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Abstract

This paper gives the classification of the Whittaker unitary dual for affine graded Hecke algebras of type E. By the Iwahori–Matsumoto involution, this is also equivalent to the classification of the spherical unitary dual for type E. Together with some results of Barbasch and Moy (D. Barbasch and A. Moy, *Unitary spherical spectrum for p-adic classical groups*, Acta Appl. Math. **44** (1996), 3–37; D. Barbasch, *The spherical unitary spectrum of split classical real and p-adic groups*, Preprint (2006), math/0609828) and Ciubotaru (D. Ciubotaru, *The Iwahori spherical unitary dual of the split group of type F4*, Represent. Theory 9 (2005), 94–137), this work completes the classification of the Whittaker Iwahori-spherical unitary dual or, equivalently, the spherical unitary dual of any split p-adic group.

1. Introduction

1.1 The present paper completes the classification of the unitary representations with Iwahori-fixed vectors and is *generic* (i.e. admitting Whittaker models) for split linear algebraic groups over p-adic fields by treating the groups of type E.

The full unitary dual for $\mathrm{GL}(n)$ was obtained in [Tad86], and for G_2 in [Mui97]. The Whittaker unitary dual with Iwahori-fixed vectors for classical split groups was determined in [BM96, Bar08]. For F_4, this is part of [Ciu05]. Using different methods, the Whittaker unitary dual for classical quasi-split groups over p-adic fields was identified in [LMT04].

It is well known that the category of representations with Iwahori fixed vectors admits an involution called the Iwahori–Matsumoto involution, denoted by IM, which takes hermitian modules to hermitian modules, and unitary modules to unitary modules [BM89]. In particular, it interchanges spherical modules with generic modules. For example, IM takes the trivial representation into the Steinberg representation. Thus, this paper also gives a classification of the spherical unitary dual of split p-adic groups of type E, completing the classification of the spherical unitary dual as well.

Let $\mathcal{G}$ be a split ($\mathbb{F}$-form of a) reductive linear algebraic group over a p-adic field $\mathbb{F}$ of characteristic zero. Recall that $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$, where $\mathcal{R}$ is the ring of integers and $\mathcal{P}$ the maximal prime ideal. Assume that the residue field $\mathcal{R}/\mathcal{P}$ has q elements. The group $\mathcal{G}$ is defined over $\mathcal{R}$, and we fix a hyperspecial maximal compact subgroup $\mathcal{K} = \mathcal{G}(\mathcal{R})$. Let $\mathcal{I}$ be an Iwahori subgroup, $\mathcal{I} \subset \mathcal{K}$. Fix also a rational Borel subgroup $\mathcal{B} = \mathcal{H}\mathcal{N}$, such that $\mathcal{G} = \mathcal{K}\mathcal{B}$. An admissible representation (π, V) is called *spherical* if $V^{\mathcal{K}} \neq (0)$. It is called *Iwahori-spherical* if $V^{\mathcal{I}} \neq (0)$.

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Let us first describe the philosophy behind our classification of unitary $\mathcal{G}$-representations. In terms of Weil homomorphisms, the Iwahori-spherical representations are parameterized by admissible maps

$$\Phi : \mathcal{W}_{\mathbb{F}}/I_{\mathbb{F}} \times \mathrm{SL}(2,\mathbb{C}) \to G, \tag{1.1.1}$$

where G is the complex group dual to $\mathcal{G}$ and $\mathcal{W}_{\mathbb{F}}$ is the Weil group with inertia group $I_{\mathbb{F}}$. Define

$$e = \log \Phi\left(\exp\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right), \quad \sigma_0 = \Phi\left(\exp\begin{pmatrix} \frac{\log q}{2} & 0 \\ 0 & -\frac{\log q}{2}\end{pmatrix}\right), \quad \sigma = \tau \cdot \sigma_0 \quad \text{where } \tau = \Phi(\varpi), \tag{1.1.2}$$

for some Frobenius element $\varpi \in \mathcal{W}_{\mathbb{F}}/I_{\mathbb{F}}$. The elements σ, τ are semisimple in G, and $e \in \mathfrak{g}_q := \{x \in \mathfrak{g} \mid \mathrm{Ad}(\sigma)x = qx\}$, where $\mathfrak{g}$ is the Lie algebra of G. Assume that G is simply connected and that Φ is real, that is, τ is a hyperbolic element (modulo the center) of G. The results in [BM89, BM93] allow a reduction for unitary representations to this setting. The centralizer of σ in G is denoted by $G(\sigma)$. This is a connected group and it acts with finitely many orbits on $\mathfrak{g}_q$. By [KL87], the irreducible Iwahori-spherical $\mathcal{G}$-representations are parameterized by G-conjugacy classes of triples (σ, e, ψ), with σ, e as above, and ψ certain representations (of 'Springer type') of the group of components $A_G(\Phi) = G(\Phi)/G(\Phi)^0$ of the centralizer $G(\Phi)$ of the image of Φ in G. By [BM94, Ree94], a representation parameterized by a triple (σ, e, ψ) is Whittaker-generic if and only if $\psi = \mathsf{triv}$ and e is in the unique open dense orbit of $G(\sigma)$ in $\mathfrak{g}_q$. Let L_Φ denote the centralizer in G of $\Phi(\{1\} \times \mathrm{SL}(2,\mathbb{C}))$. It is well known that this is a (potentially disconnected) reductive group. Let L_Φ^0 be its identity component. Clearly, we have $\tau \in L_\Phi$, and since τ was assumed hyperbolic, $\tau \in L_\Phi^0$. Thus, one may attach to Φ the following Weil homomorphism

$$\widetilde{\Phi} : \mathcal{W}_{\mathbb{F}}/I_{\mathbb{F}} \times \mathrm{SL}(2,\mathbb{C}) \to L_\Phi^0, \quad \widetilde{\Phi}|_{\mathcal{W}_{\mathbb{F}}/I_{\mathbb{F}}} = \Phi|_{\mathcal{W}_{\mathbb{F}}/I_{\mathbb{F}}} \quad \text{and} \quad \widetilde{\Phi}|_{\mathrm{SL}(2,\mathbb{C})} = 1. \tag{1.1.3}$$

Then $\widetilde{\Phi}$ parameterizes a spherical representation for the split group $\mathcal{L}_\Phi^0$ whose dual is L_Φ^0. Since Φ and therefore $\widetilde{\Phi}$ are assumed real, the centralizer of the image of $\widetilde{\Phi}$ in L_Φ^0 is connected even though L_Φ^0 may not be simply connected. If there exists nonzero $e' \in \mathfrak{l}_\Phi = \mathsf{Lie}(L_\Phi)$, such that $\mathrm{Ad}(\tau)e' = qe'$, then we would have $e + te' \in \mathfrak{g}_q$, for all $t \in \mathbb{C}$. So if e is in the dense orbit in $\mathfrak{g}_q$, then necessarily the q-eigenspace of $\mathrm{Ad}(\tau)$ in $\mathfrak{l}_\Phi$ is zero. This means that if Φ corresponds to a Whittaker-generic $\mathcal{G}$-representation, then $\widetilde{\Phi}$ must also be Whittaker-generic (and spherical) for $\mathcal{L}_\phi^0$. (The converse is false however.)

Our main result, Theorem 1.1, implies that almost always (and the few exceptions are determined explicitly), the generic $\mathcal{G}$-representation parameterized by Φ is unitary if and only if the generic spherical $\mathcal{L}_\Phi^0$-representation parameterized by $\widetilde{\Phi}$ is unitary. In addition, we give an explicit description of the generic spherical parameters for any split p-adic group, which has interesting combinatorial properties. One would naturally like to have the unitary correspondence $\Phi \to \widetilde{\Phi}$ as part of a correspondence for the larger classes of hermitian representations. However, for this, one would have to consider nonconnected groups $\mathcal{L}_\Phi$ dual to L_Φ instead of $\mathcal{L}_\Phi^0$ and L_Φ^0. This is the role played by the extended Hecke algebras that we consider in this paper (see § 4).

1.2 We recall the well-known classification of irreducible admissible spherical modules. For every irreducible spherical representation π, there is a character $\chi \in \widehat{\mathcal{H}}$ such that $\chi|_{\mathcal{H}\cap\mathcal{K}} = \mathsf{triv}$, and π is the unique spherical subquotient $L(\chi)$ of $X(\chi) = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}[\chi \otimes \mathbf{1}]$. A character χ whose restriction to $\mathcal{H} \cap \mathcal{K}$ is trivial is called *unramified*. Two modules $L(\chi)$ and $L(\chi')$ are equivalent if and only if

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there is an element in the Weyl group W such that $w\chi = \chi'$. A module $L(\chi)$ admits an invariant hermitian form if and only if there exists w such that $w\chi = \overline{\chi}^{-1}$.

More generally, by a theorem of Casselman [Cas80], an irreducible $\mathcal{G}$-representation is Iwahori-spherical if and only if it is a subquotient of an unramified $X(\chi)$. Every $X(\chi)$ has a unique irreducible subquotient which is Whittaker-generic [Rod73]. When $\mathcal{G}$ is adjoint, a subquotient is both generic and spherical if and only if it is the full $X(\chi)$, in other words, if $X(\chi)$ is irreducible [BM94].

The main theorem in [BM89] shows that in the p-adic case the classification of the Iwahori-spherical unitary dual is equivalent to the corresponding problem for the Iwahori–Hecke algebra, under the assumption of *real infinitesimal character*. In [BM93], the problem is reduced to computing the unitary dual affine graded (Iwahori–)Hecke algebras and real infinitesimal character.

We review these notions later in the paper, for now we recall the notion of real infinitesimal character. A character χ is called *real* if it takes only positive real values. An irreducible representation π is said to have real infinitesimal character if it is the subquotient of an $X(\chi)$ with χ real. So we start by parameterizing real unramified characters of $\mathcal{H}$. Since $\mathcal{G}$ is split, $\mathcal{H} \cong (\mathbb{F}^\times)^n$ where n is the rank. Define

$$\mathcal{L}(\mathcal{H})^*_{\mathbb{R}} := X^*(\mathcal{H}) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{1.2.1}$$

where $X^*(\mathcal{H})$ is the lattice of characters of the algebraic torus $\mathcal{H}$. Each element $\nu \in \mathcal{L}(\mathcal{H})^*_{\mathbb{R}}$ defines an unramified character χ_ν of $\mathcal{H}$, determined by the formula

$$\chi_\nu(\tau(a)) = |a|^{\langle \tau, \nu \rangle}, \quad a \in \mathbb{F}^\times, \tag{1.2.2}$$

where τ is an element of the lattice of one-parameter subgroups $X_*(\mathcal{H})$. Since the torus is split, each element of $X^*(\mathcal{H})$ can be regarded as a homomorphism of $\mathbb{F}^\times$ into $\mathcal{H}$. The pairing in the exponent in (1.2.2) corresponds to the natural identification of $\mathcal{L}(\mathcal{H})^*_{\mathbb{R}}$ with $\mathrm{Hom}[X_*(\mathcal{H}), \mathbb{R}]$. The map $\nu \longrightarrow \chi_\nu$ from $\mathcal{L}(\mathcal{H})^*_{\mathbb{R}}$ to real unramified characters of $\mathcal{H}$ is an isomorphism. We often identify the two sets writing simply $\chi \in \mathcal{L}(\mathcal{H})^*_{\mathbb{R}}$.

As we will be dealing exclusively with the graded affine Hecke algebra $\mathbb{H}$ (as in [Lus89]) which is defined in terms of the complex dual group, we denote by G the complex group dual to $\mathcal{G}$, and let H be the torus dual to $\mathcal{H}$. Then the real unramified characters χ are naturally identified with hyperbolic elements of the Lie algebra $\mathfrak{h}$. The infinitesimal characters are identified with orbits of hyperbolic elements (§ 2.1). We assume that all characters are real.

1.3 Next we explain in more detail the nature of our classification of the Whittaker unitary dual in the equivalent setting of affine Hecke algebras.

We attach to each χ a nilpotent orbit $\mathcal{O} = \mathcal{O}(\chi)$ satisfying the following properties. Fix a Lie triple $\{e, h, f\}$ corresponding to $\mathcal{O}$ such that $h \in \mathfrak{h}$. We write $Z(e, h, f)$, respectively $\mathfrak{z}(e, h, f)$, for the centralizer of $\{e, h, f\}$ in G, respectively $\mathfrak{g}$, and abbreviate them as $Z(\mathcal{O})$, respectively $\mathfrak{z}(\mathcal{O})$. Then $\mathcal{O}$ is such that

$$\begin{aligned} &(1) \text{ there exists } w \in W \text{ such that } w\chi = \tfrac{1}{2}h + \nu \text{ with } \nu \in \mathfrak{z}(\mathcal{O}),\\ &(2) \text{ if } \chi \text{ satisfies property (1) for any other } \mathcal{O}', \text{ then } \mathcal{O}' \subset \overline{\mathcal{O}}. \end{aligned} \tag{1.3.1}$$

The results in [BM89] guarantee that for any χ there is a unique $\mathcal{O}(\chi)$ satisfying properties (1) and (2) above. Another characterization of the orbit $\mathcal{O} = \mathcal{O}(\chi)$ is as follows. Set

$$\mathfrak{g}_1 := \{x \in \mathfrak{g} \mid [\chi, x] = x\}, \quad \mathfrak{g}_0 := \{x \in \mathfrak{g} \mid [\chi, x] = 0\}. \tag{1.3.2}$$

Then G_0, the complex Lie group corresponding to the Lie algebra $\mathfrak{g}_0$, has an open dense orbit in $\mathfrak{g}_1$. The G-saturation in $\mathfrak{g}$ of this orbit is $\mathcal{O}$.

Every generic module of $\mathbb{H}$ (and every spherical module of $\mathbb{H}$) is uniquely parameterized by a pair $(\mathcal{O}, \nu)$, $\mathcal{O} = \mathcal{O}(\chi)$ as in (1.3.1). In order to make this connection more precise, we need to recall the geometric classification of irreducible $\mathbb{H}$-modules [KL87, Lus95], and we postpone this to § 2.3. (See, in particular, Remark 2.1.)

Remark. The pair $(\mathcal{O}, \nu)$ has remarkable properties. For example, if $\nu = 0$ $(\chi = h/2)$, then the generic representation parameterized by $(\mathcal{O}, 0)$ is tempered, therefore unitary. The corresponding spherical module $L(h/2)$ is one of the parameters that the conjectures of Arthur predict to play a role in the residual spectrum. In particular, $L(h/2)$ should be unitary. This is true because it is the Iwahori–Matsumoto dual (Definition 2.2) of the generic tempered module.

Definition 1.1. The spherical modules $L(h/2)$ will be called spherical unipotent representations.

In our main result, Theorem 1.1, the tempered generic modules can be regarded as the building blocks of the Whittaker unitary dual. In the spherical unitary dual, this role is played by the spherical unipotent modules.

We partition the Whittaker (equivalently, the spherical) unitary dual into *complementary series* attached to nilpotent orbits. We say that an infinitesimal character χ as above is unitary if the generic module parameterized by χ (equivalently, the spherical $L(\chi)$) is unitary.

Definition 1.2. Fix a nilpotent G-orbit $\mathcal{O} \subset \mathfrak{g}$. The (generic or spherical) *$\mathcal{O}$-complementary series* is the set of unitary parameters χ such that $\mathcal{O} = \mathcal{O}(\chi)$ as in (1.3.1). The complementary series for the trivial nilpotent orbit is called the *0-complementary series*.

Our first result is the identification of 0-complementary series for type E in § 3. These are the irreducible principal series $X(\chi)$ which are unitary. (For a summary of the relevant results for classical groups from [Bar08], and G_2, F_4, from [Ciu05], see §§ 3.2 and 3.1.) The 0-complementary series have a nice explicit combinatorial description: they can be viewed as a union of alcoves in the dominant Weyl chamber of $\mathfrak{h}$, where the number of alcoves is a power of two, e.g. in G_2 there are two, in E_7 there are eight and in E_8 there are 16. The explicit description of the alcoves is in §§ 7.2.1–7.2.3.

The main result of the paper is the description of the complementary series for all $\mathcal{O}$ in type E, and can be summarized as follows. We use the Bala–Carter notation for nilpotent orbits in exceptional complex semisimple Lie algebras (see [Car85]).

Definition 1.3. Set

$$\mathsf{Exc} = \{\underbrace{A_1 + \widetilde{A}_1}_{\text{in } F_4}, \underbrace{A_2 + 3A_1}_{\text{in } E_7}, \underbrace{A_4 + A_2 + A_1, A_4 + A_2, D_4(a_1) + A_2, A_3 + 2A_1, A_2 + 2A_1, 4A_1}_{\text{in } E_8}\}. \tag{1.3.3}$$

Recall that $\mathfrak{z}(\mathcal{O})$ denotes the reductive centralizer in $\mathfrak{g}$ of a fixed Lie triple for $\mathcal{O}$.

Theorem 1.1. *Let $\mathbb{H}$ be the affine graded Hecke algebra for G (definition in § 2.1), and $\mathcal{O}$ be a nilpotent G-orbit in the complex Lie algebra $\mathfrak{g}$. Denote by $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ the affine graded Hecke algebra constructed from the root system of $\mathfrak{z}(\mathcal{O})$.*

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Assume that $\mathcal{O} \notin \mathsf{Exc}$ (see (1.3.3)). A (real) parameter $\chi = \frac{1}{2}h + \nu$ is in the complementary series of $\mathcal{O}$ (Definition 1.2) if and only if ν is in the 0-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

The explicit description of the complementary series, including when $\mathcal{O} \in \mathsf{Exc}$, are tabulated in § 7.

The complementary series for $\mathcal{O} \in \mathsf{Exc}$ are smaller than the corresponding 0-complementary series for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, except when $\mathcal{O} = 4A_1$ in E_8. For this one orbit, the complementary series is larger (see § 6.4.1).

The proof of the theorem for G of classical type is in [Bar08]. For types G_2 and F_4, this is part of [Ciu05]. In the present paper, we treat the case of groups of type E. The method is different from the above-mentioned papers. The main method of the proof (Proposition 5.3) consists of a *direct* comparison between the signature of hermitian forms on the generic modules for $\mathbb{H}$ parameterized in the geometric classification (see § 2.3) by $\mathcal{O}$, and the signature of hermitian forms on the spherical principal series which are irreducible (that is, representations which are both spherical and generic) for the Hecke algebra $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. This method of comparing signatures has the advantage that it explains the match-up of complementary series in Theorem 1.1. It often extends to non-generic modules (see, e.g., [Ciu062] for non-generic modules of E_6).

1.4 If one assumes the infinitesimal character (the χ above) to be real, one can use the same set for the parameter spaces for the spherical dual of a real and p-adic split group (attached to the same root datum). The main criterion for ruling out nonunitary modules is the computation of signature characters: in the real case on K-types, and in the p-adic case on W-types. So it is natural to try to compare signatures on K-types and W-types. In [Bar08, Bar04], the notion of *petite K-types* was used to transfer the results about signatures from the p-adic split group to the corresponding real split group. The methods employed there are very different from this paper. More precisely, to every petite K-type there corresponds a Weyl group representation such that the signature characters are the same. In this paper, we inherently use signature computations for *all* Weyl group representations. Since not all W-representations are known to arise from petite K-types, the results here cannot be used directly towards the spherical unitary dual of the corresponding real groups.

In different work, however, we studied the signature of petite K-types for exceptional groups of type E. The main consequence of that work is that the spherical unitary dual for a split real group $\mathcal{G}(\mathbb{R})$ is a subset of the spherical unitary dual for the corresponding p-adic group $\mathcal{G}(\mathbb{F})$ (conjecturally they are the same). Details will appear elsewhere.

1.5 To obtain the results of this paper, we made a minimal use of computer calculations, essentially for linear algebra, e.g. conjugation of semisimple elements by the Weyl group, or multiplication of matrices in a variable ν for some of the 'maximal parabolic' cases in § 5.3.

However, by the machinery presented in §§ 3.1 and 3.2, for every given Hecke algebra $\mathbb{H}$, one can reduce the identification of the unitary parameters χ to a brute-force computer calculation. More precisely, one considers sample points with rational coordinates for every facet in the arrangement of hyperplanes given by coroots equal to one in the dominant Weyl chamber. (These are the hyperplanes where $X(\chi)$ is reducible.) It is known (see [BC05]) that the signature of the long intertwining operator is constant on each facet of this arrangement.

One can then run a computer calculation of the long intertwining spherical operator (§ 3.1) on each representation of the Weyl group at every sample point. Then one finds the signature of the resulting hermitian matrices. The unitary parameters χ correspond to those facets for which these matrices are positive semidefinite for all Weyl group representations. The size of the calculation can be reduced significantly by making use of some ideas in this paper. This is not the approach we use in this paper, but we did carry out this calculation independently for exceptional groups in order to confirm the results of this paper.

