of **C** a $5'$ — δ with $0 \,\boldsymbol{+}\, 5'$ and $\alpha[0, 1, 2, 3] \,\boldsymbol{+}\, \epsilon'[0, 4, 5']$. There is $3'$ with $3'$ — (α, ϵ'). Then there must be a $6'$ with $6' \mp (0, 1, 2, 3, 4, 5)$ and $6'$ — (ϵ', η). As for the case 6 ∓ 5, we take instead of $3, 5, 6, 3', 5', 6'$ respectively and reach similarly a contradiction under the assumption β — σ.

(b) If $\beta \,\boldsymbol{+}\, \sigma$, then let $\rho[1, 4, 6]$ — σ and 6 — α. We prove that $6 = 2$. If 6 ∓ 2 there would exist $\gamma'[0, 4, 6]$ and $(4, 7)_{\mp}$ — (γ', δ). $7 \mp (0, 1, 3, 5, 6)$. By Lemma 1 (c) $\kappa'[1, 6, 7]$ — σ follows (Figure 14), a contradiction.

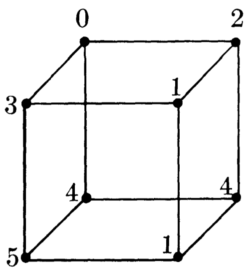

FIGURE 12

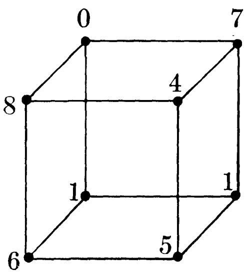

FIGURE 13

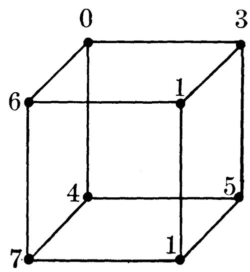

FIGURE 14

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LEMMA 5. *If* $(0, 1, 2, 3, 4, 5, 6)_{\ddagger}$, $(0, 1, 2, 3)_o$, $(0, 1, 4, 6)_o$, $(2, 4, 5, 6)_o$, $\alpha[1, 2, 4] — \beta[2, 3, 5]$, *then* $(0, 3, 5, 6)_o$ *or* $0 — 5$ *and* $3 — 6$ (Figure 7).

Proof. Assume $3 + 6$. There is $\epsilon[0, 3, 6]$. If $\epsilon + \gamma[2, 4, 5, 6]$ then (C1), (C2) and Lemma 1(c) assure $5 — \epsilon$. On the other hand $\epsilon — \gamma$ is impossible, otherwise by Lemma 4 $\delta[2, 3, 6] — \alpha$. Assume $0 + 5$. There is $\sigma[0, 5, 6]$. If $\sigma + \xi[0, 1, 2, 3]$. Then by Lemma 1(c) $(0, 3, 5, 6)_o$. If $\sigma — \xi$, there would be a 7 with $(0, 2, 5, 7)_o$ and $(1, 2, 4, 7)_o$. From this it follows by Lemma 2 $(4, 5, 6, 7)_o$ (Figure 15), thus $(0, 1, 2, 3, 4, 5, 6, 7)_o$.

LEMMA 6. *If* $(0, 1, 2, 3, 4, 5, 6)_{\ddagger}$, $(0, 1, 2, 3)_o$, $(0, 4, 5, 6)_o$, $(2, 3, 5, 6)_o$ $\alpha[0, 1, 4] — \beta[0, 3, 6]$, *then* $(1, 5, 2, 4)_\Delta$ (Figure 5).

