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LATTICES WITH DOUBLY IRREDUCIBLE ELEMENTS

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Introduction. An element x in a lattice L is *join-reducible* (*meet-reducible*) *in L* if there exist $y, z \in L$ both distinct from x such that $x=y \vee z$ ($x=y \wedge z$); x is *join-irreducible* (*meet-irreducible*) *in L* if it is not join-reducible (meet-reducible) in L; x is *doubly irreducible in L* if it is both join- and meet-irreducible in L. Let $J(L)$, $M(L)$, and $\mathrm{Irr}(L)$ denote the set of all join-irreducible elements in L, meet-irreducible elements in L, and doubly irreducible elements in L, respectively, and $\ell(L)$ the *length of L*, that is, the order of a maximum-sized chain in L minus one.

In this paper we investigate some combinatorial properties of lattices in terms of their doubly irreducible elements. First, we show (Theorem 1) that any lattice L of finite length satisfies $|L|\geq 2(\ell(L)+1)-|\mathrm{Irr}(L)|$, an inequality which, among all lattices L of finite length such that $\mathrm{Irr}(L)=\varnothing$, is best possible. This inequality is in turn useful in the computation (Corollary 1) of orders of sublattices of "small" lattices.

Next, we examine and characterize (Theorem 2) *dismantlable* lattices, that is, lattices which can be completely "dismantled" by removing one element at a time leaving a sublattice at each stage. All finite planar lattices are *dismantlable* [1]; furthermore, given a positive integer n, any large enough lattice ($|L|\geq n^{3^n}$ will do [3] [2, p. 67]) contains a *dismantlable* sublattice with precisely n elements.

Finally, if $\mathrm{Sub}(L)$ denotes the lattice of all sublattices of a lattice L, we show that every lattice L such that $\ell(\mathrm{Sub}(L))$ is finite satisfies $\ell(\mathrm{Sub}(L))=|\mathrm{Irr}(L)|+\ell(\mathrm{Sub}(L-\mathrm{Irr}(L)))$.

An inequality. Let C be a chain of maximum order in a lattice L of finite length and $x_1<x_2<\cdots<x_n$ a labelling of C. Since every element in a lattice of finite length can be represented as a join of all the join-irreducibles that it contains, there is a one-one choice function f from C into $J(L)$ defined as follows: $f(x_1)\leq x_1$; $f(x_i)\leq x_i$ and $f(x_i)\not\leq x_{i-1}$ for every $i=2, 3, \ldots n$. Thus, $|J(L)|\geq|C|$; dually, we have that $|M(L)|\geq|C|$. Combining these inequalities with the fact that $|L|\geq|J(L)|+|M(L)|-|\mathrm{Irr}(L)|$ establishes

THEOREM 1. *Every lattice L of finite length satisfies the inequality* $|L|\geq 2(\ell(L)+1)-|\mathrm{Irr}(L)|$.

Among all lattices L of finite length such that $\mathrm{Irr}(L)=\varnothing$ this inequality is best possible in the sense that for every interer $n\geq 3$ there is a lattice L_n such that $\mathrm{Irr}(L_n)=\varnothing$, $\ell(L_n)=n$ and $|L_n|=2(\ell(L)+1)$ (*see Figure* 1).

Once we observe that $L-A$ is a sublattice of L for every $A\subseteq\mathrm{Irr}(L)$ the following corollary is immediate.

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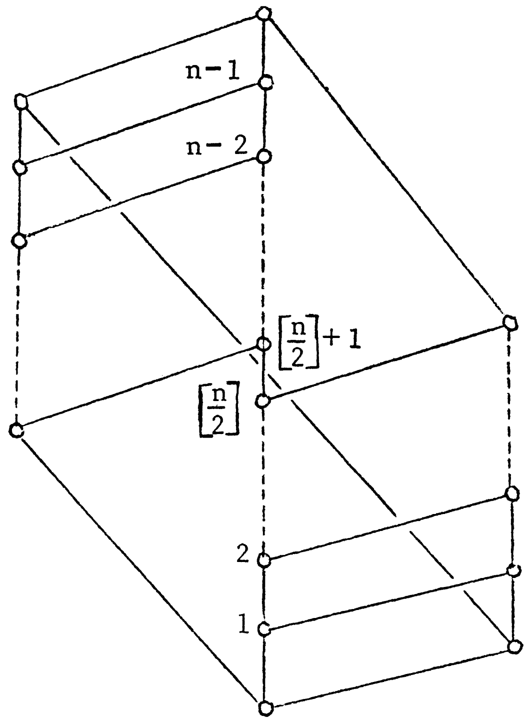