1.6 We give an outline of the paper. In § 2, we review the relevant notions about the affine graded Hecke algebra, and its representations. We introduce the construction of intertwining operators that we need for the signature computations. In § 3, we restrict to the setting of modules which are both generic and spherical, and determine the 0-complementary series. In § 4, we describe a construction of extended Hecke algebras for disconnected groups, and apply it to the setting of centralizers of nilpotent orbits. Section 5 contains the main results of the paper, Theorem 5.1, Propositions 5.2 and 5.3, and presents the method for matching signatures of intertwining operators. The explicit details and calculations needed for the proofs are in § 6. For the convenience of the reader, the results, including the exact description of the complementary series for $\mathcal{O} \in \mathsf{Exc}$, and of the 0-complementary series are tabulated in § 7.

Notation. If $\mathfrak{G}$ is an algebraic group, we denote by $\mathfrak{G}^0$ its identity component. The center will be denoted by $Z(\mathfrak{G})$. For every set of elements $\mathcal{E}$, we denote by $Z_{\mathfrak{G}}(\mathcal{E})$ the simultaneous centralizer in $\mathfrak{G}$ of all elements of $\mathcal{E}$, and by $A_{\mathfrak{G}}(\mathcal{E})$ the group of components of $Z_{\mathfrak{G}}(\mathcal{E})$.

2. Intertwining operators

2.1 As mentioned in the introduction, we work only with the Hecke algebras and the p-adic group will not play a role. Therefore, in order to simplify notation, we call the dual complex group G, its Lie algebra $\mathfrak{g}$, etc.

Let H be a maximal torus G and $B \supset H$ be a Borel subgroup. The affine Hecke algebra $\mathscr{H}$ can be described by generators and relations. Let z be an indeterminate (which can then be specialized to $q^{1/2}$). Let $\Pi \subset \Delta^+ \subset \Delta$ be the simple roots, positive roots, and roots corresponding to $H \subset B$, respectively, and let S be the simple root reflections. Set $\Delta^- = \Delta \backslash \Delta^+$. Let $G_m := \mathrm{GL}(1, \mathbb{F})$, $\check{\mathcal{X}} := \mathrm{Hom}(G_m, H)$ be the (algebraic) lattice of one-parameter subgroups, and $\mathcal{X} := \mathrm{Hom}(H, G_m)$ the lattice of algebraic characters. Then $\mathscr{H}$ will denote the Hecke algebra over $\mathbb{C}[z, z^{-1}]$ attached to the root datum $(\mathcal{X}, \check{\mathcal{X}}, \Delta, \check{\Delta}, \Pi)$. The set of generators we use is that first introduced by Bernstein.

The algebra $\mathscr{H}$ is generated over $\mathbb{C}[z, z^{-1}]$ by $\{T_w\}_{w \in W}$ and $\{\theta_x\}_{x \in \mathcal{X}}$, subject to the relations [Lus89, Proposition 3.6]:

$$\begin{gathered} T_w T_{w'} = T_{ww'} (\ell(w) + \ell(w') = \ell(ww')), \quad \theta_x \theta_y = \theta_{x+y}, \\ T_s^2 = (z^2 - 1) T_s + z^2, \quad \theta_x T_s = T_s \theta_{sx} + (z^2 - 1) \frac{\theta_x - \theta_{sx}}{1 - \theta_{-\alpha}} \quad (s = s_\alpha \in S). \end{gathered} \tag{2.1.1}$$

This realization is very convenient for determining the center of $\mathscr{H}$ and thus computing infinitesimal (central) characters of representations. Let $\mathscr{A}$ be the subalgebra over $\mathbb{C}[z, z^{-1}]$ generated by the θ_x. The Weyl group acts on this via $w \cdot \theta_x := \theta_{wx}$.

Proposition 2.1 (Bernstein–Lusztig [Lus89, Proposition 3.11]). *The center of $\mathscr{H}$ is given by $\mathscr{A}^W$, the Weyl group invariants in $\mathscr{A}$.*

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Infinitesimal characters are parameterized by W-orbits $\chi=(q,t)\in\mathbb{C}^*\times H$. We always assume that q is real or at least not a root of unity. In particular, such an infinitesimal character is called *real* if t is hyperbolic. Unless indicated otherwise, we assume from here on that *the infinitesimal character is always real*. The study of representations of $\mathscr{H}$ can be simplified by using the graded Hecke algebra $\mathbb{H}$ introduced in [Lus89]. One can identify $\mathscr{A}$ with the algebra of regular functions on $\mathbb{C}^*\times H$. Define

$$\mathscr{J}=\{f\in\mathscr{A}\mid f(1,1)=0\}. \tag{2.1.2}$$

This is an ideal in $\mathscr{A}$ and it satisfies $\mathscr{H}\mathscr{J}=\mathscr{J}\mathscr{H}$. Set $\mathscr{H}^i=\mathscr{H}\cdot\mathscr{J}^i$ (the ideal $\mathscr{J}^i$ in $\mathscr{H}$ consists of the functions which vanish to order at least i at $(1,1)$). Thus, $\mathscr{H}$ has a filtration [Lus89, Lemma 4.2]

$$\mathscr{H}=\mathscr{H}^0\supset\cdots\supset\mathscr{H}^i\supset\mathscr{H}^{i+1}\supset\cdots, \tag{2.1.3}$$

and denote the graded object by $\mathbb{H}$. It can be written as

$$\mathbb{H}=\mathbb{C}[W]\otimes\mathbb{C}[r]\otimes\mathbb{A}, \tag{2.1.4}$$

where $r\equiv z-1(\text{mod }\mathscr{J})$, and $\mathbb{A}$ is the symmetric algebra of $\mathfrak{h}^*=\mathcal{X}\otimes_{\mathbb{Z}}\mathbb{C}$. The previous relations become [Lus89, Proposition 4.4]

$$\begin{aligned} t_wt_{w'}&=t_{ww'}, \quad w,w'\in W, \quad t_s^2=1,\ s\in S,\\ \omega t_s&=t_ss(\omega)+2r\langle\omega,\check\alpha\rangle, \quad s=s_\alpha\in S,\ \omega\in\mathfrak{h}^*. \end{aligned} \tag{2.1.5}$$

The center of $\mathbb{H}$ is $\mathbb{C}[r]\otimes\mathbb{A}^W$ (see [Lus89, Proposition 4.5]). Thus, infinitesimal (central) characters are parameterized by W-orbits of elements $\overline{\chi}=(r,t)\in\mathbb{C}\oplus\mathfrak{h}$.

Theorem 2.1 (Lusztig [Lus89, Theorem 9.3 and § 10.3]). *There is a matching $\overline{\chi}\longleftrightarrow\chi$ between real infinitesimal characters $\overline{\chi}$ of $\mathbb{H}$ and real infinitesimal characters χ of $\mathscr{H}$, given by $(r,t)\mapsto(e^r,e^t)$, so that if $\mathbb{H}_{\overline{\chi}}$ and $\mathscr{H}_\chi$ are the quotients by the corresponding ideals, then there is an algebra isomorphism*

$$\mathscr{H}_\chi\cong\mathbb{H}_{\overline{\chi}}.$$

When $z=\sqrt{q}$, $\mathscr{H}$ is isomorphic with the Iwahori–Hecke algebra of a split p-adic group dual to G. Therefore, it has a natural $*$-operation coming from the usual $*$-operation on complex functions (i.e. $f^*(g):=\overline{f(g^{-1})}$). In order to transfer the question of unitarity to $\mathbb{H}$, one needs a $*$-operation on $\mathbb{H}$ which is compatible with the grading process and the isomorphism in Theorem 2.1. The $*$-action on the generators of $\mathbb{H}$ is computed in [BM93, Theorem 5.6]:

$$\begin{aligned} t_w^*&=t_{w^{-1}}, \quad w\in W,\\ \omega^*&=-\overline{\omega}+\sum_{\alpha\in\Delta^+}\langle\overline{\omega},\check\alpha\rangle t_{s_\alpha}, \quad \omega\in\mathfrak{h}^*, \end{aligned} \tag{2.1.6}$$

and the compatibility with $\mathscr{H}$ is checked in Corollary 5.2. Finally, [BM93, Theorem 4.3] shows that the isomorphism from Theorem 2.1 is analytic in (r,t).

From now on, we fix $r=1/2$, and transfer the study of the representation theory of $\mathscr{H}$ to $\mathbb{H}$.

2.2

Definition 2.1. A $\mathbb{H}$-module V is called *spherical* if $V|_W$ contains the trivial W-type. The module V is called *generic* if $V|_W$ contains the sign W-type.

The latter definition is motivated by the main theorem in [BM94].

Definition 2.2. The *Iwahori–Matsumoto involution* IM is defined as

$$\mathsf{IM}(t_w) := (-1)^{\ell(w)} t_w,\ \ w \in W, \quad \mathsf{IM}(\omega) := -\omega,\ \ \omega \in \mathfrak{h}^*. \tag{2.2.1}$$

Here IM takes spherical modules into generic modules and it preserves unitarity. In particular, IM(triv) is the Steinberg module St.

2.3 Next we need to recall certain results about the (geometric) classification of irreducible $\mathbb{H}$-modules. The results of [KL87] can be used in the setting of $\mathbb{H}$ via [Lus89]. Alternatively, the representation theory of $\mathbb{H}$ is a special case of [Lus88, Lus95, Lus02].

We parameterize irreducible representations of $\mathbb{H}$ by G-conjugacy classes (χ, e, ψ), where $\chi \in \mathfrak{g}$ is semisimple, $e \in \mathfrak{g}$ is nilpotent such that $[\chi, e] = e$, and (ψ, V_ψ) are certain irreducible representations of $A(e, \chi)$, the component group of the centralizer in G of e and χ. Embed e into a Lie triple $\{e, h, f\}$. Write $\chi = h/2 + \nu$ where ν centralizes $\{e, h, f\}$.

The constructions of [KL87, Lus95] attach to each (G-conjugacy class) (e, χ) a module $X(e, \chi)$ which decomposes under the action of $A(e, \chi)$ as a sum of standard modules $X(e, \chi, \psi)$:

$$X(e, \chi) = \bigoplus_{(\psi, V_\psi) \in \widehat{A(e,\chi)}_0} X(e, \chi, \psi) \otimes V_\psi, \tag{2.3.1}$$

where $\widehat{A(e, \chi)}_0$ will be defined below.

As a $\mathbb{C}[W]$-module,

$$X(e, \chi) \cong H^*(\mathcal{B}_e, \mathbb{C}), \tag{2.3.2}$$

where $\mathcal{B}_e$ is the variety of Borel subalgebras of $\mathfrak{g}$ containing e. (See [BM89, § 4], for a detailed explanation and references.) The action of W is the generalization of that defined by Springer. The component group $A(e, \chi)$ is naturally a subgroup of $A(e)$ because in a connected algebraic group, the centralizer of a torus is connected. The group $A(e)$ acts on the right-hand side of (2.3.2), and the action of $A(e, \chi)$ on $X(e, \chi)$ is compatible with its inclusion into $A(e)$, and the isomorphism in (2.3.2). Let $\mathcal{O}$ be the G orbit of e. According to the Springer correspondence,

$$H^*(\mathcal{B}_e, \mathbb{C}) = \bigoplus_{\phi \in \widehat{A(e)}} H^*(\mathcal{B}_e, \mathbb{C})^\phi \otimes V_\phi. \tag{2.3.3}$$

Furthermore, $H^{2\dim(\mathcal{B}_e)}(\mathcal{B}_e, \mathbb{C})^\phi$ is either zero, or it is an irreducible representation of W. Denote

$$\widehat{A(e)}_0 = \{\phi \in \widehat{A(e)} \mid H^{2\dim_{\mathbb{C}}(\mathcal{B}_e)}(\mathcal{B}_e, \mathbb{C})^\phi \neq 0\}, \tag{2.3.4}$$

and define $\widehat{A(e, \chi)}_0$ to be the representations of $A(e, \chi)$ which are restrictions of representations of $A(e)$ in $\widehat{A(e)}_0$.

For $\phi \in \widehat{A(e)}_0$, we denote the Springer representation by $\mu(\mathcal{O}, \phi)$. Each representation of W is uniquely of the form $\mu(\mathcal{O}, \phi)$ for some $(\mathcal{O}, \phi)$. The correspondence is normalized so that if e is the principal nilpotent, and ϕ is trivial, then $\mu(\mathcal{O}, \phi) = \mathsf{sgn}$.

Moreover, a result of Borho–MacPherson says that the $\mu(\mathcal{O}', \phi)$ occurring in $H^*(\mathcal{B}_e, \mathbb{C})$ all correspond to $\mathcal{O}'$ such that $\mathcal{O} \subset \overline{\mathcal{O}'}$.

Comparing with (2.3.1) and (2.3.2), we conclude that

$$\mathrm{Hom}_W[\mu(\mathcal{O}, \phi) : X(e, \chi, \psi)] = [\phi\,|_{A(e,\chi)} : \psi]. \tag{2.3.5}$$

DEFINITION 2.3. Following [BM89, Theorem 6.3], the W-representations in the set

$$\{\mu(\mathcal{O},\phi):[\phi\,|_{A(e,\chi)}\colon\psi]\neq 0\} \tag{2.3.6}$$

will be called the *lowest W-types* of $X(e,\chi,\psi)$. When $\psi=\mathsf{triv}$, we call the W-type $\mu(\mathcal{O},\mathsf{triv})$ in (2.3.5) the *generic* lowest W-type.

Clearly, the generic lowest W-type always appears with multiplicity one in $X(e,\chi,\mathsf{triv})$.

By [KL87, Theorems 8.2 and 8.3], or [Lus02, Theorem 1.21], if $\nu=0$, then $X(e,h/2,\psi)$ is *tempered irreducible*, and it has a unique lowest W-type, $\mu(\mathcal{O},\psi)$, whose multiplicity is one. (In this way, there is a one-to-one correspondence between tempered irreducible modules with real infinitesimal character and $\widehat{W}$; see [BM89, Corollary 4.8].) If, in addition, e is an element of a distinguished nilpotent orbit in the sense of Bala–Carter, then $X(e,h/2,\psi)$ is a *discrete series* module.

By [BM89, Theorem 6.3], the module $X(e,\chi,\psi)$ has a unique irreducible subquotient $\overline{X}(e,\chi,\psi)$ characterized by the fact that it contains each lowest W-type $\mu(\mathcal{O},\phi)$ with full multiplicity $[\phi\,|_{A(e,\chi)}:\psi]$.

Remark 2.1. In the geometric classification, the spherical modules are those of the form $\overline{X}(0,\chi,\mathsf{triv})$. The generic modules are $X(e,\chi,\mathsf{triv})$, such that $X(e,\chi,\mathsf{triv})$ is irreducible [BM94, Ree94]. For the generic modules, the semisimple element χ determines the nilpotent orbit $\mathcal{O}=G\cdot e$ uniquely, according to (1.3.1).

2.4 The analogous formula to (2.3.1) holds whenever the data (e,χ) factor through a Levi component M (see [BM89, § 7]). This gives the connection with the classical version of the Langlands classification, which we recall next. Let $A_M(e,\chi)$ denote the component group of the centralizer in M of e and χ. The following lemma is well known.

LEMMA 2.1. *The natural map $A_M(e,\chi)\to A_G(e,\chi)$ is an injection.*

Proof. If $T=Z(M)^0$, then $M=Z_G(T)$. We have that $Z_G(T)\cap Z_G(e)^0=Z_{Z_G(e)^0}(T)$ is connected, since the centralizer of a torus in a connected algebraic group is connected. Therefore, $M\cap Z_G(e)^0=Z_M(e)^0$.

We have

$$X(e,\chi)=\mathbb{H}\otimes_{\mathbb{H}_M}X_M(e,\chi)\quad\text{and}$$
$$\mathbb{H}\otimes_{\mathbb{H}_M}X_M(e,\chi,\tau)=\bigoplus_{\psi\in\widehat{A(e,\chi)}_0}[\psi|_{A_M(e,\chi)}:\tau]X(e,\chi,\psi). \tag{2.4.1}$$

Notation. We write $\mathrm{Ind}_M^G[\pi]$ for the induced module $\mathbb{H}\otimes_{\mathbb{H}_M}\pi$. It has a basis $\{t_x\otimes v\mid v\in\pi, x\in W/W(M)\}$.

Define

$$M(\nu):=Z_G(\nu), \tag{2.4.2}$$

and $P=M(\nu)N$ is such that $\langle\nu,\alpha\rangle>0$ for all roots $\alpha\in\Delta(\mathfrak{n})$. Write $M(\nu)=M_0(\nu)Z(M(\nu))$, where $Z(M(\nu))$ is the center.

LEMMA 2.2. *We have $A_{M(\nu)}(e,\chi)=A_G(e,\chi)$.*

Proof. This is because the centralizer of e is of the form LU with U connected unipotent, and L is the centralizer of both e and h. It follows that every component of $A_G(e,\chi)$ meets L, and

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therefore

$$A_G(e, \chi) = A_G(e, h, \nu) = A_{M(\nu)}(e, h) = A_{M(\nu)}(e, \chi). \tag{2.4.3}$$

□

For $\tau \in \widehat{A_{M(\nu)}(e, \chi)} = \widehat{A_{M_0(\nu)}(e)}$,

$$X_{M(\nu)}(e, \chi, \tau) = X_{M_0(\nu)}(e, h/2, \tau) \otimes \mathbb{C}_\nu. \tag{2.4.4}$$

The representation

$$\sigma := X_{M_0(\nu)}(e, h/2, \tau) \tag{2.4.5}$$

is a tempered irreducible module. Let $\psi \in \widehat{A_G(e, \chi)}$ be the representation corresponding to $\tau \in \widehat{A_{M(\nu)}(e, \chi)}$ via the identification in Lemma 2.2. Then

$$X(e, \chi, \psi) = \mathbb{H} \otimes_{\mathbb{H}_{M(\nu)}} X_{M(\nu)}(e, \chi, \tau). \tag{2.4.6}$$

Definition 2.4. In general, whenever σ is a tempered representation of $\mathbb{H}_M$ corresponding to the parameter $(e, h/2, \tau)$, $\tau \in \widehat{A_M(e)}$, and $\nu \in \mathfrak{z}(\mathfrak{m})$, $\chi = h/2 + \nu$, we write

$$X(M, \sigma, \nu) := \mathrm{Ind}_M^G[\sigma \otimes \nu], \tag{2.4.7}$$

and also call it a *standard* module. By (2.4.1), it decomposes as

$$X(M, \sigma, \nu) = \bigoplus_{\psi \in \widehat{A_G(e,\nu)}_0} [\psi|_{A_M(e)} : \tau] X(e, \chi, \psi). \tag{2.4.8}$$

If $M = M(\nu)$, then

$$X(M, \sigma, \nu) = X(e, \chi, \psi), \tag{2.4.9}$$

where ψ corresponds to τ as in Lemma 2.2.

The terminology is justified by the fact that $X(M, \sigma, \nu)$ is (via the Borel–Casselman correspondence) the $\mathcal{I}$-fixed vectors of an induced (standard) module in the classical form of Langlands classification for the p-adic group. In particular, tempered modules for $\mathbb{H}$ are the $\mathcal{I}$-fixed vectors of tempered representations of the p-adic group. In particular, via [BM89, Theorem 1.1], irreducible tempered $\mathbb{H}$-modules are unitary (with respect to the $*$-operation (2.1.6)). If $\langle \nu, \alpha \rangle \geqslant 0$ for all positive roots, then $X(M, \sigma, \nu)$, with $M = M(\nu)$, has a unique irreducible quotient $\overline{X}(M, \sigma, \nu)$. If $\langle \nu, \alpha \rangle \leqslant 0$ for all positive roots, then $X(M, \sigma, \nu)$, $M = M(\nu)$ has a unique irreducible submodule $\overline{X}(M, \sigma, \nu)$. In the setting of graded Hecke algebras, this form of the classification is proved in [Eve96, Theorem 2.1].