Proof. For Möbius and Laguerre plane everything is proved in Lemma 3. For the Minkowski case $1 \dot{-} 5$ implies $2 — 4$, but we still have to show that $1 \dot{-} 5$ implies $2 \underset{\cdot}{-} 4$. If $1 \dot{-} 5$ and $2 \dot{-} 4$, then there would exist 7 with $7 \ddagger (0, 1, 2, 3, 4, 5, 6)$, $2 \underset{\cdot}{-} 7$ and $7 — \delta[0, 4, 5, 6]$. We construct ϵ with $(0, 7) — \epsilon — (\alpha, \beta)$. There is $8 \ddagger (0, 1, 2, 3, 4, 5, 6, 7)$ and $8 — (\epsilon, \gamma[0, 1, 2, 3])$. Because of Lemma 3 and (Min) $5 \underset{\cdot}{-} 8$, $1 + 7$ and $4 + 8$. There are $\xi[0, 1, 7]$, $\eta[0, 4, 8]$ with $(0, 9)_{\ddagger} — (\xi, \beta)$ and $(0, 9')_{\ddagger} — (\eta, \beta)$. If $(2', 3)_{\ddagger} — (\nu[1, 3, 9], \kappa[2, 3, 5, 6])$, then due to (M) we would get $7 — 2'$ (Figure 16). But $2 \ddagger 2'$. Therefore $\nu — \kappa$. Similarly $\gamma — \mu[3, 8, 9']$ (Figure 17). If $3 + 7$,

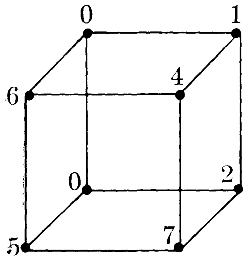

FIGURE 15

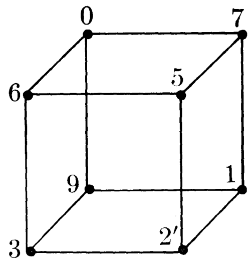

FIGURE 16

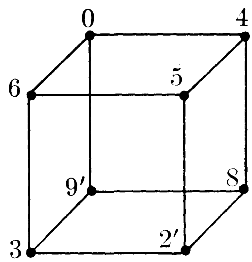

FIGURE 17

then there is $\tau[3, 5, 7]$ with $\tau + \nu$, i.e. $(1', 3)_{\ddagger} — (\tau, \nu)$. $1' \ddagger (0, 5, 6, 7)$. If $1' = 9$, then by Lemma 5 (Figure 18) $(0, 1, 3, 6)_o$ and thus $(0, 1, 2, 3, 4, 5, 6, 9)_o$, contradicts $1 \dot{-} 5$. Hence $(0, 1', 3, 5, 6, 7, 9)_{\ddagger}$. Due to Lemma 5 it follows $(0, 1', 7, 9)_o$ (Figure 19), thus $1 = 1'$. But this contradicts $1 \dot{-} 5$. Therefore $3 — 7$ and furthermore $3 \dot{-} 7$. Similarly we can get $4 \underset{\cdot}{-} 3$, $1 \underset{\cdot}{-} 6$ and $8 \dot{-} 6$. If the order of **C** is sufficient large, there are $3'$, $6'$ with $3' \ddagger (0, 1, 2, 3, 8)$, $6' \ddagger (0, 4, 5, 6, 7)$. $3' — \gamma$, $6' — \delta$ and $\beta — \beta'[0, 3', 6']$. By Lemma 3 $(2, 3', 5, 6')_o$ follows. As above we can show under the assumption $1 \dot{-} 5$ and $2 \dot{-} 4$ the relation $3' \dot{-} 7$ contradicts $3 \dot{-} 7$. Hence $1 \dot{-} 5$ and $2 \dot{-} 4$ can never happen simultaneously.

PROPOSITION 2. *If* $(0, 1, 2, 3, 4, 5, 6)_{\ddagger}$, $(0, 1, 2, 3)_o$, $(0, 4, 5, 6)_o$, $\alpha[0, 1, 4] — \beta[0, 3, 6]$, *then* $(1, 5, 2, 4)_\Delta$ *and* $(2, 6, 3, 5)_\Delta$ *are equivalent* (Figure 5).

Proof. Because of symmetry we need only to derive $(1, 5, 2, 4)_\Delta$ from $(2, 6, 3, 5)_\Delta$. If the order of **C** is sufficient large there are $3'$, $6'$ such that $(0, 1, 2, 3) \ddagger 3' — \gamma[0, 1, 2, 3]$, $(0, 4, 5, 6) \ddagger 6' — \delta[0, 4, 5, 6]$ and $(2, 3', 5, 6')_o$. Applying Lemma 6 it is easy to show $\beta[0, 3, 6] — \beta'[0, 3', 6']$ and then $(1, 5, 2, 4)_\Delta$.