L_n $(n \geq 3)$

Figure 1

COROLLARY 1. *If n is a positive integer and L is a lattice of finite length satisfying* $|L| \leq 2(\ell(L)+1)-n$ *then there is a chain* $S_n \subset S_{n-1} \subset \cdots \subset S_0 = L$ *of sublattices of L such that* $|S_i| = |S_{i-1}| - 1$ *for every* $i = 1, 2, \ldots, n$.

Dismantlable lattices. With every finite lattice L we can associate a family of sublattices defined as follows: $L_0 = L$; $L_i = L_{i-1} - \mathrm{Irr}(L_{i-1})$ for $i = 1, 2, \ldots$. (Note that $\mathrm{Irr}(L_i) \cap \mathrm{Irr}(L_j) = \varnothing$ if $i \neq j$.) In this way we obtain a descending chain $L = L_0 \supset L_1 \supset \cdots$ of sublattices of L which, since L is finite, must end; that is, there is a smallest integer n such that either $L_n = \varnothing$ or $\mathrm{Irr}(L_n) = \varnothing$. A finite lattice L is *dismantlable* if there is an integer n such that $L_n = \varnothing$ (or equivalently, $L = \bigcup_{i=0}^{n} \mathrm{Irr}(L_i)$).

It was shown in [1] that every finite planar lattice has a doubly irreducible element. Since, plainly, any sublattice of a planar lattice is planar, it follows that every finite planar lattice is dismantlable. On the other hand, the lattice of Figure 2 illustrates that not every dismantlable lattice is planar.

If $|L| \leq 5$ it is easy to verify that L is dismantlable. Now suppose that $|L| = 6$. If $\ell(L) \leq 2$ then certainly L is dismantlable; if $\ell(L) \geq 3$ then by Corollary 1, L has a 5-element sublattice (which is dismantlable) so that L is dismantlable. If $|L| = 7$ a

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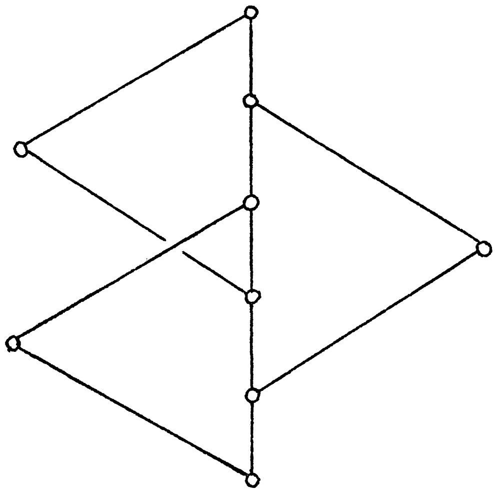

Figure 2

similar argument shows that L is dismantlable. However, for every integer $n \geq 8$ there is a lattice of order n which is not dismantlable (for example, the ordinal sum of the Boolean lattice $\mathbf{2}^3$ with a chain of order $n-8$).

G. Havas and M. Ward [3] have shown that any lattice L such that $|L| \geq n^{3^n}$ contains a sublattice of order n. In fact, their proof shows that if $|L| \geq n^{3^n}$ then L contains a *dismantlable* sublattice of order n (cf. [2, p. 67]).

THEOREM 2. *For a finite lattice L the following conditions are equivalent:*

(i) *L is dismantlable;*

(ii) *$\ell(\mathrm{Sub}(L)) = |L|$;*

(iii) *$\mathrm{Irr}(S) \neq \varnothing$ for every sublattice S of L;*

(iv) *for every chain C in L there is a positive integer n and a chain $C = S_0 \subset S_1 \subset \cdots \subset S_n = L$ of sublattices of L such that $|S_i| = |S_{i-1}| + 1$ for every $i = 1, 2, \ldots, n$.*

We shall need the following lemma.