2.5 Let $\mathfrak{z}(e, h, f)$ be the centralizer of the triple $\{e, h, f\}$, and fix $\mathfrak{a}_{BC} \subset \mathfrak{z}(e, h, f)$ a Cartan subalgebra such that $\nu \in \mathfrak{a}_{BC}$. Let $\mathfrak{m}_{BC}$ be the centralizer of $\mathfrak{a}_{BC}$, with decomposition

$$\mathfrak{m}_{BC} = \mathfrak{m}_{BC,0} + \mathfrak{a}_{BC}. \tag{2.5.1}$$

Then the Lie triple is contained in $[\mathfrak{m}_{BC}, \mathfrak{m}_{BC}] \subset [\mathfrak{m}_{BC,0}, \mathfrak{m}_{BC,0}]$. Thus, $\mathfrak{m}_{BC,0}$ is semisimple (its center centralizes the triple, so must be contained in $\mathfrak{a}_{BC}$). So $\mathfrak{m}_{BC,0}$ is the derived algebra of $\mathfrak{m}_{BC}$, and the nilpotent element e is distinguished in $\mathfrak{m}_{BC,0}$. The Levi component $\mathfrak{m}_{BC}$ is that used in the Bala–Carter classification of nilpotent orbits [Car85], hence the notation. Let M_{BC}, $M_{BC,0}$ be the corresponding groups. The triple $(e, h/2, \psi)$ with $\psi \in \widehat{A_{M_{BC,0}}(e, \chi)}$ determines a discrete series parameter on $M_{BC,0}$. Clearly, for any $\nu \in \mathfrak{a}_{BC}$, if $M(\nu)$ is as in (2.4.2), then $M_{BC} \subset M(\nu)$.

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We are interested in the question of reducibility for the induced modules $X(M, \sigma, \nu)$, where $M_{BC} \subset M$, and σ is generic. Denote by M_0 the derived subgroup of M.

Proposition 2.2. *Let M be a Levi subgroup such that $M_{BC} \subset M$. Assume that σ is a tempered generic module of $\mathbb{H}_{M,0}$ corresponding to $(e, h/2, \mathsf{triv})$, and consider the standard module $X(M, \sigma, \nu)$. Set $\chi = h/2 + \nu$ for the infinitesimal character and $\mathcal{O} = G \cdot e$. Then $X(M, \sigma, \nu)$ is reducible if and only if one of the following two conditions is satisfied:*

(1) *there exists $\mathcal{O}'$ satisfying $\overline{\mathcal{O}'} \supset \mathcal{O}$, such that $\mathcal{O}'$ has a representative e' satisfying $[\chi, e'] = e'$;*
(2) *there is no $\mathcal{O}'$ as in condition (1), but $A_M(e, \chi) \neq A_G(e, \chi)$, and there exists a nontrivial character $\psi \in \widehat{A_G(e, \chi)}_0$ satisfying $[\psi \mid_{A_M(e,\chi)} : \mathsf{triv}] \neq 0$.*

Proof. Condition (1) follows from [BM89], see also [BM96, § 2.1]. Condition (2) is an immediate consequence of formula (2.4.1). □

Remark 2.2. By (2.4.3), when $M \supset M(\nu)$, condition (1) in Proposition 2.2 is necessary and sufficient.

2.6 In the following sections we construct intertwining operators associated with elements which preserve the data (M, σ).

Assume first that M is the Levi component of an arbitrary standard parabolic subgroup, and $(\sigma, \mathcal{V}_\sigma)$ a representation of $\mathbb{H}_M$. Let

$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{a} \tag{2.6.1}$$

be the Lie algebra of M, with center $\mathfrak{a}$, and derived algebra $\mathfrak{m}_0$. Write $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ for the Cartan subalgebra, with $\mathfrak{t} \subset \mathfrak{m}_0$. If $w \in W = W(\mathfrak{g}, \mathfrak{h})$ is such that $w(\mathfrak{m}) = \mathfrak{m}'$ is another Levi subalgebra (of a standard parabolic subalgebra), choose w to be *minimal* in the double coset $W(M)wW(M')$. Let $w = s_{\alpha_1} \ldots s_{\alpha_k}$ be a reduced decomposition. In [BM96, Lemma 1.6], the elements

$$r_\alpha = t_{s_\alpha} \alpha - 1 \tag{2.6.2}$$

are introduced. Set

$$r_w := r_{\alpha_1} \cdots r_{\alpha_k}. \tag{2.6.3}$$

By [BM96, Lemma 1.6], the definition does not depend on the choice of reduced expression. As w is minimal in its double coset, it defines an isomorphism of the root data, and therefore an isomorphism $a_w : \mathbb{H}_M \longrightarrow \mathbb{H}_{M'}$. Let $(w \cdot \sigma, \mathcal{V}_{w \cdot \sigma})$ be the representation of $\mathbb{H}_{M'}$ obtained from σ by composing with a_w^{-1}. Then r_w defines an intertwining operator

$$\begin{gathered} A_w(\sigma, \nu) : \operatorname{Ind}_M^G[\sigma \otimes \nu] \longrightarrow \operatorname{Ind}_{M'}^G[w \cdot \sigma \otimes w \cdot \nu], \\ t \otimes v \mapsto t r_w \otimes a_w^{-1}(v), \quad t \in W, v \in \mathcal{V}_\sigma. \end{gathered} \tag{2.6.4}$$

For each $(\mu, V_\mu) \in \widehat{W}$, $A_w(\sigma, \nu)$ induces an intertwining operator

$$A_{w,\mu}(\sigma, \nu) : \operatorname{Hom}_W[V_\mu : \operatorname{Ind}_M^G[\sigma \otimes \nu]] \longrightarrow \operatorname{Hom}_W[V_\mu : \operatorname{Ind}_{M'}^G[w \cdot \sigma \otimes w \cdot \nu]], \tag{2.6.5}$$

which by Frobenius reciprocity can be written as

$$A_{w,\mu}(\sigma, \nu) : \operatorname{Hom}_{W(M)}[V_\mu : \mathcal{V}_\sigma] \longrightarrow \operatorname{Hom}_{W(M')}[V_\mu : \mathcal{V}_{w \cdot \sigma}]. \tag{2.6.6}$$

These operators are defined for all ν not just real ones. We assume that ν is complex for the rest of the section. If $\beta \in \Delta^+$ (respectively Δ^-), we also use the notation $\beta > 0$ (respectively $\beta < 0$).

Define the element $\kappa_w \in \mathbb{A}$:

$$\kappa_w = \prod_{\beta>0, w\beta<0} (\beta^2 - 1). \tag{2.6.7}$$

Proposition 2.3. *The operators $A_w(\sigma, \nu)$ have the following properties:*

1. *$A_w(\sigma, \nu)$ is polynomial in ν;*
2. *$A_{w^{-1}}(w \cdot \sigma, w \cdot \nu) \circ A_w(\sigma, \nu) = (\sigma \otimes \nu)(\kappa_w)$, where κ_w is defined in (2.6.7); furthermore κ_w is an element of the center of $\mathbb{H}_M$, so the right-hand side is a scalar multiple of the identity;*
3. *assume that σ is hermitian, then the hermitian dual of $A_w(\sigma, \nu)$ is $A_w(\sigma, \nu)^* = A_{w^{-1}}(w \cdot \sigma, -w \cdot \overline{\nu})$.*

Proof. Part (1) is clear from the definition. Part (2) follows from the fact that in the Hecke algebra,

$$r_\alpha^2 = (t_\alpha \alpha - 1)^2 = \alpha^2 - 1 \quad \text{for } \alpha \text{ a simple root.} \tag{2.6.8}$$

The fact that κ_w is in the center of $\mathbb{H}_M$ follows from the fact that w is shortest in the double coset. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ and $\mathfrak{p}' = \mathfrak{m}' + \mathfrak{n}'$ be the standard parabolic subalgebras. If $\beta \notin \Delta(\mathfrak{m})^+$ is such that $w\beta < 0$, then $-w\beta \in \Delta(\mathfrak{n}')$. If $\alpha \in \Pi(\mathfrak{m})$ is a simple root, then $-w(s_\alpha(\beta)) = -s_{w(\alpha)}(w\beta) \in \Delta(\mathfrak{n}')$, because $w\alpha$ is a simple root of $\mathfrak{m}'$, and so preserves $\mathfrak{n}'$.

For part (3) we recall from [BM96, Corollary 1.4], that the hermitian dual of $\mathrm{Ind}_M^G[\sigma \otimes \nu]$ is $\mathrm{Ind}_M^G[\sigma \otimes -\overline{\nu}]$ with the pairing given by

$$\langle t_x \otimes v_1 \mathbf{1}_\nu, t_y \otimes v_2 \mathbf{1}_{-\overline{\nu}} \rangle = \langle \sigma(\epsilon_M(t_y^{-1} t_x) v_1), v_2 \rangle. \tag{2.6.9}$$

In this formula, $x, y \in W/W(M)$, $v_1, v_2 \in \mathcal{V}_\sigma$, and ϵ_M is the projection of $\mathbb{C}[W]$ onto $\mathbb{C}[W(M)]$ (see [BM96, Equation (1.4.1)]). We omit the rest of the proof. $\square$

Remark 2.3. The formula for the pairing in (2.6.9) follows from [BM96, § 1.4], which uses the fact that $\mathbb{H}$ has the $*$ operation given by (2.1.6).

2.7 We still assume that ν is complex. We specialize to the case when σ is a generic discrete series for $\mathbb{H}_{M_0}$. So in this case $M = M_{BC}$, and the induced module in the previous section is $X(M, \sigma, \nu)$. As in (2.4.1), this decomposes into a direct sum of standard modules $X(M, \sigma, \nu, \psi)$, with $\psi \in \widehat{A(e, \nu)}_0$. In particular, $X(M, \sigma, \nu, \mathsf{triv})$ is the only generic summand.

In this case we normalize the operators A_w so that they are Id on $\mu(\mathcal{O}, \mathsf{triv})$, and restrict them to the subspace $X(M, \sigma, \nu, \mathsf{triv})$. We denote the normalized operators $\mathcal{A}_w(\sigma, \nu)$ (on $X(M, \sigma, \nu)$), respectively $\mathcal{A}_w(\sigma, \nu, \mathsf{triv})$ (on $X(M, \sigma, \nu, \mathsf{triv})$). They define, by restriction to Hom spaces as in (2.6.6), operators $\mathcal{A}_{w,\mu}$ for $\mu \in \widehat{W}$.

Assume that w decomposes into $w = w_1 w_2$, such that $\ell(w) = \ell(w_1) + \ell(w_2)$ ($\ell(w)$ is the length of w). The fact that $r_w = r_{w_1} r_{w_2}$ (see [BM96, Lemma 1.6]) implies one of the most important properties of the operators $\mathcal{A}_w$, the factorization:

$$\begin{aligned} &\mathcal{A}_{w_1 w_2}(\sigma, \nu) = \mathcal{A}_{w_1}(w_2 \cdot \sigma, w_2 \cdot \nu) \circ \mathcal{A}_{w_2}(\sigma, \nu) \quad \text{and similarly} \\ &\mathcal{A}_{w_1 w_2}(\sigma, \nu, \mathsf{triv}) = \mathcal{A}_{w_1}(w_2 \cdot \sigma, w_2 \cdot \nu, \mathsf{triv}) \circ \mathcal{A}_{w_2}(\sigma, \nu, \mathsf{triv}). \end{aligned} \tag{2.7.1}$$

Proposition 2.4. *Assume that $\mathfrak{m}$ is the Levi component of a maximal standard parabolic subalgebra. Then $\mathcal{A}_w(\sigma, \nu, \mathsf{triv})$ does not have any poles in the region of ν satisfying $\langle \nu, \beta \rangle \geqslant 0$ for all $\beta > 0$ such that $w\beta < 0$.*

Proof. Either $w\nu = \nu$ or otherwise $\langle w\nu, \beta\rangle < 0$ for all $\beta \in \Delta(\mathfrak{n})$. By [Eve96, Theorem 2.1], if $\langle \beta, \nu\rangle > 0$ for all $\beta \in \Delta(\mathfrak{n})$, then $X(M, \sigma, \nu, \mathsf{triv})$ has a unique irreducible quotient, while if $\langle \nu', \beta\rangle < 0$ for all $\beta \in \Delta(\mathfrak{n}')$, then $X(M', \sigma', \nu', \mathsf{triv})$ has a unique irreducible submodule. By the results reviewed in § 2.3, this is the unique subquotient containing $\mu(\mathcal{O}, \mathsf{triv})$. Thus, $\mathcal{A}_w$ maps $X(M, \sigma, \nu, \mathsf{triv})$ onto $\overline{X}(M, \sigma, \nu, \mathsf{triv})$.

Assume that $\mathcal{A}_w$ has a pole of order $k > 0$ at ν_0 with $\Re\nu_0 > 0$. Then $(\nu - \nu_0)^k \mathcal{A}(\sigma, \nu, \mathsf{triv})$ extends analytically to $\nu = \nu_0$, and is nonzero. Its image is disjoint from $\overline{X}(M, \sigma, \nu, \mathsf{triv})$, which contradicts the fact that $\overline{X}(w \cdot M, w \cdot \sigma, w \cdot \nu)$ is the unique irreducible submodule of $X(w \cdot M, w \cdot \sigma, w \cdot \nu, \mathsf{triv})$.

Now suppose that $\mathcal{A}_w$ has a pole at ν_0 with $\Re\nu_0 = 0$. We use the analogues of relations (1)–(3) from Proposition 2.3; the relation (2) implies that for the normalized operators we have

$$\mathcal{A}_{w^{-1}} \circ \mathcal{A}_w = \mathsf{Id}. \tag{2.7.2}$$

Write

$$\begin{aligned} \mathcal{A}_w(\sigma, \nu, \mathsf{triv}) &= (\nu - \nu_0)^k [A_0 + (\nu - \nu_0) A_1 + \cdots] \quad \text{where } A_0 \neq 0, \text{ and} \\ \mathcal{A}_{w^{-1}}(w \cdot \sigma, w \cdot \nu, \mathsf{triv}) &= (\mathcal{A}_w(\sigma, -\overline{\nu}, \mathsf{triv}))^* = (-\nu + \nu_0)^k [A_0^* + (-\nu + \nu_0) A_1^* + \cdots]. \end{aligned} \tag{2.7.3}$$

Then if $k < 0$, relation (2) in Proposition 2.3 implies $A_0^* A_0 = 0$, which is a contradiction.

2.8 We present a standard technique for factorizing intertwining operators (see [SV80, § 3], for the setting of real reductive groups).

Definition 2.5. We say that two Levi components $\mathfrak{m}, \mathfrak{m}'$ are adjacent, if either $\mathfrak{m} = \mathfrak{m}'$ or there is a Levi component Σ such that $\mathfrak{m}, \mathfrak{m}' \subset \Sigma$ are maximal Levi components conjugate by $W(\Sigma)$.

Lemma 2.3. *Let w be such that $w(\mathfrak{m}) = \mathfrak{m}'$, and w minimal in the double coset $W(M)wW(M')$. Then there is a chain of adjacent Levi components $\mathfrak{m}_0 = \mathfrak{m}, \ldots, \mathfrak{m}_k = \mathfrak{m}'$.*

Proof. We perform an induction on the length of w. If $\mathfrak{m} = \mathfrak{m}'$ and $w = 1$, there is nothing to prove. Otherwise there is α simple such that $w\alpha < 0$. Then let Σ_1 be the Levi component with simple roots $\Delta(\mathfrak{m}) \cup \{\alpha\}$. Then ww_1^{-1} has shorter length, and the induction hypothesis applies. □

We always consider minimal length chains of Levi subalgebras. The main reason for these notions is the following. Let $w^0_{\Sigma_i}$ be the longest element in $W(\Sigma_i)$, and w_i be the shortest element in $W(\mathfrak{m}_{i-1}) w^0_{\Sigma_i} W(\mathfrak{m}_i)$. Then we can write

$$w = w_k \cdot \cdots \cdot w_1, \quad \mathcal{A}_w = \mathcal{A}_{w_k} \circ \cdots \circ \mathcal{A}_{w_1}. \tag{2.8.1}$$

The $\mathcal{A}_{w_i}$ are induced from the corresponding operators for maximal Levi components, and so Proposition 2.4 applies.

Theorem 2.2. *The intertwining operators $\mathcal{A}_w$ have the following properties:*

1. $\mathcal{A}_w(\sigma, \nu, \mathsf{triv})$ *is analytic for ν such that $\langle \Re\nu, \beta\rangle \geqslant 0$ for all $\beta > 0$ such that $w\beta < 0$;*
2. $\mathcal{A}_{w^{-1}}(w \cdot \sigma, w \cdot \nu, \mathsf{triv}) \circ \mathcal{A}_w(\sigma, \nu, \mathsf{triv}) = \mathsf{Id}$*;*
3. $\mathcal{A}_w(\sigma, \nu, \mathsf{triv})^* = \mathcal{A}_{w^{-1}}(w \cdot \sigma, -w \cdot \overline{\nu}, \mathsf{triv})$.

Proof. This follows from Proposition 2.4 and Lemma 2.3. □

Remark 2.4. If there exists an isomorphism $\tau : w \cdot \sigma \longrightarrow \sigma$, we compose the intertwining operators $\mathcal{A}_w$ with $(1 \otimes \tau)$. For simplicity, we denote these operators by $\mathcal{A}_w$ also. If in fact,

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$w\cdot\sigma\cong\sigma$, $w\cdot\nu=-\overline{\nu}$, then the operator $\mathcal{A}_w$ gives rise to a hermitian form. This is because, as recalled before, $\mathrm{Ind}_M^G[\sigma\otimes-\overline{\nu}]$ is the hermitian dual of $\mathrm{Ind}_M^G[\sigma\otimes\nu]$.

2.9 We assume that ν is *real*. Let $x\in G$ stabilize $\{e,h,f\}$. Then we can choose the Cartan subalgebra $\mathfrak{a}_{BC}$ of $\mathfrak{z}(e,h,f)$ so that it is stabilized by x. Furthermore, since x stabilizes $\mathfrak{m}_{BC}$ and $\mathfrak{m}_{BC,0}$, by a classical result of Steinberg, there is a Cartan subalgebra $\mathfrak{t}\subset\mathfrak{m}_{BC,0}$, stabilized by x. Let

$$\mathfrak{h}:=\mathfrak{t}+\mathfrak{a}_{BC} \tag{2.9.1}$$

be the Cartan subalgebra of $\mathfrak{m}_{BC}$. We can also choose a Borel subalgebra of $\mathfrak{m}_{BC}$ containing $\mathfrak{h}$ which is stabilized by x. So x gives rise to a Weyl group element w_x, the shortest element in the double coset $W_{M_{BC}}xW_{M_{BC}}$. Thus, we obtain an intertwining operator $\mathcal{A}_{w_x}$ by the construction in §§ 2.6–2.8.

If $x\cdot\nu=-\nu$ and $\tau:x\cdot\sigma\xrightarrow{\cong}\sigma$, by Remark 2.4, $\mathcal{A}_{w_x}$ gives rise to a hermitian form.

2.10 We apply the construction of § 2.9 in the following special case. Let $\overline{\alpha}$ be a simple root of $\mathfrak{a}_{BC}\subset\mathfrak{z}(e,h,f)$. Let $x_{\overline{\alpha}}\in Z(e,h,f)^0$ be an element inducing the reflection $s_{\overline{\alpha}}$ on $\mathfrak{h}$. Then $x_{\overline{\alpha}}$ stabilizes $\mathfrak{m}_{BC}$. The element $x_{\overline{\alpha}}$ may need to be modified by an element in $M_{BC,0}$ so as to stabilize $\mathfrak{t}$ as well. Then it gives rise to a Weyl group element $w_{\overline{\alpha}}$, shortest in the double coset $W_{M_{BC}}x_{\overline{\alpha}}W_{M_{BC}}$, and to an intertwining operator $\mathcal{A}_{w_{\bar{\alpha}}}$. The new $x_{\overline{\alpha}}$ may not fix the Lie triple. However, since it modified the original element by one in $M_{BC,0}$, there is an isomorphism $\tau_{\overline{\alpha}}:w_{\overline{\alpha}}\sigma\xrightarrow{\cong}\sigma$.

Then, as in Remark 2.4, we have a normalized intertwining operator

$$\mathcal{A}_{\overline{\alpha}}:X(M_{BC},\sigma,\nu,\mathsf{triv})\longrightarrow X(M_{BC},\sigma,w_{\overline{\alpha}}\nu,\mathsf{triv}). \tag{2.10.1}$$

2.11 We construct intertwining operators for another class of elements normalizing σ. We consider an $M\supset M_{BC}$, and write $\mathfrak{m}=\mathfrak{m}_0+\mathfrak{a}$, $\mathfrak{a}\subset\mathfrak{a}_{BC}$, as in (2.6.1). Let A and H be the Cartan groups corresponding to $\mathfrak{a}$, $\mathfrak{h}$, and let σ be a tempered representation of $\mathbb{H}_{M,0}$. Define

$$\begin{aligned}N(\mathfrak{a})&:=\{w\in W\mid w\mathfrak{a}=\mathfrak{a}\},\\ C(\mathfrak{a},M)&:=\{w\in N(\mathfrak{a})\mid w(\Delta^+(\mathfrak{m}))=\Delta^+(\mathfrak{m})\}.\end{aligned} \tag{2.11.1}$$

The following formula is a particular case (which we need here for the construction of intertwining operators) of a more general result that we postpone to § 4.1.