LEMMA 7. *If* $(0, 1, 2, 3, 4, 5)_{\ddagger}$ *and* $(0, 1, 2, 3)_o$, *then each of the following conditions is necessary for the other two*:
(1) $(2, 5, 3, 4)_\Delta$,
(2) $\delta[1, 2, 4] — \epsilon[0, 4, 5]$ *and*
(3) $\alpha(0, 1, 4) — \beta[0, 3, 5]$ (Figure 20).

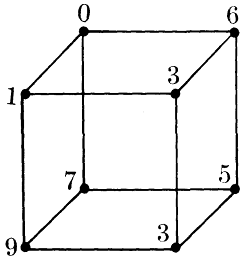

FIGURE 18

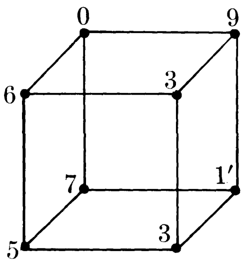

FIGURE 19

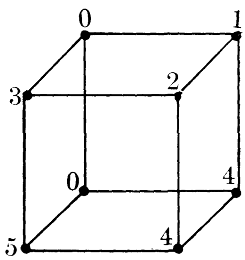

FIGURE 20

Proof. Applying Proposition 2 it is easy to see that (3) is implied by the other two conditions. Now assume (2) and (3). If $3 + 4$, then by (C2) there is 6 with $(4, 6)_{\ddagger} — (\epsilon, k(2, 3, 4])$ and $6 \ddagger (0, 1, 2, 3)$. Using the fact that (1) and (2) imply (3) we get $\alpha[0, 1, 4] — \beta[0, 3, 6]$. It follows $5 = 6$, i.e. $(2, 3, 4, 5)_o$. If $3 \dot{-} 4$, then we choose a $3'$ with $3' + 4$ and $(1, 2) \ddagger 3' — \gamma[0, 1, 2, 3]$. There exist $\beta'[0, 3', 5'] — \alpha$ with $5' — \epsilon$ and 6 with $(4, 6)_{\ddagger} — (\epsilon, k[2, 3', 4])$. By applying what has just been proved we get $(2, 3', 4, 5')_o$. From this by proposition 2 $(2, 5, 3, 4)_\Delta$ follows. We skip the easy proof that (1) and (3) are sufficient for (2).

By Proposition 2 and Lemma 7 one gets

LEMMA 8. *If* $(0, 1, 2, 3, 4, 5, 6)_{\ddagger}$, $(0, 1, 2, 3)_o$, $(0, 4, 5, 6)_o$ *then any three of the following conditions imply the last:*

(1) $(1, 5, 2, 4)_\Delta$,
(2) $(2, 6, 3, 5)_\Delta$,
(3) $\xi[0, 1, 5] — \eta[0, 2, 6]$,
(4) $\zeta[0, 2, 4] — \theta[0, 3, 5]$ (cf. **L5** in [**8**]).

Definition 6. *The special proposition of Pappus with respect to* 0 in a touch plan **C** denoted (SP0). There is a point 0 in **C** such that under the assumptions $(0, 1, 2, 3, 4, 5, 6)_{\neq}$, $\gamma[0, 1, 2, 3] \neq \epsilon[0, 4, 5, 6]$,

(1) if $\xi[0, 1, 5] — \eta[0, 2, 6]$ and $\zeta[0, 2, 4] — \theta[0, 3, 5]$, then either $1 \dot{-} 4$ and $3 \dot{-} 6$ or $1 \underset{\cdot}{-} 4$ and $3 \underset{\cdot}{-} 6$ or $\alpha[0, 1, 4] — \beta[0, 3, 6]$;

(2) if $1 \dot{-} 5$, $2 \dot{-} 6$ and $\zeta[0, 2, 4] — \theta[0, 3, 5]$ then either $1 \underset{\cdot}{-} 4$ and $3 \underset{\cdot}{-} 6$ or $\alpha[0, 1, 4] — \beta[0, 3, 6]$;

(3) if $1 \dot{-} 5$, $2 \dot{-} 6$, $2 \underset{\cdot}{-} 4$, $3 \underset{\cdot}{-} 5$ then $\alpha[0, 1, 4] — \beta[0, 3, 6]$.