LEMMA 1. *Let C be a maximal chain in a lattice L of finite length and S a subset of L disjoint from $\mathrm{Irr}(L) \cap C$. Then S is a sublattice of $L - (\mathrm{Irr}(L) \cap C)$ containing $C - (\mathrm{Irr}(L) \cap C)$ if and only if $S \cup (\mathrm{Irr}(L) \cap C)$ is a sublattice of L containing C.*

Proof. The "if" part is obvious. Let S be a sublattice of $L - (\mathrm{Irr}(L) \cap C)$ containing $C - (\mathrm{Irr}(L) \cap C)$. It suffices to show that for every $x \in \mathrm{Irr}(L) \cap C$ and $y \in S$ such that x is incomparable with y, $x \vee y$, $x \wedge y \in S \cup (\mathrm{Irr}(L) \cap C)$. Now take $x = x_0 \prec x_1 \prec \cdots \prec x_r = x \vee y$ to be a covering chain between x and $x \vee y$ (x_i covers x_{i-1} for every $i = 1, 2, \ldots, r$). Since x is doubly irreducible in L, x_1 is its unique cover and since C is a maximal chain, $x_1 \in C$. If x_1 is not doubly irreducible in L then $x_1 \in C - (\mathrm{Irr}(L) \cap C)$, otherwise $x_2 \in C$. Iterating, there exists a positive integer $i \leq r$ such that $x_i \in C-(\mathrm{Irr}(L) \cap C)$. Thus, $x \vee y \leq x_i \vee y \leq x \vee y$ and since $x_i, y \in S$ we have that $x \vee y = x_i \vee y \in S$. A dual argument shows that $x \wedge y \in S$.

Proof of Theorem 2. That each of (ii), (iii), and (iv) implies (i) is obvious, as is (i) implies (ii).

(i) implies (iii): Let S be an arbitrary sublattice of a dismantlable lattice L. We show that $\mathrm{Irr}(S) \neq \varnothing$. Let m be the smallest integer such that $S \cap (\bigcup_{i=0}^{m} \mathrm{Irr}(L_i)) \neq \varnothing$. If x is join-reducible in S then there exist $y, z \in S$ both distinct from x such that $x = y \vee z$. Now if $y \in \mathrm{Irr}(L_i)$ and $z \in \mathrm{Irr}(L_j)$, for $i, j \geq m$, then $y, z \in L_m$, which is impossible since $x \in \mathrm{Irr}(L_m)$. Otherwise, either $i < m$ or $j < m$, which, however, contradicts the minimality of m. In any case then, x must be join-irreducible in S and dually, x must be meet-irreducible in S, that is, $x \in \mathrm{Irr}(L)$.

(i) implies (iv): Let C be a chain in a dismantlable lattice L. Without loss of generality we may take C to be a maximal chain in L. We proceed by induction on $|L|$. By assumption $\mathrm{Irr}(L) \neq \varnothing$.

If $\mathrm{Irr}(L) \cap C = \varnothing$ and $x \in \mathrm{Irr}(L)$ then clearly $L-\{x\}$ is a dismantlable sublattice of L containing C. Applying the inductive hypothesis to $L-\{x\}$ we are done.

If $\mathrm{Irr}(L) \cap C \neq \varnothing$ then $L-(\mathrm{Irr}(L) \cap C)$ is a dismantlable sublattice of L. Now take B a maximal chain in $L-(\mathrm{Irr}(L) \cap C)$ containing $C-(\mathrm{Irr}(L) \cap C)$. Applying the inductive hypothesis we get a chain $B = S'_0 \subset S'_1 \subset \cdots \subset S'_m = L-(\mathrm{Irr}(L) \cap C)$ of sublattices of L such that $|S'_i| = |S'_{i-1}|+1$ for every $i = 1, 2, \ldots, m$. Now let $B - C = \{b_1, b_2, \ldots, b_k\}$ ($B-C$ may be empty) and define a chain of subsets of L as follows: $S_0 = C$; $S_j = C \cup \{b_1, b_2, \ldots, b_j\}$ for every $j = 1, 2, \ldots, k$; $S_{k+i} = S_k \cup S'_i$ for every $i = 1, 2, \ldots, m$. Finally, in view of Lemma 1, $S_0, S_1, \ldots, S_{k+m}$ are all sublattices of L. The proof of the theorem is now complete.