Lemma 2.4. *We have the following results*

$$(1)\quad N(\mathfrak{a})=C(\mathfrak{a},M)\ltimes W(M). \tag{2.11.2}$$

$$(2)\quad N_G(\mathfrak{a})/M\cong C(\mathfrak{a},M). \tag{2.11.3}$$

Proof. (1) From (2.11.1), we see that

$$N(\mathfrak{a})=C(\mathfrak{a},M)\cdot W(M). \tag{2.11.4}$$

In fact, as in the proof of Lemma 4.1,

$$C(\mathfrak{a},M)\cap W(M)=\{1\}, \tag{2.11.5}$$

and $W(M)$ is a normal subgroup, because any element xmx^{-1} with $x\in N_G(\mathfrak{a})$ centralizes $\mathfrak{a}$, so must be in M.

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(2) Clearly M is normal in $N_G(\mathfrak{a})$. Let $n \in N_G(\mathfrak{a})$. Then $n\mathfrak{a} = \mathfrak{a}$, and $n\mathfrak{h} = \mathfrak{h}' = \mathfrak{t}' + \mathfrak{a}$. There is an element $m \in M$ such that

$$mn\mathfrak{h} = \mathfrak{h}, \quad mn(\Delta^+(\mathfrak{m})) = \Delta^+(\mathfrak{m}). \tag{2.11.6}$$

Thus, the M-coset of mn is in $C(\mathfrak{a}, M)$. This map is a group homomorphism, and an isomorphism onto $C(\mathfrak{a}, M)$. □

If $c \in C(\mathfrak{a}, M)$ is such that $c \cdot \sigma \cong \sigma$, then by the construction in § 2.7, in particular Remark 2.4, there is a normalized intertwining operator

$$\mathcal{A}_c(\sigma, \nu) : X(M, \sigma, \nu) \to X(M, \sigma, c \cdot \nu). \tag{2.11.7}$$

and for every $(\mu, V_\mu) \in \widehat{W}$, this induces an operator $\mathcal{A}_{c,\mu}(\sigma, \nu)$ as in (2.6.5) and (2.6.6).

2.12 We put the constructions in the previous sections together. We consider the case when $M = M_{BC}$.

Let $W(\mathfrak{z}, \mathfrak{a}_{BC})$ denote the Weyl group of $\mathfrak{a}_{BC}$ in $\mathfrak{z} := \mathfrak{z}(e, h, f)$. Denote by $\mathbb{H}(\mathfrak{z})$ the graded Hecke algebra constructed from the root system of $\mathfrak{z}$. In this section we study the relation of $W(\mathfrak{z}, \mathfrak{a}_{BC})$ with $C(\mathfrak{a}_{BC}, M_{BC})$, in particular we show that $C(\mathfrak{a}_{BC}, M_{BC})$ contains naturally a subgroup isomorphic to $W(\mathfrak{z}, \mathfrak{a})$. Elements in this subgroup give rise to $\mathbb{H}(\mathfrak{z})$-intertwining operators of the (spherical) principal series $X_{\mathbb{H}(\mathfrak{z})}(0, \nu)$ of $\mathbb{H}(\mathfrak{z})$, as well as $\mathbb{H}$-intertwining operators for $X(M, \sigma, \nu)$ by (2.11.7).

Set

$$\widetilde{A}_{BC} := Z_{Z(e,h,f)}(\mathfrak{a}_{BC}). \tag{2.12.1}$$

Then $A_{BC} \subset \widetilde{A}_{BC}$, so there is a surjection

$$N_{Z(e,h,f)}(\mathfrak{a}_{BC})/A_{BC} \longrightarrow N_{Z(e,h,f)}(\mathfrak{a}_{BC})/\tilde{A}_{BC}. \tag{2.12.2}$$

Furthermore, there is an injective group homomorphism,

$$W(\mathfrak{z}, \mathfrak{a}_{BC}) = N_{Z(e,h,f)}(\mathfrak{a}_{BC})/\tilde{A}_{BC} \longrightarrow N_G(\mathfrak{a}_{BC})/Z_G(\mathfrak{a}_{BC}) = N_G(\mathfrak{a}_{BC})/M_{BC}. \tag{2.12.3}$$

Proposition 2.5. *We have the following results.*

(1) *The composition of the map in Lemma 2.11.3 with the map in (2.12.3) gives an injective homomorphism*

$$W(\mathfrak{z}, \mathfrak{a}_{BC}) \hookrightarrow C(\mathfrak{a}_{BC}, M_{BC}).$$

(2) *The composition of the map in Lemma 2.11.3 with the map in (2.12.2)*

$$A_G(e) \ltimes W(\mathfrak{z}, \mathfrak{a}_{BC}) \cong N_{Z(e,h,f)}(\mathfrak{a}_{BC})/A_{BC} \longrightarrow N_G(\mathfrak{a}_{BC})/M_{BC} \xrightarrow{\cong} C(\mathfrak{a}_{BC}, M_{BC})$$

is onto.

Proof. Part (1) is clear. For part (2), let $n \in N_G(\mathfrak{a}_{BC})$ be given. Then n induces an automorphism of $\mathfrak{m}_{BC}$. So it maps the Lie triple $\{e, h, f\}$ into another Lie triple $\{e', h', f'\}$. The Levi component is of the form

$$\mathfrak{m}_{BC} \cong \mathfrak{m}_1 \times gl(a_1) \times \cdots \times gl(a_r), \tag{2.12.4}$$

with $\mathfrak{m}_1$ simple, not type A. The nilpotent orbit is a distinguished one on $\mathfrak{m}_1$, and the principal nilpotent on the $gl(a_i)$ factors. Since any automorphism of a simple (or even a reductive algebra with simple derived algebra) maps a distinguished orbit into itself, there is $m \in M_{BC}$, such that

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mn stabilizes the triple $\{e,h,f\}$. Thus, every M_{BC} coset of $N_G(\mathfrak{a}_{BC})$ contains a representative in $N_{Z(e,h,f)}(\mathfrak{a}_{BC})$, which is the claim of the proposition. □

The image of the map in part (2) consists of elements which stabilize σ. Thus, each element in $x \in A_G(e) \ltimes W(\mathfrak{z}, \mathfrak{a}_{BC})$ gives rise to an intertwining operator

$$\mathcal{A}_x(\sigma,\nu) : X(M,\sigma,\nu) \longrightarrow X(M,\sigma,w_x\cdot\nu) \tag{2.12.5}$$

normalized to be Id on $\mu(\mathcal{O},\mathsf{triv})$. In particular, we obtain an action of $A_G(e,\nu)$ on $X(M,\sigma,\nu)$. This action should coincide with that defined geometrically, but we have not been able to verify this abstractly.

2.13 Denote by $W(\mathfrak{z}(\mathcal{O})) \cong W(\mathfrak{z},\mathfrak{a}_{BC})$ the abstract Weyl group of $\mathfrak{z}(e,h,f)$, and similarly $A(\mathcal{O})$ for the component group, and set $W(Z(\mathcal{O})) := A(\mathcal{O}) \ltimes W(\mathfrak{z}(\mathcal{O}))$.

We restrict now to the case of hermitian Langlands parameters, (M,σ,ν), where σ is a discrete series for M. Recall that this means that $M = M_{BC}$, but in order to simplify notation, we drop the subscript in this section. As before, there must exist $w \in W$ such that

$$wM = M, \quad w\sigma \cong \sigma \quad \text{and} \quad w\nu = -\nu. \tag{2.13.1}$$

For $(\mu, V_\mu) \in \widehat{W}$, § 2.11 defines an operator $\mathcal{A}_\mu(\sigma,\nu)$ (by Frobenius reciprocity) on the space $\mathrm{Hom}_{W(M)}(V_\mu,\sigma)$. The group $C(\mathfrak{a},M)$ acts on $W(M)$, and therefore on $\widehat{W(M)}$, and preserves σ. Let $\mu_M(\mathcal{O},\mathsf{triv})$ be the unique lowest $W(M)$-type of σ. Then

$$\begin{aligned}&\mathrm{Hom}_W[\mu(\mathcal{O},\mathsf{triv}) : X(M,\sigma,\nu)]\\ &\quad = \mathrm{Hom}_{W(M)}[\mu(\mathcal{O},\mathsf{triv}) : \sigma] = \mathrm{Hom}_{W(M)}[\mu(\mathcal{O},\mathsf{triv}) : \mu_M(\mathcal{O},\mathsf{triv})] = 1.\end{aligned}$$

In the calculations in § 6, we only consider W-types μ in $X(M,\sigma,\nu)$ with the property that

$$\mathrm{Hom}_{W(M)}[\mu : \sigma] = \mathrm{Hom}_{W(M)}[\mu : \mu_M(\mathcal{O},\mathsf{triv})].$$

We need the fact that $C(\mathfrak{a},M)$ preserves $\mu_M(\mathcal{O},\mathsf{triv})$. By [BM89, Corollary 4.8], since σ is tempered, this is equivalent to the fact that $C(\mathfrak{a},M)$ preserves σ.

Definition 2.6. Let σ be a discrete series for $\mathbb{H}_M$ parameterized by $\mathcal{O}$, where $M = M_{BC}$ of $\mathcal{O}$. The space $\mathrm{Hom}_{W(M)}(\mu,\sigma)$ has the structure of a representation of $C(\mathfrak{a},M)$ and via the map from Proposition 2.5, it is a $W(\mathfrak{z}(\mathcal{O}))$-representation and a $W(Z(\mathcal{O}))$-representation, which we denote by $\rho(\mu)$ and $\rho'(\mu)$, respectively.

2.14 In view of Lemma 2.1, for every Levi subgroup $M_{BC} \subset M \subset G$, one has $A_M(e) \subset A_G(e)$. In a large number of cases, $A_G(e) = A_{M_{BC}}(e)$, and analyzing the standard modules $X(M_{BC},\sigma,\nu)$ with σ a discrete series is sufficient. In the other cases, we also need intermediate Levi components M' with the property that $A_G(e,\nu) = A_{M'}(e)$.

Consider the Levi subgroups M with Lie algebras $\mathfrak{m}$ subject to the conditions:

(1) $e \in \mathfrak{m}$;

(2) $A_G(e) = A_M(e)$.

We call the nilpotent orbit $\mathcal{O}$ *quasi-distinguished* if the minimal subalgebra with respect to conditions (1) and (2) is $\mathfrak{g}$. Note that every distinguished $\mathcal{O}$ is also quasi-distinguished.

Proposition 2.6. *If $\mathcal{O}$ is a quasi-distinguished nilpotent orbit, then $\mathfrak{z}(\mathcal{O})$ is a torus.*

Proof. It is easy to verify the statement case by case using the Bala–Carter [Car85] classification of nilpotent orbits. □

Table 1. Limits of discrete series.

Type of $\mathfrak{g}$	Levi subalgebra $\mathfrak{m} \subset \mathfrak{g}$	Nilpotent in $\mathfrak{m}$
E_6	E_6	$D_4(a_1)$
	D_4	$A_2 = (3311)$
E_7	E_7	$E_6(a_1)$
	E_7	$A_4 + A_1$
	D_6	$D_5(a_1) = (7311)$
	D_6	$A_4 = (5511)$
	D_6	$A_3 + A_2 = (5331)$
E_8	E_8	$D_7(a_1)$
	E_8	$E_6(a_1) + A_1$
	E_8	$D_7(a_2)$
	E_8	$D_5 + A_2$
	D_7	$D_6(a_1) = (9311)$
	D_7	$D_6(a_2) = (7511)$
	D_7	$D_4 + A_2 = (7331)$
	D_7	$A_4 + 2A_1 = (5531)$

Definition 2.7. If σ is a tempered irreducible module parameterized by a quasi-distinguished $\mathcal{O}$, we call σ a *limit of discrete series.*

With this definition, any discrete series is a limit of discrete series. In Table 1 we list the limits of discrete series, which are not discrete series, and appear for various Levi subalgebras of E_6, E_7, and E_8. Clearly, if σ is a limit of discrete series for $\mathfrak{m}$ in E_6, it will also be considered in E_7 and E_8. Therefore, to eliminate redundancy, we list a pair $(\mathfrak{m}, \mathcal{O})$ only for the smallest algebra for which this pair appears. For $\mathfrak{m}$ of type D, we also give the notation of the orbit as a partition. In type A, the only quasi-distinguished orbit is the principal orbit.

As before, consider the module $X(M_{BC}, \sigma, \nu)$, σ generic discrete series. For the calculations in § 6, whenever $A_{M_{BC}}(e, \nu) \neq A_G(e, \nu)$, we can find a pair (M', σ'), where M' is a Levi component $M' \supset M_{BC}$, with the following properties:

(1) $A_{M'}(e, \nu) = A_G(e, \nu)$;

(2) σ' is the generic summand of $\mathrm{Ind}_{M_{BC}}^{M'}[\sigma]$ and σ' is a limit of discrete series for M';

(3) $X(M_{BC}, \sigma, \nu, \mathsf{triv}) = X(M', \sigma', \nu)$.

3. The 0-complementary series

3.1 We specialize to the case of spherical principal series. Some of these results were already presented in the introduction in the setting of the split p-adic group. Consider the *principal series module*

$$X(\chi) = \mathbb{H} \otimes_{\mathbb{H}} \mathbb{C}_{\chi}, \quad \chi \in \mathfrak{h}. \tag{3.1.1}$$

As a W-representation, $X(\chi)$ is isomorphic to $\mathbb{C}[W]$. In particular, the module $X(\chi)$ has a unique generic subquotient and a unique spherical subquotient. The latter is denoted by $\overline{X}(\chi)$. We refer to a semisimple element χ as unitary if $\overline{X}(\chi)$ is unitary.

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The construction of intertwining operators as presented in §§ 2.6–2.11 becomes simpler in this setting. Consider the intertwining operator given by r_{w_0}, where w_0 is the longest element in the Weyl group, and normalized so that it is Id on the trivial W-type. Since the operator only depends on χ, we simply denote it by $\mathcal{A}_{w_0}(\chi) : X(\chi) \to X(w_0\chi)$.

If χ is dominant (i.e. $\langle \chi, \alpha \rangle \geqslant 0$ for all roots $\alpha \in \Delta^+$) the image of $\mathcal{A}_{w_0}(\chi)$ is $\overline{X}(\chi)$. Moreover, $\overline{X}(\chi)$ is hermitian if and only if $w_0\chi = -\chi$. The principal series $X(\chi)$ is reducible if and only if $\langle \alpha, \chi \rangle = 1$ for some $\alpha \in \Delta^+$ (see (3.1.3) below). The generic subquotient is also spherical if and only if $X(\chi)$ is irreducible.

Note that $r_{w_0} = r_{\alpha_1} \cdots r_{\alpha_k}$ acts on the right and, therefore, each α_j in the decomposition into r_{α_j} can be replaced by the scalar $\langle \alpha_j, w_j\chi \rangle$, where $w_j = s_{j+1}s_{j+2} \cdots s_k$ in the intertwining operator $\mathcal{A}_{w_0}(\chi)$. For every $(\mu, V_\mu) \in \widehat{W}$, we use the notation

$$a_\mu(\chi) = \mathcal{A}_{w_0,\mu}(\chi) : V_\mu^* \longrightarrow V_\mu^*. \tag{3.1.2}$$

Remark 3.1. Assume $w_0\chi = -\chi$. The hermitian form on $\overline{X}(\chi)$ is positive definite if and only if all of the operators $a_\mu(\chi)$ are positive semidefinite.

More precisely, the operators $a_\mu(\chi)$ are characterized by the fact that, in the decomposition $a_\mu(\chi) = a_{\mu,\alpha_1}(w_1\chi) \cdots a_{\mu,\alpha_k}(w_k\chi)$ coming from the reduced expression for w_0 as above (see also § 2.8),

$$a_{\mu,\alpha_j}(\nu) = \begin{cases} 1 & \text{on the } (+1)\text{-eigenspace of } s_{\alpha_j} \text{ on } V_\mu^*, \\ \dfrac{1 - \langle \alpha_j, \nu \rangle}{1 + \langle \alpha_j, \nu \rangle} & \text{on the } (-1)\text{-eigenspace of } s_{\alpha_j} \text{ on } V_\mu^*. \end{cases} \tag{3.1.3}$$

If α is a simple root, we have the formula [BM96, § 1.6] $t_{s_\alpha} r_w = r_w t_{s_{w^{-1}\alpha}}$. From this, since $s_{w^{-1}\alpha} = w^{-1}s_\alpha w$, it follows that

$$t_w r_w = r_w t_w \quad \text{for any } w \in W. \tag{3.1.4}$$

In particular, for $w = w_0$, we obtain that every $a_\mu(\chi)$ preserves the $(+1)$ (respectively (-1)) eigenspaces of w_0 on μ^*.

3.2 Consider χ in the (-1)-eigenspace of w_0. In order to determine whether χ is unitary, one would have to compute the operators $a_\mu(\chi)$ on the W-type μ. An operator $a_\mu(\chi)$ has constant signature on any facet in the arrangement of hyperplanes

$$\langle \chi, \alpha \rangle = 1, \quad \alpha \in \Delta^+ \quad \text{and} \quad \langle \chi, \alpha \rangle = 0, \quad \alpha \in \Pi, \tag{3.2.1}$$

in the dominant Weyl chamber $\mathcal{C}$ of $\mathfrak{h}$ (see [BC05, Theorem 2.4]).

The 0-complementary series (Definition 1.1) is a union of open regions in this arrangement of hyperplanes.

Recall that the *fundamental alcove* $\mathcal{C}_0$ is the set

$$\mathcal{C}_0 = \{\chi \in \mathcal{C} : 0 < \langle \alpha, \chi \rangle < 1, \text{ for all } \alpha \in \Pi\}. \tag{3.2.2}$$

If W_{aff} denotes the affine Weyl group, an *alcove* is, by definition, any open region in $\mathcal{C}$ which is W_{aff}-conjugate with $\mathcal{C}_0$. Clearly, any alcove is a simplex.

The main results of this section are summarized next.

Theorem 3.1. *The* 0*-complementary series are:*

- *as in Theorem 3.2 for types* $A,\ B,\ C,\ D$*;*
- *as in Proposition 3.1 for types* $G_2,\ F_4$*;*

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– *the hermitian* χ *(*$w_0\chi = -\chi$*) in the union of the two alcoves in § 7.2.1 for type* E_6*;*
– *the union of the eight alcoves in § 7.2.2 for type* E_7*;*
– *the union of the* 16 *alcoves in § 7.2.3 for type* E_8*.*

3.3 We recall the description of the 0-complementary series for Hecke algebras of classical types.

THEOREM 3.2 (Barbasch [Bar08, Theorem 3.1]). *The parameters* $\chi = (\nu_1, \nu_2, \ldots, \nu_n)$ *in the 0-complementary series are:*

A: $\chi = (\nu_1, \ldots, \nu_k, -\nu_k, \ldots, -\nu_1)$ *or* $(\nu_1, \ldots, \nu_k, 0, -\nu_k, \ldots, -\nu_1)$*, with* $0 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_k < \frac{1}{2}$*;*
C: $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n < \frac{1}{2}$*;*
B: *there exists* i *such that* $0 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_i < 1 - \nu_{i-1} < \nu_{i+1} < \cdots < \nu_n < 1$*, and between any* $\nu_j < \nu_{j+1}$*,* $i \leqslant j < n$*, there is an odd number of* $(1 - \nu_l)$*,* $1 \leqslant l < i$*;*
D: *same conditions as for type* B*, with* ν_1 *replaced by* $|\nu_1|$*; in addition, for* χ *to be hermitian, if* n *is odd, then* $\nu_1 = 0$*.*

Example. If the Hecke algebra is of type B_6 (which means the p-adic group is the split form of PSp(12)), the 0-complementary series in the dominant Weyl chamber is a union of four alcoves, in coordinates:

- $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \nu_3 \leqslant \nu_4 \leqslant \nu_5 \leqslant \nu_6 < 1 - \nu_5$;
- $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \nu_3 \leqslant \nu_4 \leqslant \nu_5 < 1 - \nu_4 < \nu_6 < 1 - \nu_3$;
- $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \nu_3 \leqslant \nu_4 \leqslant \nu_5 < 1 - \nu_4 < 1 - \nu_3 < 1 - \nu_2 < \nu_6 < 1 - \nu_1$;
- $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \nu_3 \leqslant \nu_4 < 1 - \nu_3 < \nu_5 < 1 - \nu_2 < \nu_6 < 1 - \nu_1$.