PROPOSITION 3. (SP0) *holds.*

Proof. Consider (1) at first. If $1 \mathbf{+} 4$, then there is $\rho[1, 2, 4]$. We have four cases to study i.e. (a) $\rho — \epsilon[0, 4, 5, 6]$; (b) $\rho — (4, 5)$; (c) $\rho — (4, 6)$; (d) there is 7 with $(4, 5, 6) \neq 7 — (\rho, \epsilon)$. By the method given in [**8**] **S2** and applying Proposition 2 Lemma 7, 8 instead of **L2**, **L4**, **L5** of [**8**] respectively we get $\alpha[0, 1, 4] — \beta[0, 3, 6]$. If $1 \dot{-} 4$, then there is $5'$ on ϵ with $(1, 4, 2, 5')_\Delta$ if $5' = 5$ then we have (1) by means of Lemma 8. If $5' \neq 5$ we get by Proposition 2 $(1, 5, 3, 5')_\Delta$ and further more $(2, 5', 3, 6)_\Delta$. Hence $3 \dot{-} 6$. Similarly $1 \underset{\cdot}{-} 4$ and $3 \underset{\cdot}{-} 6$ are equivalent. Therefore (1) is proved. Now we consider (2). If $1 \mathbf{+} 4$, then we construct β' with $(0, 3, 6') — \beta' — \alpha$. Due to (1) we get $2 \dot{-} 6'$ and thus $6' = 6$. Similarly we get $\alpha — \beta[0, 3, 6]$ for $3 \mathbf{+} 6$, $1 \dot{-} 5$ and $2 \dot{-} 6$. Hence if $1 \dot{-} 5$, $2 \dot{-} 6$ and $1 \underset{\cdot}{-} 4$, then $3 \underset{\cdot}{-} 6$ i.e. (2) is true. Finally if $1 \dot{-} 5$, $2 \dot{-} 6$, $2 \underset{\cdot}{-} 4$ and $3 \underset{\cdot}{-} 5$, then $1 \mathbf{+} 4$. Let $\alpha — \beta'[0, 3, 6']$ and $6' — \epsilon$. By (2) $2 \dot{-} 6'$, which implies $6' = 6$. This completes the proof of (3).

In order to prove that a Miquelian touch plane is a $\mathbf{C}(\mathfrak{A})$ over an algebra $\mathfrak{A}$ one has to show (1) the induced affine plane is Pappian; (2) every chain is either a straight line or a conic; (3) the conics corresponding to the chains are quadradic forms $ax^2 + bxy + cy^2 + \ldots$ with fixed ratio $a : b : c$. Since an affine plane is Pappian if it is Pappian with respect to a certain two lines (G. Pickert [**24**]), Proposition 3 assures the statement (1). (In (SP0) the two lines γ, ϵ are arbitrary except any line of a touching class.) For (2) we use the Steiner's definition of conic, namely, a conic is the set of points of intersection of corresponding lines of two projectively related coplanar pencils of lines. It is necessary to separate the study of the Möbius case from that of the other two. Referring to Figures 21, 22 in order to show $\xi[1, 2, 3, 4, 5, \ldots]$ is a conic we have to prove that $\alpha[0, 1, 3]$, $\beta[0, 1, 4]$, $\gamma[0, 1, 5) \ldots$ are projective to $\alpha'[0, 2, 3]$, $\beta'[0, 2, 4]$, $\gamma'[0, 2, 5] \ldots$ denoted $\alpha\beta\gamma \ldots \barwedge \alpha'\beta'\gamma' \ldots$. To show this projectivity let us take any $6 \neq (1, 2, 3, 4, 5)$ with $6 — \xi$ (if there is no such 6, then ξ is certainly a conic) and take $\delta[0, 4, 6]$ and $\delta'[0, 5, 7]$ with $\delta — \delta'$. 7 may be one of 1, 2, 3, 4, 5, 6. If $\{\alpha, \delta\}$, $\{\beta, \beta''\}$, $\{\gamma, \gamma''\}$ are three pairs of an involution i.e. $\alpha\delta\beta\beta''\gamma\gamma'' \;\overline{\wedge}\; \delta\alpha\beta''\beta\gamma''\gamma$ (Figure 21) and similarly $\{\alpha', \delta\}$, $\{\beta', \beta''\}$, $\{\gamma', \gamma''\}$ (Figure 21) are three pairs of an involution, then the product of the involutions is the projectivity we desire. We now prove $\alpha\delta\beta\beta''\gamma\gamma'' \;\overline{\wedge}\; \delta\alpha\beta''\beta\gamma''\gamma$. Let 8 — ($\beta$[0, 1, 4], θ[0, 3, 5]), 9 — (δ''[0, 8], β''[0, 3, 6]) where δ'' — δ. It is worth emphasizing that we need **S**3 in [**8**] to get $(0, 1, 7, 9)_o$ after applying Proposition 2 to obtain $(1, 3, 8, 9)_o$ (Figure 22). We make σ[0, 1, 7, 9]. Since $\alpha\delta''\beta\beta''\theta\sigma \;\overline{\wedge}\; \delta''\alpha\beta''\beta\sigma\theta$ and $\alpha\delta'\gamma\gamma''\theta\sigma \;\overline{\wedge}\; \delta'\alpha\gamma''\gamma\sigma\theta$ (noting $\delta = \delta' = \delta''$ in the