COROLLARY, 2. *Every sublattice and epimorphic image of a dismantlable lattice is dismantlable.*

Proof. The first part follows at once from Theorem 2(iii).

That epimorphic images of a dismantlable lattice are dismantlable we prove in the more convenient terminology of congruence relations. Let L be dismantlable and Θ be a congruence relation on L. We show that the quotient L/Θ is dismantlable. Since every sublattice of L/Θ is of the form S/Θ_S, where S is a sublattice of L and Θ_S is the restriction of Θ to S, it suffices by Theorem 2(iii) to prove that $\mathrm{Irr}(S/\Theta_S) \neq \varnothing$ for every sublattice S of L. This we do by induction on $|S|$.

Let S be a sublattice of L. By the first part S is dismantlable so in particular there is an $x \in \mathrm{Irr}(S)$. Again $S-\{x\}$ is a sublattice of L and therefore, by the inductive hypothesis $\mathrm{Irr}(S-\{x\}/\Theta_{S-\{x\}}) \neq \varnothing$. If the congruence class $[x]\Theta_S$ has at least two elements then $S/\Theta_S \simeq S-\{x\}/\Theta_{S-\{x\}}$ and we are done. Otherwise $[x]\Theta_S = \{x\}$. If $[x]\Theta_S = [y]\Theta_S \vee [z]\Theta_S$, where $y, z \in S$, then $x \equiv y \vee z(\Theta_S)$ which implies that $x = y \vee z$. But $x \in \mathrm{Irr}(S)$ so that $x = y$ or $x = z$, that is, $[x]\Theta_S = [y]\Theta_S$ or $[x]\Theta_S = [z]\Theta_S$. Thus, $[x]\Theta_S$ is join-irreducible in S/Θ_S, and by a dual argument, $[x]\Theta_S$ is meet-irreducible in S/Θ_S as well. Thus, $\mathrm{Irr}(S/\Theta_S) \neq \varnothing$ and the induction is complete.

REMARK. If L is a dismantlable lattice then there is a positive integer n such that $L=\bigcup_{i=0}^{n}\mathrm{Irr}(L_i)$ and in fact, $\ell(\mathrm{Sub}(L))=|\bigcup_{i=0}^{n}\mathrm{Irr}(L_i)|$. An analogous result holds in a more general context.

Any lattice L such that $\ell(\mathrm{Sub}(L))$ is finite satisfies $\ell(\mathrm{Sub}(L))=|\mathrm{Irr}(L)|+\ell(\mathrm{Sub}(L-\mathrm{Irr}(L)))$.

We show by induction on $\ell(\mathrm{Sub}(L))$ that if $\mathrm{Irr}(L)\neq\varnothing$ then $\ell(\mathrm{Sub}(L))=1+\ell(\mathrm{Sub}(L-\{x\}))$ for every $x\in\mathrm{Irr}(L)$. Observe that

$$\ell(\mathrm{Sub}(L)) = 1+\max(\ell(\mathrm{Sub}(M))\mid M \text{ maximal proper sublattice of } L). \tag{1}$$

Suppose that the maximum in (1) is attained by some maximal proper sublattice M which is not of the form $L-\{x\}$ where $x\in\mathrm{Irr}(L)$. Since M is maximal $\mathrm{Irr}(L)\subseteq M$. In particular, $\mathrm{Irr}(L)\subseteq\mathrm{Irr}(M)$ and $\mathrm{Irr}(M)\neq\varnothing$. By the inductive hypothesis

$$\ell(\mathrm{Sub}(M)) = 1+\ell(\mathrm{Sub}(M-\{x\})) \quad \text{for every } x\in\mathrm{Irr}(M). \tag{2}$$

Now if x is an arbitrary doubly irreducible element in L, $M-\{x\}\subset L-\{x\}$, so that

$$\ell(\mathrm{Sub}(M-\{x\})) \leq \ell(\mathrm{Sub}(L-\{x\}))-1. \tag{3}$$

Combining (2) and (3), and bearing in mind the choice of M in (1) we get that $\ell(\mathrm{Sub}(M))=\ell(\mathrm{Sub}(L-\{x\}))$ and we are done.

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