In general, for type B_n, there are $2^{\lfloor \frac{n-1}{2} \rfloor}$ unitary alcoves of this type.

3.4 We also need the description of the 0-complementary series for the Hecke algebras of type G_2 and F_4. We use the roots $\alpha_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\alpha_2 = (-1, 1, 0)$ for G_2 and $\alpha_1 = (1, -1, -1, -1)$, $\alpha_2 = (0, 0, 0, 2)$, $\alpha_3 = (0, 0, 1, -1)$, $\alpha_4 = (0, 1, -1, 0)$ for F_4.

PROPOSITION 3.1 [Ciu05, § 3.3, Appendix B]. *We have the following results.*

(1) *If* $\mathbb{H}$ *is of type* G_2 *and* $\chi = (\nu_1, \nu_1 + \nu_2, -2\nu_1 - \nu_2)$*,* $\nu_1 \geqslant 0$*,* $\nu_2 \geqslant 0$*, is a spherical parameter, the 0-complementary series is*

$$\{3\nu_1 + 2\nu_2 < 1\} \cup \{2\nu_1 + \nu_2 < 1 < 3\nu_1 + \nu_2\}. \tag{3.4.1}$$

(2) *If* $\mathbb{H}$ *is of type* F_4 *and* $\chi = (\nu_1, \nu_2, \nu_3, \nu_4)$*,* $\nu_1 - \nu_2 - \nu_3 - \nu_4 \geqslant 0$*,* $\nu_2 \geqslant \nu_3 \geqslant \nu_4 \geqslant 0$*, is a spherical parameter, the 0-complementary series is*

$$\{2\nu_1 < 1\} \cup \{\nu_1 + \nu_2 + \nu_3 - \nu_4 < 1 < \nu_1 + \nu_2 + \nu_3 + \nu_4\}. \tag{3.4.2}$$

Part (1) of Proposition 3.1 was first established in [Mui97] in the setting of the split p-adic group G_2.

3.5 In the rest of this section, we determine the 0-complementary series for types E_7 and E_8. (The method also applies in type D_n, where we recover known results of [BM96, Bar08].) For E_6, the argument needs to be modified slightly due to the fact that $w_0 \neq -1$, but it is essentially the same. It is presented in detail in [Ciu062, § 3.5].

Assume that G is of type D_{2m}, E_7 or E_8. The notation for W-types is as in [Car85]. One important nonunitarity criterion that we use is the following. Let M be a Levi subgroup

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of type A_2. The nilpotent orbit A_2 has that two lowest W-types, $\mu(A_2,\mathsf{triv})$ and $\mu(A_2,\mathsf{sgn})$ as follows:

$$\begin{array}{ll} D_{2m}: & (2m-2,1)\times(1),\ (2m-2)\times(11)\\ E_7: & 56'_a,\ 21_a\\ E_8: & 112_z,\ 28_x, \end{array} \tag{3.5.1}$$

on which operators $\mathcal{A}(\mathsf{St},\nu)$ for the standard module $X(M,\mathsf{St},\nu)$, $M=A_2$, have opposite signature whenever $Z_G(\nu)=M$. (The details for this type of calculation are in Lemma 5.1 and in § 6.) This means that for all ν such that $Z_G(\nu)=M=A_2$, the module $\overline{X}(A_2,\mathsf{St},\nu)$ is not unitary. Therefore, we have the following.

Lemma 3.1. *The generic module $X(A_2,\mathsf{St},\nu)$ is not unitary for all parameters ν such that $Z_G(\nu)=A_2$.*

3.6 Recall the hyperplane arrangement (3.2.1). The connected components of the complement of this hyperplane arrangement in $\mathcal{C}$ will be called *regions*. Inside any region $\mathcal{F}$, the intertwining operators $a_\mu(\chi)$ are isomorphisms, therefore their signature is constant in $\mathcal{F}$.

We recall first that the unbounded regions are not unitary. This is a well-known result. A proof in the setting of the Hecke algebra can be found in [BC05, § 3.3].

Lemma 3.2. *If the open region $\mathcal{F}$ is unbounded, and $\chi\in\mathcal{F}$, then the operator $a_\mu(\chi)$, for μ the reflection representation, is not positive definite.*

3.7 Recall the relation of partial order on Δ^+:

$$\beta_1>\beta_2 \quad \text{if } \beta_1-\beta_2 \text{ is a sum of positive roots.} \tag{3.7.1}$$

If $\beta_1\geqslant\beta_2$ or $\beta_2>\beta_1$, then β_1,β_2 are said to be *comparable*, otherwise they are *incomparable*. A subset of incomparable positive roots is called an *antichain*. Two roots in an antichain, being incomparable, must have nonpositive inner product.

If $\Pi=\{\alpha_1,\ldots,\alpha_n\}$ are the simple roots and a positive root β is $\beta=\sum_{i=1} m_i\alpha_i$, call $\sum_{i=1} m_i$ the *height of* β. We consider the positive roots ordered in (3.7.1) on *levels* given by the height. The simple roots are level one and the highest root is level $h-1$, where h is the Coxeter number ($h=2(n-1)$ in D_n, $h=18$ in E_7 and $h=30$ in E_8).

Any region $\mathcal{F}$ is an intersection of half-spaces $\langle\beta,\chi\rangle>1$ or $\langle\beta,\chi\rangle<1$, for all $\beta\in\Delta^+$, and $\langle\alpha,\chi\rangle\geqslant 0$, for all $\alpha\in\Pi$. Let

$$\begin{array}{c} \delta(\mathcal{F}) \text{ be the set of maximal roots among the roots } \beta<1 \text{ on } \mathcal{F}, \quad \text{and}\\ \delta'(\mathcal{F}) \text{ be the set of minimal roots among the roots } \beta'>1 \text{ on } \mathcal{F}. \end{array} \tag{3.7.2}$$

The following proposition is clear (and well known).

Proposition 3.2. *For every region $\mathcal{F}$, both $\delta(\mathcal{F})$ and $\delta'(\mathcal{F})$ are antichains in Δ^+. Moreover, the correspondences $\mathcal{F}\to\delta(\mathcal{F})$ and $\mathcal{F}\to\delta'(\mathcal{F})$ are bijections between the set of regions and the set of antichains of positive roots.*

Remark 3.2. A region $\mathcal{F}$ is infinite if and only if $\delta'(\mathcal{F})\cap\Pi\neq\emptyset$.

Proof. Let $\chi \in \mathcal{F}$ and assume that $\langle \alpha, \chi \rangle > 1$, for some simple root α. If ω_α is the corresponding coweight, for all $t \geqslant 0$, $\langle \beta, \chi + t\omega_\alpha \rangle > 1 + t \geqslant 1$, if $\beta > \alpha$, and $\langle \beta', \chi + t\omega_\alpha \rangle = \langle \beta', \chi \rangle$, for all β' incomparable to α. This implies that $\chi + t\omega_\alpha$ is in $\mathcal{F}$, for all $t \geqslant 0$. □

The walls of the region $\mathcal{F}$ (regarded as a convex polytope) are given by the hyperplanes $\beta = 1$, for $\beta \in \delta(\mathcal{F}) \cup \delta'(\mathcal{F})$, and possibly by $\alpha = 0$, for some simple roots α.

Note that a simple root α does not give a wall $\alpha = 0$ of $\mathcal{F}$ if and only there exists a root $\beta \in \delta(\mathcal{F})$ such that $\beta + \alpha$ is also a root (in the simply laced case, equivalently, $\langle \beta, \alpha \rangle = -1$). This is because in this case, for all $\chi \in \mathcal{F}$, $\langle \beta, \chi \rangle < 1 < \langle \beta + \alpha, \chi \rangle$, so one cannot set $\langle \alpha, \chi \rangle = 0$ without crossing a hyperplane $\beta = 1$. Similarly, one can formulate such a condition with the roots in $\delta'(\mathcal{F})$.

3.8 The signature of intertwining operators $a_\mu(\chi)$ on the walls of the dominant Weyl chamber is known by unitary induction from smaller groups. In D_{2n}, by setting a simple root equal to zero, we obtain a parameter unitarily induced irreducible from $D_{2n-2} + A_1$, in E_7 from D_6, and in E_8 from E_7. In particular, a region $\mathcal{F}$, which has a wall $\alpha = 0$, for some simple root α, is unitary if and only if the parameters on the wall $\alpha = 0$ are induced from a unitary region in the smaller group. This is a well-known argument, see Lemma 5.4.

We need the following information about the antichains formed of mutually orthogonal roots. We call such subsets *orthogonal antichains*.

LEMMA 3.3. *If Δ is a simply laced root system, the maximal cardinality of an orthogonal antichain in Δ^+ equals the number of positive roots at level $[(h(\Delta) + 1)/2]$, where $h(\Delta)$ is the Coxeter number.*

Proof. We verified this assertion case by case. It also follows from the main theorem in [Som05], which states that every antichain is W-conjugate to a subset of the Dynkin diagram of Δ. □

PROPOSITION 3.3. *Any unitary region $\mathcal{F}$ has a wall of the form $\alpha = 0$, for some simple root α.*

Proof. In view of Lemma 3.2, we may assume that $\mathcal{F}$ is a finite region, that is, a convex polytope. Assume by contradictions that all the walls of $\mathcal{F}$ are $\beta = 1$, for $\beta \in \delta(\mathcal{F}) \cup \delta'(\mathcal{F})$. There are two cases which we treat separately:

(a) $\mathcal{F}$ has a dihedral angle of $2\pi/3$; and

(b) all dihedral angles of $\mathcal{F}$ are non-obtuse.

(a) Let $\beta_1 \in \delta(\mathcal{F}), \beta_2 \in \delta'(\mathcal{F})$ be such that $\langle \beta_1, \beta_2 \rangle = -1$ and they give adjacent walls of $\mathcal{F}$. Let χ_0 be a parameter such that $\chi_0 \in (\beta_1 = 1) \cap (\beta_2 = 1) \cap \overline{\mathcal{F}}$, but $\langle \beta, \chi_0 \rangle \neq 1$, for any $\beta \notin \{\beta_1, \beta_2\}$. This is possible, otherwise there should exist a positive root β such that $\beta_1 = 1, \beta_2 = 1$ implies necessarily $\beta = 1$. In particular, $\{\beta_1, \beta_2, \beta\}$ are linearly dependent over $\mathbb{Z}$. Since we are in the simply laced case, one must be a sum of the other two roots, but then they cannot all be equal to one simultaneously.

The principal series $X(\chi_0)$ is reducible. The generic factor is parameterized by the nilpotent orbit A_2. By Lemma 3.1, this factor is not unitary, and therefore the region $\mathcal{F}$ is also nonunitary.

(b) Assume that all dihedral angles of $\mathcal{F}$ are non-obtuse. A classical theorem of Coxeter implies in our case that $\mathcal{F}$ must, in fact, be a simplex. We are therefore in the following situation:

$$\begin{aligned} \langle \beta_1, \beta_2 \rangle = 0 \quad &\text{if } \beta_1, \beta_2 \in \delta(\mathcal{F}) \text{ or } \beta_1, \beta_2 \in \delta'(\mathcal{F}), \\ \langle \beta, \beta' \rangle \in \{0, 1\} \quad &\text{if } \beta \in \delta(\mathcal{F}) \text{ and } \beta' \in \delta'(\mathcal{F}). \end{aligned} \tag{3.8.1}$$

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The antichains $\delta(\mathcal{F})$ and $\delta'(\mathcal{F})$ are orthogonal. Set $k=|\delta(\mathcal{F})|$, $k'=|\delta'(\mathcal{F})|$, and $k+k'=n+1$, where n is the rank of Δ. By Lemma 3.3, $k\leqslant m+1$ for D_n $(n=2m)$ and $k\leqslant 4$ for E_7, E_8, and same for k'. This immediately gives a contradiction for E_8 $(k+k'\leqslant 8<9)$. In E_7, the only possibility is $k=k'=4$, and in D_{2m}, $k=m+1, k'=m$ $(k=m, k'=m+1$ is analogous). It remains to analyze these cases.

Fix $\beta'\in\delta'(\mathcal{F})$. For all $\beta\in\delta(\mathcal{F})$, $\langle\beta,\beta'\rangle\in\{0,1\}$. If for all $\beta\in\delta(\mathcal{F})$, β' is not comparable to β (in particular, $\langle\beta,\beta'\rangle=0$), $\{\beta_1,\dots,\beta_k,\beta'\}$ would be an antichain of $k+1$ orthogonal roots, which is a contradiction. Thus, there exists β such that $\beta'>\beta$. Let α be a simple root such that $\langle\beta,\alpha\rangle=-1$, and $\beta'\geqslant\beta+\alpha>\beta$ (this is always possible in the simply laced case). Since $\beta<1$ is a wall, $\beta+\alpha>1$, so necessarily $\beta'=\beta+\alpha$ (otherwise, $\beta'>1$ would not be a wall).

To summarize, for each $\beta'\in\delta'(\mathcal{F})$, there exists $\beta\in\delta(\mathcal{F})$ such that $\beta'-\beta$ is a simple root. Similarly, for each $\beta\in\delta(\mathcal{F})$ there exists $\beta'\in\delta'(\mathcal{F})$ with $\beta'-\beta$ a simple root.

If α is a simple root, $\alpha=0$ is not a wall of $\mathcal{F}$ if and only if there exists $\beta\in\delta$ such that $\beta<1<\beta+\alpha$ in $\mathcal{F}$. From the discussion above, the region $\mathcal{F}$ is not adjacent to the walls of the dominant chamber if and only if for any α simple root, there exists $\beta\in\delta(\mathcal{F})$ and $\beta'\in\delta'(\mathcal{F})$ such that $\beta'-\beta=\alpha$.

If this is the case, we are looking at a bipartite graph with $k+k'$ vertices (roots) $\delta(\mathcal{F})\cup\delta'(\mathcal{F})$ and at least $n=k+k'-1$ edges (simple roots), such that any vertex has degree at least one. We would like to claim that this graph is connected. The only way to fail connectedness is if there exists a complete (bipartite) subgraph $\{\beta_1,\beta_2\}\cup\{\beta_1',\beta_2'\}$. This means that there exist simple roots $\alpha_1,\dots,\alpha_4$ such that

$$\beta_1'=\beta_1+\alpha_1=\beta_2+\alpha_2,\quad \beta_2'=\beta_1+\alpha_3=\beta_2+\alpha_4. \tag{3.8.2}$$

Then

$$1=\langle\beta_1',\beta_2\rangle=\langle\beta_1+\alpha_1,\beta_2\rangle=\langle\alpha_1,\beta_2\rangle, \tag{3.8.3}$$

and similarly $\langle\alpha_4,\beta\rangle=1$. However, then

$$0=\langle\beta_1',\beta_2'\rangle=\langle\beta_1+\alpha_1,\beta_2+\alpha_4\rangle=2+\langle\alpha_1,\alpha_4\rangle, \tag{3.8.4}$$

so $\langle\alpha_1,\alpha_4\rangle=-2$, which gives a contradiction (simply laced case).

If the graph is connected, it means that $\delta(\mathcal{F})$, respectively $\delta'(\mathcal{F})$, are formed from the positive roots on the same level of the root system, and moreover the two levels are consecutive. However, this is false by inspection.

Corollary 3.1. *A parameter χ is in the 0-complementary series if and only if χ can be deformed irreducibly to a point χ_0, such that $X(\chi_0)$ is unitarily and irreducibly induced from a parameter in the 0-complementary series on a proper Levi component.*

We also remark that part (b) of the proof of Proposition 3.3 can be applied to the regions $\mathcal{F}$ for which the antichains $\delta(\mathcal{F})$ and $\delta'(\mathcal{F})$ are formed only of roots at levels greater than or equal to $h(\Delta)/2$ (since the sum of two such roots cannot be a root, their inner product is non-negative). Then, all such regions are adjacent to the walls of the dominant Weyl chamber. By induction, we find that all unitary regions are of this form.

3.9 An important fact is that for the determination of the 0-complementary series, one only needs to know the signature of intertwining operators on a small number of W-types (and not on all of $\mathbb{C}[W]$). In addition to its intrinsic interest, we need this information in § 5 and for the calculations in § 6. (See § 5.8 for the explanation.)

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Table 2. The 0-relevant W-types.

Type	0-relevant W-types
A	$\{(n-1,1)\}$
B,C,D	$\{(n-1)\times(1),(1,n-1)\times(0)\}$ or $\{(n-1)\times(1),(n-2)\times(2)\}$
G_2	$\{2_1,2_2\}$ or $\{2_1,1_2,1_3\}$
F_4	$\{4_2,9_1\}$
E_6	$\{6_p,20_p\}$
E_7	$\{7'_a,27_a\}$ or $\{7'_a,21'_b\}$
E_8	$\{8_z,35_x\}$

Definition 3.1. Assume that the root system of $\mathbb{H}$ is simple. The W-types in Table 2 are called *0-relevant.*

Proposition 3.4. *A parameter χ is in the* 0*-complementary series if and only if the operators $a_\mu(\chi)$ are positive definite on all* 0*-relevant μ.*

In every list of 0-relevant W-types, the reflection representation refl appears. Recall Lemma 3.2 which states that the signature of refl in any infinite region is not positive definite. Note also that for every type of W not type A, the second W-type in a possible list of 0-relevant appears in $\mathsf{Sym}^2(\mathsf{refl})$. (In fact, for exceptional groups, this is the unique nontrivial W-type in $\mathsf{Sym}^2(\mathsf{refl})$.)

Proof. For type A, the claim follows easily from the fact that, in this case, every region § 3.6 is adjacent to a wall of the dominant Weyl chamber.

For types B,C,D, the proof is in [BC05, Proposition 3.3 and Theorem 3.4]. The proof is conceptual, and it is based on some simple calculations of determinants of intertwining operators. An essential step in the proof is the fact that the centralizer $\mathfrak{z}(\mathcal{O})$ of the nilpotent orbit $\mathcal{O}=A_1=(2,2,1,\dots,1)$ has a factor of type A_1.

Types G_2 and F_4 can be found in [Ciu05, Corollary 3.6 and Appendix B], and type E_6 is in [Ciu062, Corollary 3.5]. A similar argument as in the classical types works here as well; the argument uses the fact that the centralizer of $\mathcal{O}=A_1$ is of type A, more precisely, A_1 for G_2, A_3 for F_4, and A_5 for E_6.

For E_7 and E_8 one cannot use the same argument. The difference is that the centralizers $\mathfrak{z}(\mathcal{O})$ for $\mathcal{O}=A_1$ do not contain a factor of type A. The proof of the proposition and corollary in § 3.8 shows that a spherical parameter χ is in the 0-complementary if and only if the operators $a_\mu(\chi)$ are positive definite on:

(i) $\mu\in\{7'_a,27_a,56'_a,21_a\}$ or $\mu\in\{7'_a,21'_b,56'_a,21_a\}$ for E_7;

(ii) $\mu\in\{8_z,35_x,112_z,28_x\}$ for E_8.

In other words, on a strictly larger set than what we called 0-relevant in Table 2. In order to show that, in fact, it is sufficient to consider only the signatures of the 0-relevant W-types for E_7, E_8, we used a computer calculation. We only need to use this finer information for E_7 in one place in this paper, namely in § 6.3.3, for the nilpotent $A_1\subset E_8$ (whose centralizer is E_7). Proposition 3.4 for E_8 will not be needed in the sequel. □

4. Extended Hecke algebras

4.1 The goal is to construct graded Hecke algebras for certain disconnected groups.

Suppose that $\mathfrak{G}$ is an arbitrary linear algebraic group with connected component $\mathfrak{G}^0$, and component group $R := \mathfrak{G}/\mathfrak{G}^0$. Let $\mathbb{H}$ denote the graded Hecke algebra associated with $\mathfrak{G}^0$. Choose a pair (B, H), where B is a Borel subgroup, and $H \subset B$ a Cartan subgroup in $\mathfrak{G}^0$. Denote by $W := N_{\mathfrak{G}^0}(H)/H$, the Weyl group of $\mathfrak{G}^0$.