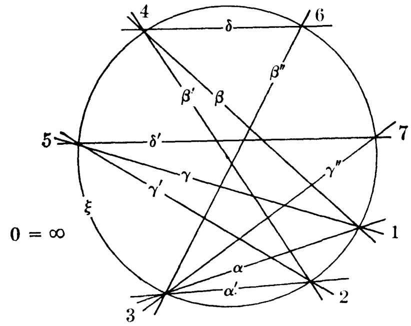

FIGURE 21

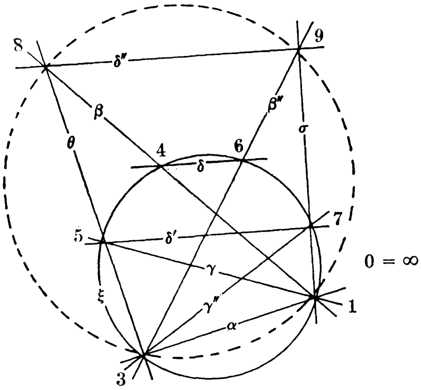

FIGURE 22

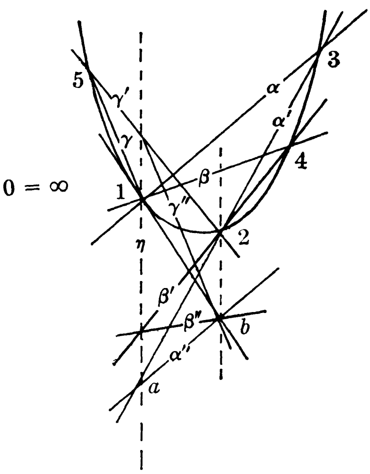

FIGURE 23

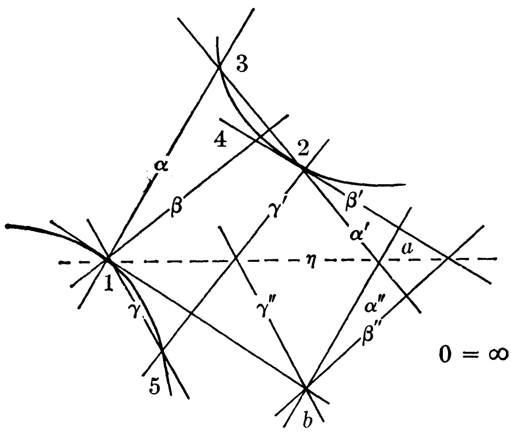

FIGURE 24

sense of projectivity) have two common pairs, they represent the same involution; that is, $\alpha\delta\beta\beta''\gamma\gamma'' \barwedge \delta\alpha\beta''\beta\gamma''\gamma$. For the Laguerre and Minkowski plane we refer to Figures 23, 24. The projectivity $\alpha\beta\gamma \ldots \barwedge \alpha'\beta'\gamma' \ldots$ is obviously the product of perspectivities $\alpha\beta\gamma \ldots \barwedge \alpha''\beta''\gamma'' \ldots$ and $\alpha''\beta''\gamma'' \ldots \barwedge \alpha'\beta'\gamma' \ldots$. The first perspectivity with the improper line as axis and centre 0 mapping 1 to b is due to $\alpha — \alpha''$, $\beta — \beta''$, $\gamma — \gamma'' \ldots$ which are obtained by Lemma 7. The second perspectivity with axis $\eta[0, 1, a]$ and centre 0 maps b to 2 etc. Now we consider (3). If 1, 2, 3, 4 are four points of a chain not passing through 0, where $1 \neq 3$, $2 \neq 3$, then the four lines $\alpha[0, 1, 2]$, $\beta[0, 2, 3]$, $\gamma[0, 3, 4]$, $\delta[0, 4, 1]$ form a secant quadrilateral. On the one hand since three points determine one chain and this by (2) the conic, a quadrilateral determines a chain. On the other hand by Lemma 1(a) two chains with two common points possess the same quadrilaterals and thus their quadratic forms have the same ratio $a : b : c$. For any pair of chains α, β we can find chains γ, δ such that each intersection $\alpha \cap \beta$, $\beta \cap \gamma$, $\gamma \cap \delta$, $\delta \cap \alpha$ has two elements. Hence (3) is true. Therefore we have

THEOREM 3. *Every Miquelian MLM touch-plane* $\mathbf{C}$ *is a* $\mathbf{C}(\mathfrak{A})$.

Combining the above theorem with results of W. Benz etc. we state

THEOREM 4. *For an MLM touch plane* $\mathbf{C}$ *the following conditions are equivalent*

(1) $\mathbf{C}$ *is isomorphic to a* $\mathbf{C}(\mathfrak{A})$;

(2) (SM0) *is valid;*