Lemma 4.1. *We have*

$$N_{\mathfrak{G}}(H)/H \cong R \ltimes W.$$

Proof. Let

$$R' := \{g \in \mathfrak{G} \mid gH = H,\ gB = B\}. \tag{4.1.1}$$

We show that $R \cong R'/H$. It is clear that $H \subset R'$ and $R' \cap \mathfrak{G}^0 = H$. Furthermore,

$$N_{\mathfrak{G}}(H) = R \cdot N_{\mathfrak{G}^0}(H). \tag{4.1.2}$$

Finally, R' meets every component of $\mathfrak{G}$. Indeed, if $g \in \mathfrak{G}$, then $g \cdot B = B',\ g \cdot H = H'$, where (B', H') is another pair of the same type as (B, H). Then there is $g_0 \in \mathfrak{G}^0$ such that $(g_0B, g_0H) = (B, H)$. Then $g_0g \in R'$, and belongs to the same component as g. The proof follows. □

If $g \in \mathfrak{G}$, then $(g \cdot B, g \cdot H)$ is another pair of Borel and Cartan subgroups. Thus, there exist an element $x \in \mathfrak{G}^0$ such that xg stabilizes the pair (B, H). Then xg determines an automorphism a_g of the based root datum. If $g \in \mathfrak{G}^0$, then $a_g = \mathsf{Id}$. Suppose that $g_1, g_2 \in \mathfrak{G}$, and $x_1, x_2 \in \mathfrak{G}^0$ are such that $x_1g_1,\ x_2g_2$ stabilize the pair (B, H). Then the fact that

$$x_1g_1x_2g_2 = (x_1g_1x_2g_1^{-1})(g_1g_2), \quad x_1g_1x_2g_1^{-1} \in \mathfrak{G}^0, \tag{4.1.3}$$

implies that

$$a_{g_1}a_{g_2} = a_{g_1g_2}. \tag{4.1.4}$$

Thus, the group $R \cong R'/H$ maps to the group of automorphisms of the root datum for $\mathfrak{G}^0$, and therefore maps to the automorphism group of $\mathbb{H}$, the corresponding affine graded Hecke algebra. We identify R with this automorphism group.

Definition 4.1. Let $\mathbb{H}$ denote the graded Hecke algebra for the root datum of $\mathfrak{G}^0$ (as in (2.1.4)). We define $\mathbb{H}'$ to be the semidirect product

$$\mathbb{H}' := \mathbb{C}[R] \ltimes \mathbb{H}, \tag{4.1.5}$$

where the action of R on $\mathbb{H}$ is induced by the a_g defined earlier.

4.2 We are interested in the spherical representations of $\mathbb{H}'$. This is a special case of Mackey induction. Set

$$\mathcal{K}' := R \ltimes W \quad \text{and} \quad \mathcal{K} := W. \tag{4.2.1}$$

A representation of $\mathbb{H}'$ is called *spherical* if it contains the trivial representation of $\mathcal{K}'$.

Lemma 4.2. *The center of* $\mathbb{H}'$ *is* $\mathbb{A}^{\mathcal{K}'}$.

Proof. This is clear from Proposition 2.1. □

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For every $\nu \in \mathfrak{h}^*$, we use the following notation:

$$R(\nu) = \text{the centralizer of } \nu \text{ in } R, \quad \mathbb{A}'(\nu) = \mathbb{C}[R(\nu)] \ltimes \mathbb{A}, \quad \mathbb{H}'(\nu) = \mathbb{C}[R(\nu)] \ltimes \mathbb{H}, \tag{4.2.2}$$

where $\mathbb{A}$ is the abelian part of $\mathbb{H}$ (as in (2.1.4)), and the action of $R(\nu) \subset R$ is as in Definition 4.1.

Consider

$$X'(\nu) = \mathbb{H}' \otimes_{\mathbb{A}'(\nu)} \mathbb{C}_\nu. \tag{4.2.3}$$

PROPOSITION 4.1. *Assume that (π, V) is a spherical irreducible representation of $\mathbb{H}'$. The multiplicity of the trivial representation of $\mathcal{K}'$ is one.*

Proof. Let ν be a weight of V under $\mathbb{A}$, spanned by v_ν, and define

$$R_\nu := \{r \in R \mid \pi(r)v_\nu = v_\nu\}. \tag{4.2.4}$$

Set $\mathbb{A}'_\nu := \mathbb{C}[R_\nu] \ltimes \mathbb{A}$. Then V is a quotient of $\mathbb{H}' \otimes_{\mathbb{A}_\nu} \mathbb{C}_\nu$, *via* the map $x \otimes \mathbf{1}_\nu \mapsto \pi(x)v_\nu$. However, as a $\mathcal{K}'$ module,

$$\mathbb{H}' \otimes_{\mathbb{A}_\nu} \mathbb{C}_\nu = \sum_{\mu \in \widehat{\mathcal{K}}} V_\mu \otimes (V_\mu^*)^{R_\nu}. \tag{4.2.5}$$

Thus, the trivial representation occurs exactly once in $\mathbb{H}' \otimes_{\mathbb{A}_\nu} \mathbb{C}_\nu$, and the claim follows. □

COROLLARY 4.1. *We have*

$$R_\nu = R(\nu).$$

Proof. Let V denote the spherical irreducible quotient of $\mathbb{H}' \otimes_{\mathbb{A}_\nu} \mathbb{C}_\nu$, as in the proof of Proposition 4.1. Consider the subspace

$$\left\{ \sum_{y \in R(\nu)} ky \otimes \mathbf{1}_\nu \right\}_{k \in \mathcal{K}'} \subset \mathbb{H}' \otimes_{\mathbb{A}_\nu} \mathbb{C}_\nu. \tag{4.2.6}$$

This is $\mathbb{H}'$-invariant, and isomorphic to $X'(\nu)$ from (4.2.3). Since by the analogue of (4.2.5) $X'(\nu)$ is spherical, we obtain a nontrivial homomorphism (hence, surjective)

$$X'(\nu) \longrightarrow V. \tag{4.2.7}$$

The claim follows from the fact that the stabilizer of $\mathbf{1}_\nu$ in R is $R(\nu)$. □

4.3 There is a natural extension of the Langlands classification for spherical modules to $\mathbb{H}'$. We do not make use of it in an essential way in this paper, rather it is listed here in order to make clearer the analogy between the description of $\mathcal{O}$-complementary series (§ 5, especially §§ 5.5–5.7) and the spherical unitary dual of the extended Hecke algebra constructed from the centralizer $Z(\mathcal{O})$ (see § 4.5).

PROPOSITION 4.2. *Every irreducible spherical module of $\mathbb{H}'$ is of the form*

$$L'(\nu) := \mathbb{H}' \otimes_{\mathbb{H}'(\nu)} L(\nu).$$

Two such modules $L'(\nu)$ and $L'(\nu')$ are equivalent, if and only if ν and ν' are in the same orbit under $\mathcal{K}'$.

If $\nu \geqslant 0$, then $X'(\nu)$ has a unique irreducible quotient $L'(\nu)$, if $\nu \leqslant 0$, then $X'(\nu)$ has a unique irreducible submodule $L'(\nu)$.

Proof. The proof is based on the Langlands classification for $\mathbb{H}$ and the restriction formulas listed below. We omit the details of the proof. Corollary 4.1 implies that the restriction to $\mathbb{H}$ of $X'(\nu)$ is

$$X'(\nu)|_{\mathbb{H}} = \sum_{r\in R/R(\nu)} \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_{r\nu}. \tag{4.3.1}$$

Moreover, if L_0 is any spherical $\mathbb{H}$-module in the restriction $L'(\nu)|_{\mathbb{H}}$, then

$$L'(\nu)|_{\mathbb{H}} = \sum_{r\in R/R(\nu)} r\cdot L_0. \tag{4.3.2}$$ □

Corollary 4.2. *We have the following results.*

(1) *If $L'(\nu)$ is hermitian, but $L(\nu)$ is not, then the form on $L'(\nu)$ is indefinite.*

(2) *The module $L'(\nu)$ is unitary if and only if $L(\nu)$ is unitary.*

Proof. If $L'(\nu)$ is unitary, then so is every factor of its restriction to $\mathbb{H}$; these are the $L(k\nu)$ with $k\in\mathcal{K}'$. Also, if a factor $L(k\nu)$ is not hermitian, its hermitian dual occurs in the decomposition, and necessarily the hermitian form on $L'(\nu)$ cannot be positive definite. If, on the other hand, $L(\nu)$ is unitary, then all of the $L(k\nu)$ occurring in the decomposition (4.3.1) are unitary as well. □

4.4 We can extend the definition of intertwining operators to this setting. Assume $\xi w\in R\ltimes W$. Then, similarly to § 3.1, we define a spherical $\mathbb{H}'$-operator

$$A'_{\xi w}(\nu): X'(\nu)\to X'(\xi w\nu), \quad x\otimes \mathbf{1}_\nu \mapsto x\xi r_w\otimes \mathbf{1}_{\xi w\nu}. \tag{4.4.1}$$

The operator $\mathcal{A}'_{\xi w}$ is $A'_{\xi w}$ normalized to be the identity on the trivial $\mathcal{K}'$-type. For every $\mathcal{K}'$-type μ', this induces an operator

$$a'_{\xi w,\mu'}(\nu): \mathrm{Hom}_{\mathcal{K}'}[\mu' : X'(\nu)] \longrightarrow \mathrm{Hom}_{\mathcal{K}'}[\mu' : X'(\xi w\nu)]. \tag{4.4.2}$$

Remark 4.1. When $w\nu=-\nu$, the $\mathbb{H}'$-operator $\mathcal{A}'_w(\nu)$ gives rise to a hermitian form on $\mathrm{Hom}_{\mathcal{K}'}[\mu' : X'(\pm\nu)]$ which can be naturally identified with the form induced by the $\mathbb{H}$-operator $\mathcal{A}_w(\nu)$ on $\mathrm{Hom}_{R(\nu)}[\mu' : \mathsf{triv}] = ((\mu')^*)^{R(\nu)}$.

4.5 The definitions in the previous sections can be applied to centralizers of nilpotent orbits. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$, and $Z(\mathcal{O})$ be the centralizer in G of a Lie triple $\{e,h,f\}$ of $\mathcal{O}$, with identity component $Z(\mathcal{O})^0$. We denote by $\mathbb{H}(Z(\mathcal{O}))$ (respectively $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$) the Hecke algebras $\mathbb{H}'$ (respectively $\mathbb{H}$) from Definition 4.1. In this particular case, we have

$$\mathcal{K}=W(\mathfrak{z}(\mathcal{O})),\quad \mathcal{K}'=W(Z(\mathcal{O})),\quad R=A_G(e),\quad R(\nu)=A_G(e,\nu). \tag{4.5.1}$$

By Corollary 4.2, one can identify the spherical unitary dual of $\mathbb{H}(Z(\mathcal{O}))$ with that of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

4.6 We present an interesting instance of the construction. Assume that the root system Δ is simple and it has roots of two lengths. Let $c:\Pi\to\mathbb{Z}_{\geqslant 0}$ be a function, such that $c(\alpha)=c(\alpha')$ whenever α and α' are W-conjugate. One defines the graded Hecke algebra $\mathbb{H}_c$ with parameter c as in § 2.1, in particular (2.1.5), but with commutation relation

$$\omega t_s = t_s s(\omega) + c(\alpha)\langle\omega,\check\alpha\rangle,\quad s=s_\alpha,\ \omega\in\mathfrak{h}^*. \tag{4.6.1}$$

Consider the case

$$c(\alpha)=\begin{cases}1, & \alpha \text{ long root,}\\ 0, & \alpha \text{ short root.}\end{cases} \tag{4.6.2}$$

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Denote the corresponding graded Hecke algebra by $\mathbb{H}_{1,0}$, and let $\Delta_l \subset \Delta$ denote the subset of long roots, which is a root (sub)system, and Π_l be the simple roots in Δ_l. (Note that $\Pi_l \not\subset \Pi$, in fact rank $\Delta_l =$ rank Δ.) Let $W(\Delta_l)$ be the corresponding Weyl group, and let W_s denote the reflection subgroup of W generated by the simple short roots in Π. Then W_s acts on Δ_l, and on $W(\Delta_l)$ by conjugation.

Lemma 4.3. *We have* $W = W_s \ltimes W(\Delta_l)$.

Proof. This follows from the classification of simple root systems. □

Let $\mathbb{H}(\Delta_l)$ denote the graded Hecke algebra corresponding to the root datum $(\mathcal{X}, \check{\mathcal{X}}, \Delta_l, \check{\Delta}_l, \Pi_l)$. We can apply the construction in (4.1.5) with $\mathbb{H} = \mathbb{H}(\Delta_l)$ and $R = W_s$.

Proposition 4.3. *We have* $\mathbb{H}_{1,0} \cong \mathbb{C}[W_s] \ltimes \mathbb{H}(\Delta_l)$.

Proof. In view of the definitions with generators and relations, one only needs to check that if $\beta \in \Pi_l$, then (4.6.1) holds with $s = s_\beta$. There exists a reflection s in a simple short root and $\alpha \in \Pi$ (long root) such that $\beta = s(\alpha)$, therefore $t_\beta = t_s t_{s_\alpha} t_s$. Using this, it is straightforward to check that $\omega t_{s_\beta} = t_{s_\beta} s_\beta(\omega) + \langle \omega, \check{\beta} \rangle$. □

Remark 4.2. If Δ is simple, the possible cases are:

(1) $\mathbb{H}(C_n)_{1,0} = \mathbb{C}[S_n] \ltimes \mathbb{H}(A_1^n)$;

(2) $\mathbb{H}(B_n)_{1,0} = \mathbb{C}[S_2] \ltimes \mathbb{H}(D_n)$;

(3) $\mathbb{H}(G_2)_{1,0} = \mathbb{C}[S_2] \ltimes \mathbb{H}(A_2)$;

(4) $\mathbb{H}(F_4)_{1,0} = \mathbb{C}[S_3] \ltimes \mathbb{H}(D_4)$.

The cases (1), with $n \leqslant 3$, and (2)–(4) all appear as Hecke algebras $\mathbb{H}(Z(\mathcal{O}))$.

5. Main results

In this section we present the main results of this paper. The explicit calculations (for type E_8) are presented in § 6. We only consider modules with real infinitesimal characters.

5.1 Recall $\mathcal{O} \subset \mathfrak{g}$, where $\mathfrak{g}$ is of type E_6, E_7, E_8. Let $\{e, h, f\}$ be a Lie triple for $\mathcal{O}$, and let $X(e, \chi, \mathsf{triv})$ be a generic hermitian representation. Recall the centralizer $Z(\mathcal{O})$ with Lie algebra $\mathfrak{z}(\mathcal{O})$, and the decomposition $\chi = h/2 + \nu$. The algebra $\mathfrak{z}(\mathcal{O})$ is a product of simple algebras and a torus.

By Definition 1.2, the complementary series attached to $\mathcal{O}$ is the set of all $\chi = \frac{1}{2}h + \nu$ such that the generic module $X(e, \chi, \mathsf{triv})$ is unitary (and irreducible). The parameter $\nu \in \mathfrak{z}(\mathcal{O})$ parameterizes a spherical module for the Hecke algebra $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, and by § 4, also a spherical module for the Hecke algebra $\mathbb{H}(Z(\mathcal{O}))$.

Theorem 5.1. *The parameter $\chi = h/2 + \nu$ is in the complementary series attached to $\mathcal{O}$ if and only if the corresponding parameter ν is in the* 0*-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. The* 0*-complementary series for the Hecke algebras of simple types are listed in Proposition 3.1.*

The following exceptions occur:

- $\mathcal{O} = A_1 + \widetilde{A}_1$ *in* F_4;
- $\mathcal{O} = A_2 + 3A_1$ *in* E_7;
- $\mathcal{O} \in \{A_4 + A_2 + A_1, A_4 + A_2, D_4(a_1) + A_2, A_3 + 2A_1, A_2 + 2A_1, 4A_1\}$ *in* E_8.

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In all of the exceptions, except $\mathcal{O} = 4A_1$ in E_8, the complementary series attached to $\mathcal{O}$ is smaller than the 0-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. The explicit description is recorded in § 7.

In the rest of this section, we present the elements of the proof.

5.2 The starting case is that of intertwining operators for induced modules from Levi components of *maximal* parabolic subalgebras. We would like to relate these operators with operators for Hecke algebras of rank one.

First we need to record some results about the reducibility of standard modules. Let $P = MN$ ($\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$) be a maximal parabolic, and $X(M, \sigma, \nu)$ be a standard module. Using Proposition 2.2, we can easily find the reducibility points of $X(M, \sigma, \nu)$, $\nu > 0$. The answer is given in Theorem 5.2 below. Its nature is related to conjectures of Langlands.

Let $\{e, h, f\} \subset \mathfrak{m}$ be a Lie triple parameterizing the tempered module σ. Then $\mathfrak{n}$ is a module for the $\mathfrak{sl}(2, \mathbb{C})$ generated by $\{e, h, f\}$. Let α be the unique simple root not in $\Delta(\mathfrak{m})$, and $\check{\omega}$ the corresponding coweight, which commutes with $\{e, h, f\}$. The eigenvalues of $\check{\omega}$ on $\mathfrak{n}$ are of the form $1, 2, \ldots, k$, where k is the multiplicity of α in the highest root. (For classical groups, $k \leqslant 2$.) Let

$$\mathfrak{n} = \bigoplus_{i=1}^{k} \mathfrak{n}_i \tag{5.2.1}$$

be the corresponding decomposition into eigenspaces, and decompose each $\mathfrak{n}_i$ into simple $\mathfrak{sl}(2)$ modules. The following statement follows from the geometric classification (and Proposition 2.2), and it is also known as a consequence of the main result of [MS98].

Theorem 5.2 (Muić and Shahidi [MS98, Proposition 3.3]). *Assume that σ is a generic tempered module. Let $\mathfrak{n}_i = \bigoplus_j (d_{ij})$ be the decomposition of $\mathfrak{n}_i$, $i = 1, k$, into simple $\mathfrak{sl}(2) = \mathbb{C}\langle e, h, f\rangle$ modules, where (d) denotes the simple module of dimension d. Then the reducibility points of $X(M, \sigma, \nu)$, with $\nu > 0$, are*

$$\left\{\frac{d_{ij} + 1}{2i}\right\}_{i,j}.$$

Now we restrict to the case when σ is a generic discrete series, and set $\mathcal{O} = G \cdot e$. Moreover, since $\mathfrak{m} = \mathfrak{m}_{BC}$ is a maximal Levi component, the algebra $\mathfrak{z}(\mathcal{O})$ is either $\mathfrak{sl}(2)$ or a one-dimensional torus [Car85]. If the trivial $\mathfrak{sl}(2)$ module appears in the decomposition (5.2.1), let $i(\sigma)$ denote the eigenvalue i for which it appears. This is the case precisely when $\mathfrak{z}(\mathcal{O}) = A_1$. It turns out that $i(\sigma) \in \{1, 2\}$.

Proposition 5.1. *We have the following results.*

(1) *If $\mathfrak{z}(\mathcal{O}) = T_1$ (i.e. a one-dimensional torus), then $X(M, \sigma, \nu)$ is reducible at $\nu = 0$.*
(2) *If $\mathfrak{z}(\mathcal{O}) = A_1$, then $X(M, \sigma, \nu)$ is irreducible at $\nu = 0$.*
(3) *When $\mathfrak{z}(\mathcal{O}) = A_1$, and $\mathcal{O} \neq A_4 + A_2 + A_1$ in E_8, the first reducibility point of $X(M, \sigma, \nu)$, $\nu \geqslant 0$ is*

$$\nu_0 = \frac{1}{i(\sigma)}. \tag{5.2.2}$$

(4) *When $\mathcal{O} = A_4 + A_2 + A_1$ in E_8, the first reducibility point of $X(M, \sigma, \nu)$, $\nu \geqslant 0$, is $\nu_0 = 3/10$ (while $1/i(\sigma) = 1/2$).*

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Proof. This follows from the conditions in Proposition 2.2. Alternatively, for the reducibility points $\nu > 0$, one can use Theorem 5.2 which has a different proof. When $\mathcal{O} = A_4 + A_2 + A_1$, we have $k = 6$. The trivial $\mathfrak{sl}(2)$-module appears in $\mathfrak{n}_2$, so $\nu_0 = 1/i(\sigma) = 1/2$. However, in this case, $X(M, \sigma, \nu)$ is reducible at $\frac{3}{10}$ because $\mathcal{O}' = A_4 + A_3$ is as in Proposition 2.2(1). Equivalently, because there is a two-dimensional $\mathfrak{sl}(2)$-module in $\mathfrak{n}_5$, Theorem 5.2 gives a reducibility point $\frac{3}{10}$. □

5.3 Assume that (M, σ, ν) is hermitian with σ a generic discrete series, and let $w \in W$ be such that $w(M) = M$, $w\sigma \cong \sigma$, $w\nu = -\nu$. Recall that $\mu_M(\mathcal{O}, \mathsf{triv})$ is the lowest $W(M)$-type of σ, and $\mu(\mathcal{O}, \mathsf{triv})$ is the generic lowest W-type of $X(M, \sigma, \nu)$. As in § 2, the element w gives rise to intertwining operators $\mathcal{A}_{w,\mu}(\sigma, \nu)$ on each W-type μ appearing in $X(M, \sigma, \nu)$. Recall that these operators are normalized so that $\mathcal{A}_{w,\mu(\mathcal{O},\mathsf{triv})}(\sigma, \nu)$ is the identity operator.

The following result is [BM96, Proposition 2.4]. For $\nu \gg 0$, $X(M, \sigma, \nu)$ is irreducible, so the signature on any W-type is constant. We call this the *signature at* ∞.

LEMMA 5.1. *Assume that the W-type μ satisfies the conditions:*

$$\dim \mathrm{Hom}_W[\mu : X(M, \sigma, \nu)] = 1 \quad \text{and} \quad \mathrm{Hom}_{W(M)}[\mu : \sigma] = \mathrm{Hom}_{W(M)}[\mu : \mu_M(\mathcal{O}, \mathsf{triv})].$$

Then the signature at ∞ of the operator $\mathcal{A}_{w,\mu}(\sigma, \nu)$ is

$$d(\mu) = (-1)^{\deg \mu + \deg \mu(\mathcal{O},\mathsf{triv})},$$

where $\deg \mu$ denotes the lowest harmonic degree of μ.

We now turn to the unitarity of $X(M, \sigma, \nu)$.

PROPOSITION 5.2. *Let (M, σ, ν), $\nu > 0$, be hermitian maximal parabolic data attached to a nilpotent orbit $\mathcal{O}$, with σ a generic discrete series, and $e \in \mathcal{O}$.*

(1) *Assume that $\mathfrak{z}(\mathcal{O}) = T_1$. Then there exists a lowest W-type $\mu(\mathcal{O}, \psi)$, $\psi \neq \mathsf{triv}$, of $X(M, \sigma, \nu)$, occurring with multiplicity one, such that*

$$\mathcal{A}_{w,\mu(\mathcal{O},\mathsf{triv})}(\sigma, \nu) = +\mathsf{Id} \quad \text{and} \quad \mathcal{A}_{w,\mu(\mathcal{O},\psi)}(\sigma, \nu) = -\mathsf{Id}, \quad \text{for } \nu > 0. \tag{5.3.1}$$

(2) *Assume that $\mathfrak{z}(\mathcal{O}) = A_1$. Let $\mathcal{O}'$ be the nilpotent orbit in $\mathfrak{g}$ which meets $\mathfrak{m} \times \mathfrak{z}(\mathcal{O})$ in e on $\mathfrak{m}$ and the principal orbit on $\mathfrak{z}(\mathcal{O})$. Then $\mu(\mathcal{O}', \mathsf{triv})$ occurs with multiplicity one in $X(M, \sigma, \nu)$, and*

$$\mathcal{A}_{w,\mu(\mathcal{O},\mathsf{triv})}(\sigma, \nu) = \mathsf{Id} \quad \text{and} \quad \mathcal{A}_{w,\mu(\mathcal{O}',\mathsf{triv})}(\sigma, \nu) = \left(\frac{1 - i(\sigma)\nu}{1 + i(\sigma)\nu}\right)^{\ell} \mathsf{Id}, \quad \text{for } \nu \geqslant 0, \tag{5.3.2}$$

where ℓ is some odd positive integer (which may depend on (M, σ)).

For uniformity, in case (1) of the proposition, or if (M, σ, ν) is never hermitian for $\nu > 0$, set $\mathcal{O}' = \mathcal{O}$. (This notation will be used in § 5.4.)

Proof. We give an outline of the argument. Complete details for type E_8 are presented in § 6.2.

If $\mathfrak{z}(\mathcal{O}) = T_1$, and (M, σ, ν), $\nu > 0$, is hermitian, then $A(\mathcal{O}) \neq 1$ [Car85]. The standard module $X(M, \sigma, \nu)$ has two lowest W-types $\mu(\mathcal{O}, \mathsf{triv})$ and $\mu(\mathcal{O}, \psi)$ both appearing with multiplicity one and having lowest harmonic degrees of opposite parity. At $\nu = 0$, $X(M, \sigma, 0)$ is reducible and each factor is a tempered module, therefore unitary. If $\nu > 0$, $\mu(\mathcal{O}, \mathsf{triv})$ and $\mu(\mathcal{O}, \psi)$ always occur in $X(M, \sigma, \nu)$. Having opposite signature at ∞, they have opposite signature for all $\nu > 0$.

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If $\mathfrak{z}(\mathcal{O}) = A_1$, then $\overline{X}(M, \sigma, \nu)$ has a unique lowest W-type $\mu(\mathcal{O}, \mathsf{triv})$ (see [Car85]). The module $X(M, \sigma, 0)$ is irreducible and tempered. At $\nu = 1/i(\sigma)$, all factors other than $\overline{X}(M, \sigma, \nu)$ are parameterized by strictly larger nilpotent orbits. One of the factors corresponds to the orbit $\mathcal{O}'$ and lowest W-type $\mu(\mathcal{O}', \mathsf{triv})$. We verify in every case that $\mu(\mathcal{O}', \mathsf{triv})$ satisfies the conditions of Lemma 5.1. Moreover, $\mu(\mathcal{O}', \mathsf{triv})$ has harmonic degree of opposite parity to $\mu(\mathcal{O}, \mathsf{triv})$. The claim follows then from the fact that for $\nu > \nu_0$, the two W-types $\mu(\mathcal{O}, \mathsf{triv})$ and $\mu(\mathcal{O}', \mathsf{triv})$ occur in the factor $\overline{X}(M, \sigma, \nu)$. □

We summarize this in the following corollary.

Corollary 5.1. *We have the following results.*

(1) *If $\mathfrak{z}(\mathcal{O})$ is of type T_1, then $\overline{X}(M, \sigma, \nu)$ is not unitary for $\nu > 0$.*

(2) *If $\mathfrak{z}(\mathcal{O})$ is of type A_1, then $X(M, \sigma, \nu)$ is unitary and irreducible if and only if $0 \leqslant \nu < \nu_0$, where ν_0 is the first reducibility point of $X(M, \sigma, \nu)$ on the half-line $\nu > 0$.*

Proof. Part (1) follows directly from Proposition 5.2. For part (2), we also immediately have that $X(M, \sigma, \nu)$ can only be unitary in the interval $[0, 1/i(\sigma))$. Since $X(M, \sigma, \nu)$ is irreducible and unitary at $\nu = 0$, it stays unitary until the first point of reducibility ν_0. When $\mathcal{O} \neq A_4 + A_2 + A_1$, we have $\nu_0 = 1/i(\sigma)$ (see Proposition 5.1), so this completes the argument. For $\mathcal{O} = A_4 + A_2 + A_1$ in E_8, we need an extra argument to rule out the segment $(\nu_0, 1/i(\sigma)) = (\frac{3}{10}, \frac{1}{2})$. The details of this case appear in § 6.2.4.

Remark. Note that (5.3.2) only tells us that, at the reducibility point $\nu = 1/i(\sigma)$, the order of the zero for the operator $\mathcal{A}_{w,\mu(\mathcal{O}',\mathsf{triv})}(\sigma, \nu)$ is an odd integer ℓ. This is of course sufficient to conclude that Corollary 5.1 holds. However, it is natural to expect that $\ell = 1$ for all (M, σ), where M is a Levi of a maximal parabolic, and σ is generic. We verified this conjecture by computing $\mathcal{A}_{w,\mu(\mathcal{O}',\mathsf{triv})}(\sigma, \nu)$ explicitly in all cases (M, σ) as above, when G is simply laced of rank at most seven.

5.4 Fix a nilpotent orbit $\mathcal{O}$ with Bala–Carter Levi $\mathfrak{m}_{BC}$, and let σ be the generic discrete series of $\mathbb{H}_{M_0}$ ($M = M_{BC}$) parameterized by $\mathcal{O}$.

Definition 5.1. If $\mathfrak{m}_{BC}$ is a maximal (proper) Levi subalgebra, recall the orbit $\mathcal{O}'$ constructed in Proposition 5.2. We say that the W-type μ is *σ-petite* if μ is a lowest W-type for $\mathcal{O}$ or for $\mathcal{O}'$.

If $\mathfrak{m}_{BC}$ is not maximal, let $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ be all of the Levi subalgebras, not necessarily of a standard parabolic subalgebra, such that $\mathfrak{m}_{BC} \subset \mathfrak{m}_j$, and $\mathfrak{m}_{BC}$ is a maximal Levi subalgebra of $\mathfrak{m}_j$, $j = 1, \dots, k$. For every j, let $\{\mu_{ij}\}_i$ denote the set of $W(M_j)$-types which are σ-petite in $\mathfrak{m}_j$. We say that the W-type μ is *σ-petite* (in $\mathfrak{g}$) if for every j, the only $W(M_j)$-types of σ contained in the restriction $\mu|_{W(M_j)}$ are the petite $W(M_j)$-types μ_{ij}.

Clearly, every lowest W-type of $X(M, \sigma, \nu)$ which contains $\mu_M(\mathcal{O}, \mathsf{triv})$ in its restriction to $W(M)$ is σ-petite.

Example. For the spherical principal series, that is, $\sigma = \mathsf{triv}$, $M = H$, this definition is a tautology: every W-type is σ-petite. The other extremal case is when M is maximal parabolic; then there are exactly two σ-petite W-types, those from Proposition 5.2.

An intermediate example (for $\mathbb{H}$ of simply laced type) is when $\mathcal{O} = A_1$, the minimal nilpotent orbit. Then $\sigma = \mathsf{St}$ for $M = A_1$, and a W-type μ is σ-petite if and only if $\mu|_{W(A_2)}$ does not contain the sign representation.

5.5 The following lemma should be compared with Corollary 4.2.

Lemma 5.2. *We have the following results.*

(1) *The Langlands parameter* (M, σ, ν) *is hermitian if and only if* ν *is a hermitian (spherical) parameter for* $\mathbb{H}(Z(\mathcal{O}))$.

(2) *If* (M, σ, ν) *is hermitian, but either* ν *is not a hermitian (spherical) parameter for* $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ *or* ν *is not in the semisimple part of* $\mathfrak{z}(\mathcal{O})$, *then* $\overline{X}(M, \sigma, \nu)$ *is not unitary.*

Proof. We verify these assertions in § 6. For part (2), the method is the same as in Proposition 5.2(1): we find the two lowest W-types $\mu(\mathcal{O}, \mathsf{triv})$ and $\mu(\mathcal{O}, \psi)$, $\psi \neq \mathsf{triv}$ of $\overline{X}(M, \sigma, \nu)$, occurring with multiplicity one, such that the operators $\mathcal{A}_{\mu(\mathcal{O},\mathsf{triv})}(\sigma, \nu)$ and $\mathcal{A}_{\mu(\mathcal{O},\psi)}(\sigma, \nu)$ have opposite signatures. □

5.6 The main result, Theorem 5.1, is a consequence of the construction in this section, which also provides an explanation of why such a result should hold. The method of calculation is uniform, but the details need to be checked in each case. (In § 6, we only present the detailed calculations in type E_8.) To help orient the reader, we give an outline of the method.

Recall that $X(M, \sigma, \nu)$ is an induced module, where σ is a generic discrete series parameterized by a Lie triple $\{e, h, f\} \subset \mathfrak{m}$. Also from § 2.12, recall that $\mathfrak{a}$ denotes a Cartan subalgebra of $\mathfrak{z}(\mathcal{O})$ with $\nu \in \mathfrak{a}$ and $C(\mathfrak{a}, M) \subset W$ is defined by (2.11.1). For simplicity we drop the subscript BC here. By Proposition 2.5, $C(\mathfrak{a}, M)$ is the image of a homomorphism of $W(Z(\mathcal{O}))$ to W. If w is an element of $W(Z(\mathcal{O}))$, we denote by $\overline{w}$ its image in W under this homomorphism.

By Lemma 5.2, we may assume that (M, σ, ν) is hermitian and that ν is hermitian (spherical) for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ and in the semisimple part of $\mathfrak{z}(\mathcal{O})$. This means that there exists

$$w_Z \in W(\mathfrak{z}(\mathcal{O})) \quad \text{such that } w_Z \nu = -\nu. \tag{5.6.1}$$

Let $\mathcal{A}_{\overline{w}_Z}(\sigma, \nu)$ be the $\mathbb{H}$-intertwining operator (see § 2.7) which induces the operators $\mathcal{A}_{\overline{w}_Z,\mu}(\sigma, \nu)$, $\mu \in \widehat{W}$.

The element $w_Z \in \mathfrak{z}(\mathcal{O})$ defines a spherical $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$-intertwining operators (3.1.2) $a_{\rho(\mu)}(\nu)$, $\rho(\mu) \in \widehat{W(\mathfrak{z}(\mathcal{O}))}$. We would like to show that for μ a σ-petite W-type, these two operators defined by w_Z actually *coincide*.

The idea is to decompose $\mathcal{A}_{\overline{w}_Z,\mu}(\sigma, \nu)$ into a product of factors similar to the usual decomposition of the spherical long intertwining operator (as in § 3.1) for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, such that each factor in $\mathcal{A}_{\overline{w}_Z,\mu}(\sigma, \nu)$ is identical to the corresponding simple factor in the spherical intertwining operator of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

For each simple root $\bar{\alpha} \in \Pi(\mathfrak{z}(\mathcal{O}), \mathfrak{a})$, we find an element $\bar{s}_\alpha \in C(\mathfrak{a}, M)$, which induces the corresponding simple reflection on $\mathfrak{a}$. Then the $\bar{s}_\alpha$ generate a subgroup of $C(\mathfrak{a}, M)$ isomorphic to $W(\mathfrak{z}(\mathcal{O}))$. Let $\overline{w}_Z$ be the image in $C(\mathfrak{a}, M)$ of w_Z.

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We apply the construction in § 2.7. First, the operators $\mathcal{A}_{\overline{w}_Z,\mu}(\sigma,\nu)$ decompose into a product of the form

$$\mathcal{A}_{\bar{s}_{\alpha_1},\mu}(\sigma,\nu_{\bar{\alpha}_1})\cdots\mathcal{A}_{\bar{s}_{\alpha_k},\mu}(\sigma,\nu_{\bar{\alpha}_k}), \tag{5.6.2}$$

corresponding to a decomposition $\overline{w}_Z=\bar{s}_{\alpha_1}\cdots\bar{s}_{\alpha_k}$.

Fix an $\bar{\alpha}$. The reflection $\bar{s}_\alpha$ preserves (M,σ). By Lemma 2.3 there exists a chain of adjacent Levi components $\mathfrak{m}=\mathfrak{m}_0,\dots,\mathfrak{m}_k=\mathfrak{m}$, such that $\bar{s}_\alpha$ can be decomposed into a product

$$\bar{s}_\alpha=w_k\cdots w_1, \tag{5.6.3}$$

as in (2.8.1). The operator $\mathcal{A}_{\bar{s}_\alpha,\mu}(\sigma,\nu_{\bar{\alpha}})$ acquires a decomposition accordingly into a product of maximal parabolic factors of the form $\mathcal{A}_{w_{m_j},\mu}(\widetilde{w}_j\sigma,\widetilde{w}_j\nu_{\bar{\alpha}})$, where $\widetilde{w}_j=w_{m_{j+1}}\dots w_{m_1}$.

Recall from Definition 2.6 that the space $\mathrm{Hom}_{W(M)}[\mu,\sigma]$ has a natural structure of a $W(\mathfrak{z}(\mathcal{O}))$-type, which is denoted $\rho(\mu)$, and a structure of $W(Z(\mathcal{O})$-type, which is denoted $\rho'(\mu)$.

Lemma 5.3. *With the notation above, if μ is a σ-petite W-type (Definition 5.4), and $\bar{\alpha}$ is a simple root of $\mathfrak{z}(\mathcal{O})$, then*

$$\mathcal{A}_{\bar{s}_\alpha,\mu}(\nu)=a_{\rho(\mu),\bar{\alpha}}(\nu),$$

where $a_{\rho(\mu),\bar{\alpha}}(\nu)$ is given by (3.1.3).

Proof. In the discussion above, we have decomposed $\mathcal{A}_{\bar{s}_\alpha,\mu}(\sigma,\nu_{\bar{\alpha}})$ into a product of factors, $\mathcal{A}_{w_{m_j},\mu}(\widetilde{w}_j\sigma,\widetilde{w}_j\nu_{\bar{\alpha}})$, each induced from some maximal parabolic case $\mathfrak{m}_j\subset\Sigma_j$. As such, for every j, the discrete series $\widetilde{w}_j\sigma$ is parameterized in $\mathbb{H}_{M_{j,0}}$ by a nilpotent element whose reductive centralizer $\mathfrak{z}_{\Sigma_j}$ in Σ_j is either an $\mathfrak{sl}(2)$ or a one-dimensional torus.

By inspection, in § 6, we find that in the decomposition induced by (5.6.3), there exists j_0 such that $\mathfrak{z}_{\Sigma_{j_0}}=\mathfrak{sl}(2)$, and if $j\neq j_0$, then $\mathfrak{z}_{\Sigma_j}$ is a torus. By the definition of σ-petite in the maximal parabolic case, and Proposition 5.2, the factors $j\neq j_0$ do not contribute, while the factor $j=j_0$ is identical with $a_{\rho(\mu),\bar{\alpha}}(\nu)$. □

We summarize the construction in the following proposition. Retain the previous notation, and let $X_{\mathbb{H}(\mathfrak{z}(\mathcal{O}))}(\nu)$ denote the spherical principal series for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

Proposition 5.3. *Assume that (M,σ,ν) (where $M=M_{BC}$) is hermitian with ν hermitian for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ and w_Z as in (5.6.1). If μ is a σ-petite W-type (Definition 5.4), let $\rho(\mu)$ be the corresponding $W(\mathfrak{z}(\mathcal{O}))$-type (§ 2.13).*

The $\mathbb{H}$-intertwining operator $\mathcal{A}_{\overline{w}_Z,\mu}(\sigma,\nu)$ on the space

$$\mathrm{Hom}_W[\mu:X(M,\sigma,\nu)]$$

coincides with the spherical $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$-intertwining operator $a_{\rho(\mu)}(\nu)$ on the space

$$\mathrm{Hom}_{W(\mathfrak{z}(\mathcal{O}))}[\rho(\mu):X_{\mathbb{H}(\mathfrak{z}(\mathcal{O}))}(\nu)]=\rho(\mu)^*.$$

In this matching, the generic lowest W-type $\mu(\mathcal{O},\mathsf{triv})$ corresponds to the trivial $W(\mathfrak{z}(\mathcal{O}))$-type.

5.7 Assume that we are in the setting of Proposition 5.3. If the parameter ν is such that $A_G(e,\nu)\neq A_M(e,\nu)$, then the image of the intertwining operator $\mathcal{A}_{\overline{w}_Z}(\sigma,\nu)$ is *not* irreducible. In this case, by Proposition 2.5(2) (see also the remark after (2.12.5)), we have a decomposition under the action of $A_G(e,\nu)$

$$X(M,\sigma,\nu)=\bigoplus_{(\psi,V_\psi)\in\widehat{A_G(e,\nu)}}X(M,\sigma,\nu,\psi)\otimes V_\psi, \tag{5.7.1}$$

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which induces

$$\mathrm{Hom}_W[\mu : X(M,\sigma,\nu)] = \bigoplus_{(\psi,V_\psi)\in \widehat{A_G(e,\nu)}} \mathrm{Hom}_W[\mu : X(M,\sigma,\nu,\psi)] \otimes V_\psi. \tag{5.7.2}$$

Recall that the intertwining operators are normalized so that the operator on the generic lowest W-type is identically one. Then, as in § 2.7, $\mathcal{A}_{\overline{w}_z}(\sigma,\nu)$ induces operators

$$\mathcal{A}_{\overline{w}_Z,\mu}(\sigma,\nu,\mathsf{triv}) : \mathrm{Hom}_W[\mu : X(M,\sigma,\nu,\mathsf{triv})] \to \mathrm{Hom}_W[\mu : X(M,\sigma,-\nu,\mathsf{triv})]. \tag{5.7.3}$$

Recall that in § 4 we constructed the spherical principal series $X'_{\mathbb{H}(Z(\mathcal{O}))}(\nu)$ (see (4.2.3)) for the extended Hecke algebra $\mathbb{H}(Z(\mathcal{O}))$, as well as the operators $a'_{\rho(\mu')}(\nu)$ (see (4.4.2)).