(3) *There is an automorphism group of G which is sharply transitive with respect to all triples of non-touching points and if* τ *is an element of this group and* $1^\tau = 2$, $2^\tau = 1$, $3^\tau = 4$ *then* $4^\tau = 3$ *and if* $1^\tau — 1$, $2^\tau — 2$, $3^\tau — 3$ *then* $x^\tau — x$ *for all points* x [**2**; **5**; **21**].

(4) *There is a group G. For four points* 1, 2, 3, 4 *such that either* $(1, 2, 3)_0$ *and* $(1, 2, 4)_0$ *or* $(3, 4, 1)_0$ *and* $(3, 4, 2)_0$ *there exists an automorphism mapping* 3 *to* 4 *and keeping* 1, 2 *invariant. This automorphism corresponds to exactly one element of G, denoted by* [1, 2; 3, 4]. *We require* [1, 2; 3, 4] = [3, 4; 1, 2] [**11**].

W. Benz [**1**] characterized the Möbius planes by a "cross ratio" function. This can be generalized for all three geometries of chains considered here [**10**]. Another essentially different characterization is given by means of a polarity or a quasipolarity, as done by G. Ewald [**12**] for Möbius plane, A. Uhl (**26**] for Laguerre plane and the present author [**9**] for Minkowski plane. The field in the former two cases is Euclidean and in the last case is of characteristic unequal to two.

As in [**8**] one can consider Proposition 2 as a degenerated proposition of Miquel and find all other possible degenerations and prove after defining a Miquelian dice etc. the generalized complete proposition of Miquel ((VMM) in [**8**]). Since we are dealing with $(1, 2, 3, 4)_\Delta$ instead of $(1, 2, 3, 4)_0$ the conditions here are somewhat more complicated than those in [**8**]. It seems not worth publishing them.

Remark at revision. In regard to a problem given in the paragraph behind Definition 5 we remark that we have in the meantime succeeded in proving the following equivalent properties in an MLM-touch plane **C** without using coordinate: (1) (SM0), (2) Angle axioms (W. Benz [**3**], G. Kaerlein [**18**], L. J. Smid [**25**]), (3) The existences of an automorphism group stated in Theorem 4 (3). This result will be published in the forthcoming article: *Involutions in the geometries of chains.*

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