Corollary 5.2. *Retain the notation from Proposition 5.3 and (5.7.3). The $\mathbb{H}$-intertwining operator $\mathcal{A}_{\overline{w}_Z,\mu}(\sigma,\nu,\mathsf{triv})$ on $\mathrm{Hom}_W[\mu : X(M,\sigma,\nu,\mathsf{triv})]$ is identical with the $\mathbb{H}(Z(\mathcal{O}))$-intertwining operator $a'_{\rho'(\mu)}(\nu)$ (defined in (4.4.2)) on the space $\mathrm{Hom}_{W(Z(\mathcal{O}))}[\rho'(\mu) : X'_{\mathbb{H}(Z(\mathcal{O}))}(\nu)]$, which in turn is equivalent with the $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$-intertwining operator $a_{\rho(\mu)}(\nu)$ restricted to the subspace $((\rho'(\mu)^*)^{A_G(e,\nu)}$.*

Proof. Follows from Proposition 5.3 and § 4.4. □

5.8 Fix $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$, and let $M = M_{BC}$, $\{e,h,f\}$, and σ be as before. Let $\mathfrak{S}(\mathcal{O})$ denote the set of σ-petite W-types (Definition 5.1). Set

$$\rho(\mathfrak{S}(\mathcal{O})) = \{\rho(\mu) \in \widehat{W(\mathfrak{z}(\mathcal{O}))} \mid \mu \in \mathfrak{S}(\mathcal{O})\}, \tag{5.8.1}$$

where $\rho(\mu)$ is defined in Definition 2.6.

By comparison with the spherical intertwining operators in $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, the matching of intertwining operators in §§ 5.3 and 5.2 tells us the signature of the hermitian form on the σ-petite W-types.

By § 3.9, one knows a very small subset of $\widehat{W(\mathfrak{z}(\mathcal{O}))}$, the 0-relevant $W(\mathfrak{z}(\mathcal{O}))$-types (Definition 3.1), which are sufficient to detect the unitarity of the 0-complementary series. Call this set $\mathfrak{B}(\mathfrak{z}(\mathcal{O}))$.

Definition 5.2. We say that $\mathcal{O}$ satisfies the *signature criterion* if $\mathfrak{B}(\mathfrak{z}(\mathcal{O})) \subset \rho(\mathfrak{S}(\mathcal{O}))$.

Our main criterion of nonunitarity follows from this discussion.

Corollary 5.3. *We have the following results.*

1. *If $\mathcal{O}$ satisfies the signature criterion, then necessarily a parameter $\chi = h/2 + \nu$ is in the $\mathcal{O}$-complementary series of $\mathbb{H}$ if and only if ν is the 0-complementary series for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.*
2. *If $\mathbb{H}$ is of type E, the only nilpotent orbits which do not satisfy the signature criterion are $4A_1$ in E_7, and $D_4 + A_1$, $2A_2 + 2A_1$, $4A_1$ in E_8.*

Proof. Part (1) is clear. Part (2) is established by computing the σ-petite W-types. The calculations for type E_8 are in § 6. □

Note that the nilpotent $4A_1$ in E_8 is one of the exceptions in Theorem 5.1, and in fact the complementary series turns out to be larger than the 0-complementary series for the centralizer $\mathfrak{z}(\mathcal{O}) = C_4$.

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For the other cases, $4A_1$ in E_7, D_4+A_1, $2A_2+2A_1$ in E_8, we use *ad-hoc* additional arguments involving the signature of some other W-types which appears with small multiplicity (and by Springer's correspondence belong to nilpotent orbits close to $\mathcal{O}$ in the closure ordering), to prove the inclusion of the $\mathcal{O}$-complementary series of $\mathbb{H}$ into the 0-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

5.9 Case $\mathcal{O}\neq \mathbf{4A_1}$ in $\mathbf{E_8}$. Let us assume that, by the previous discussion, we know that the complementary series of $\mathcal{O}$ is included in the 0-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$.

Using the method of decomposing intertwining operators into factors coming from maximal parabolic cases (§ 2.7), and the reducibility points for maximal parabolic cases (§ 5.2), we can determine the hyperplanes of reducibility of standard modules $X(M,\sigma,\nu,\mathsf{triv})$.

We check whether any of these hyperplanes of reducibility intersects the 0-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. When this happens, we are in one of the exceptions of Theorem 5.1. In these cases, we need some extra arguments involving the signature of operators on W-types which are not σ-petite, but they rule out the nonunitary parameters $\chi=h/2+\nu$, with ν inside the 0-complementary series of $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$. The details are given in §§ 6.2.4–6.4.5.

We consider the cases when the reducibility hyperplanes do not intersect the 0-complementary series; in this case we need to show that the parameters in the 0-complementary series for $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$ are unitary for $\mathbb{H}$. Every parameter $\chi=h/2+\nu$ in this set can be deformed continuously and irreducibly to a parameter $\chi_0=h/2+\nu_0$, for which the corresponding standard module is unitarily and irreducibly induced from a unitary module on a Levi subgroup. The unitarity is a consequence of the following well-known result.

Lemma 5.4. *For $0\leqslant t\leqslant 1$, let $\xi_t\in\mathfrak{z}(\mathfrak{m})$ be a family of characters which depend continuously on t, and ξ_0 is unitary. Assume that $\mathrm{Ind}_M^G[\mathcal{V}\otimes\xi_t]$ is irreducible, where $\mathcal{V}$ is a module for $\mathbb{H}_M$. If $\mathcal{V}\otimes\xi_t$ is hermitian for all $0\leqslant t\leqslant 1$, then $\mathrm{Ind}_M^G[\mathcal{V}\otimes\xi_1]$ is unitary if and only if $\mathcal{V}$ is unitary.*

Case $\mathcal{O}=\mathbf{4A_1}$ in $\mathbf{E_8}$. Here $\mathfrak{z}(\mathcal{O})=C_4$, $M=4A_1$ and $\sigma=\mathsf{St}$. The details are in § 6.4.1. Using the signature of the σ-petite W-types, we find that the $4A_1$-complementary series is formed of parameters $\chi=h/2+\nu$, where ν must lie in one of two regions.

The first region corresponds to ν in the 0-complementary series of the $\mathbb{H}(\mathfrak{z}(\mathcal{O}))$, and we can show that χ is unitary by the same deformation argument as in Lemma 5.4.

If χ is in the second region, called $\mathcal{R}$ in § 6.4.1, a more delicate argument is needed. First we analyze the signature of other W-types, which are not σ-petite, and find that there exists only one possible unitary subregion $\mathcal{R}_3$ of $\mathcal{R}$. (The notation and explicit description are given in (6.4.1).) Now assume $\nu\in\mathcal{R}_3$. We deform ν continuously to ν_0, such that $X(4A_1,\mathsf{St},\nu)$ is irreducible for $\nu\neq\nu_0$, but $X(4A_1,\mathsf{St},\nu_0)$ is reducible. We find that $X(4A_1,\mathsf{St},\nu_0)$ has two composition factors, and that they are both unitary. Then we use a signature filtration argument (cf. [Vog84]) to conclude that $X(4A_1,\mathsf{St},\nu)$ must be unitary.

6. Explicit calculations for type E_8

The simple roots α_i and coweights $\check{\omega}_i$, $i=1,8$ in type E_8 are as in [Bou02]. The W-types for E_8 were classified in [Fra70], and we use the same labeling of the irreducible characters. (See also [Car85].) The W-structure of standard modules is given by the Green polynomials calculated in [BS84]; we also used the (unpublished) tables in [Alv05]. For restrictions of W-types and for the computation of the associated $W(\mathfrak{z}(\mathcal{O}))$-type $\rho(\mu)$ to a given W-type μ (with notation as in § 5.6), we used the software 'GAP'. For some of the explicit computations with

Table 3. Embeddings of discrete series.

Type	Nilpotent	Lowest W-type	Levi component
D_4	$(5,3)$	$1^3\times 1$	D_3
D_5	$(7,3)$	$1^4\times 1$	D_4
D_6	$(9,3)$	$1^5\times 1$	D_5
	$(7,5)$	$1^4\times 1^2$	A_5
D_7	$(11,3)$	$1^6\times 1$	D_6
	$(9,5)$	$1^5\times 1^2$	A_6
E_6	$E_6(a_1)$	$6'_p$	D_5
	$E_6(a_3)$	$30'_p$	A_5
E_7	$E_7(a_1)$	7_a	E_6
	$E_7(a_2)$	$27'_a$	E_6
	$E_7(a_3)$	56_a	D_6
	$E_7(a_4)$	189_b	D_5+A_1
	$E_7(a_5)$	315_a	A_5+A_1

intertwining operators in the maximal parabolic cases for exceptional groups (see the remark after Proposition 5.2), we used integer matrix models of W-types, and the software 'Mathematica'. The classification and labeling of nilpotent orbits is as in [Car85].

6.1 If a nilpotent orbit is distinguished, it parameterizes discrete series and, in particular, exactly one generic discrete series. The corresponding infinitesimal characters are in the tables of § 7.

For the explicit calculations of intertwining operators that we need (see the remark at the end of § 5.3), when the standard module is not induced from a Steinberg representation on a Levi subalgebra, we embed it into an induced from the Steinberg representation from a smaller subalgebra, such that the generic lowest W-type appears with multiplicity one. This is possible because the rank is small. Table 3 lists embeddings for discrete series. We give the distinguished non-principal nilpotent orbit $\mathcal{O}$, the lowest W-type μ_0 corresponding to the trivial representation in $\widehat{A(\mathcal{O})}$, and a Levi component M such that $\dim \mathrm{Hom}_W[\mu_0 : \mathrm{Ind}_M^G(\mathsf{St})] = 1$.

6.2 For the maximal parabolic cases, we verify all of the details of the argument outlined in the proof of Proposition 5.2. Depending on the details of the discussion, there are three types of arguments that we consider. For each type, we present the details in one example, then list the other nilpotents for which the same argument applies. The only exception is the nilpotent $A_4+A_2+A_1$, which we treat separately.

To simplify notation, we denote by μ_0, μ_0', ..., the lowest W-types $\mu(\mathcal{O},\mathsf{triv})$, $\mu(\mathcal{O},\psi)$, ..., and by $\mu_1,\mu_2,\ldots$, the W-types of the form $\mu(\mathcal{O}',\mathsf{triv})$.

6.2.1 *Nilpotent* $\mathbf{E_7}$. The centralizer is $\mathfrak{z}(\mathcal{O})=A_1$, the lowest W-type is $\mu_0=84'_x$, and the infinitesimal character is $\chi=(0,1,2,3,4,5,-17/2,17/2)+\nu\check{\omega}_8$, with $\nu\geqslant 0$.

The standard module corresponding to $\mathcal{O}=E_7$ is $X(E_7,\mathsf{St},\nu)$. The first reducibility point is at $\nu_0=\frac{1}{2}$, where the generic factor is parameterized by the nilpotent orbit $\mathcal{O}'=E_8(a_3)$ and

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Table 4. Maximal parabolic cases, $\mathfrak{z}(\mathcal{O}) = A_1$, type one.

$\mathcal{O}$	χ	$\mu_{\mathbf{0}}$	$\mathcal{O}'$	$\mu_{\mathbf{1}}$
E_7	$(0, 1, 2, 3, 4, 5, -\frac{17}{2}, \frac{17}{2}) + \nu\check{\omega}_8$	$84'_x$	$E_8(a_3)$	$112'_z$
$E_7(a_1)$	$(0, 1, 1, 2, 3, 4, -\frac{13}{2}, \frac{13}{2}) + \nu\check{\omega}_8$	$567'_x$	$E_8(b_4)$	$560'_z$
D_7	$(0, 1, 2, 3, 4, 5, 6, 0) + \nu\check{\omega}_1$	$400'_z$	$E_8(a_5)$	$700'_x$
$E_7(a_2)$	$(0, 1, 1, 2, 2, 3, -\frac{11}{2}, \frac{11}{2}) + \nu\check{\omega}_8$	$1344'_x$	$E_8(b_5)$	$1400'_z$
A_7	$(-\frac{17}{4}, -\frac{13}{4}, -\frac{9}{4}, -\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{7}{4}) + \nu\check{\omega}_2$	$1400'_{zz}$	$E_8(b_6)$	$2240'_x$
$E_7(a_5)$	$(0, 0, 1, 1, 1, 2, -\frac{5}{2}, \frac{5}{2}) + \nu\check{\omega}_8$	7168_w	$E_8(a_7)$	4480_y

Table 5. Maximal parabolic cases, $\mathfrak{z}(\mathcal{O}) = A_1$, type two.

$\mathcal{O}$	χ	$\mu_{\mathbf{0}}$	$\mathcal{O}'$	$\mu_{\mathbf{1}}$	$\mu_{\mathbf{2}}$
$E_6 + A_1$	$(0, 1, 2, 3, 4, -\frac{9}{2}, -\frac{7}{2}, 4) + \nu\check{\omega}_7$	$448'_z$	$E_7(a_2)$	$1344'_x$	$1008'_z$
$E_7(a_3)$	$(0, 0, 1, 1, 2, 3, -\frac{9}{2}, \frac{9}{2}) + \nu\check{\omega}_8$	$2268'_x$	$D_7(a_1)$	$3240'_z$	$1050'_x$
$E_7(a_4)$	$(0, 0, 1, 1, 1, 2, -\frac{7}{2}, \frac{7}{2}) + \nu\check{\omega}_8$	$6075'_x$	$D_5 + A_2$	$4536'_z$	$840'_x$
$A_6 + A_1$	$(\frac{13}{4}, -\frac{9}{4}, -\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \frac{1}{4}) + \nu\check{\omega}_3$	$2835'_x$	$D_5 + A_2$	$4536'_z$	$840'_x$
$E_6(a_3)A_1$	$(0, 0, 1, 1, 2, -\frac{5}{2}, -\frac{3}{2}, 2) + \nu\check{\omega}_7$	3150_y	$E_7(a_5)$	7168_w	1680_y
$D_5(a_1)A_2$	$(0, 1, 1, 2, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}) + \nu\check{\omega}_6$	1344_w	$E_6(a_3)A_1$	1134_y	448_w
$A_4 + A_3$	$(0, 1, 2, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, 1) + \nu\check{\omega}_5$	420_y	$D_5(a_1)A_2$	1344_w	1134_y

lowest W-type $\mu_1 = 112'_z$. The W-types μ_0 and μ_1 have opposite signs at infinity. Since the nilpotent $\mathcal{O}'$ is distinguished, there cannot be another factor with lowest W-type μ_1 for $\nu > \nu_0$. Therefore, μ_0 and μ_1 stay in the same factor for $\nu > \nu_0$. The complementary series is $0 \leqslant \nu < \frac{1}{2}$. The other cases of this type are listed in Table 4.

6.2.2 *Nilpotent* $\mathbf{E_6} + \mathbf{A_1}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = A_1$, the lowest W-type is $\mu_0 = 448'_z$, and the infinitesimal character is $(0, 1, 2, 3, 4, -\frac{9}{2}, -\frac{7}{2}, 4) + \nu\check{\omega}_7$.

The standard module is $X(E_6 + A_1, \mathsf{St}, \nu)$, $\nu \geqslant 0$. The first reducibility point is at $\nu_0 = \frac{1}{2}$, where the generic factor is parameterized by the nilpotent orbit $E_8(b_5)$ and lowest W-type $1400'_z$. (However, the argument from the nilpotent E_7 does not apply here since $448'_z$ and $1400'_z$ have the same signature at ∞.)

At $\nu = \nu_0$ there may also be a factor parameterized by the nilpotent orbit $\mathcal{O}' = E_7(a_2)$ with lowest W-type $\mu_1 = 1344'_x$. The W-types μ_0 and μ_1 may only be separate for $\nu = \nu_0$. The reason is that for $\nu > \nu_0$, any irreducible factor with lowest W-type μ_1 must also contain the W-type $\mu_2 = 1008'_z$. However, μ_2 does not appear in $X(E_6 + A_1, \mathsf{St}, \nu)|_W$ at all. Moreover, since μ_0 and μ_1 have opposite signs at infinity, they must be separate at least once, so they are separate exactly at $\nu = \nu_0$. The complementary series is $0 \leqslant \nu < \frac{1}{2}$. The other cases of this type are listed in Table 5.

6.2.3 *Nilpotent* $\mathbf{D_7(a_1)}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = T_1$, and the infinitesimal character is $(0, 1, 1, 2, 3, 4, 5, 0) + \nu\check{\omega}_1$.

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TABLE 6. Maximal parabolic cases, $\mathfrak{z}(\mathcal{O}) = T_1$.

$\mathcal{O}$	χ	$\mu_{\mathbf{0}}$	$\mu'_{\mathbf{0}}$
$D_7(a_1)$	$(0, 1, 1, 2, 3, 4, 5, 0) + \nu\check{\omega}_1$	$3240'_z$	$1050'_x$
$E_6(a_1) + A_1$	$(0, 1, 1, 2, 3, -\frac{7}{2}, -\frac{5}{2}, 3) + \nu\check{\omega}_7$	$4096'_z$	$4096'_x$
$D_7(a_2)$	$(0, 1, 1, 2, 2, 3, 4, 0) + \nu\check{\omega}_1$	$4200'_x$	$3360'_z$
$D_5 + A_2$	$(0, 1, 2, 3, -3, -2, -1, 2) + \nu\check{\omega}_6$	$4536'_z$	$840'_x$

The standard module $X(D_7, \sigma, \nu)$, where σ is the generic discrete series parameterized by the nilpotent orbit $(11, 3)$ in the Hecke algebra of type D_7, is reducible at $\nu = 0$, and it has two lowest W-types for $\nu > 0$, $\mu_0 = 3240'_z$ and $\mu'_0 = 1050'_x$.

At $\nu = 0$, X breaks into the sum of tempered modules, each containing one lowest W-type, which are unitary. For $\nu > 0$, μ_0 and μ'_0 stay in the same factor, and they have opposite signs at infinity. There is no complementary series. The generic module is unitary only at $\nu = 0$. The other cases of this type are listed in Table 6.

6.2.4 *Nilpotent* $\mathbf{A_4 + A_2 + A_1}$. The centralizer is A_1, the lowest W-type is $\mu_0 = 2835_x$, and the infinitesimal character is $(0, 1, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}) + \nu\check{\omega}_4$. The standard module is $X(A_4 + A_2 + A_1, \mathsf{St}, \nu)$, $\nu \geqslant 0$. The first reducibility point is at $\nu_0 = \frac{3}{10}$, where the generic factor is parameterized by the nilpotent $\mathcal{O}' = A_4 + A_3$ and the W-type $\mu_1 = 420_y$. There are exactly two composition factors at this point, one parameterized by $\mathcal{O}$ (with lowest W-type μ_0), and the generic factor. Then, either all of the W-types in the generic factor change sign at $\nu = \nu_0$, or none of them do. A direct calculation shows that the determinant of the operator on the W-type $35'_x$ has opposite sign to the scalar on the sign representation $1'_x$, in this interval. It follows that on the interval $(\frac{3}{10}, \frac{1}{2})$, also μ_1 has negative sign. The next reducibility point is at $\nu = \frac{1}{2}$. A similar argument as for the nilpotent $E_6 + A_1$ (§ 6.2.2), shows that $\overline{X}(A_4 + A_2 + A_1, \mathsf{St}, \nu)$ is not unitary for $\nu > 1/2$. The complementary series is $0 \leqslant \nu < \frac{3}{10}$.

For the rest of the nilpotent orbits in E_8, we check the details of the argument outlined in the proof of Proposition 5.3 in every case, and determine the correspondences between intertwining operators on W-types and spherical operators on $W(\mathfrak{z}(\mathcal{O}))$-types. The exceptions (i.e. the nilpotent orbits for which the complementary series is not the same as the 0-complementary series of the centralizer) are discussed separately. If Δ_1 is a root system, and $\Delta_2 \subset \Delta_1$ is a subsystem, we denote by $w_m(\Delta_1, \Delta_2)$, the element $w_0(\Delta_1) \cdot w_0(\Delta_2)$.

6.3 Single lowest W-type orbits

We begin with two representative examples.

6.3.1 *Nilpotent* $\mathbf{E_6}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = G_2$, the lowest W-type is $\mu_0 = 525'_x$, and the infinitesimal character is $(0, 1, 2, 3, 4, -4, -4, 4) + \nu_2(0, 0, 0, 0, 0, 0, 1, 1) + \nu_1(0, 0, 0, 0, 0, 1, 1, 2)$, with $\nu_1 \geqslant 0$, $\nu_2 \geqslant 0$. The standard module is $X(E_6, \mathsf{St}, \nu)$, $\nu = (\nu_1, \nu_2)$.

The subgroup $W(\mathfrak{z}) \cong W(G_2) \subset W$ is generated by

$$\bar{s}_1 = w_m(E_7, E_6), \quad \bar{s}_2 = s_8. \tag{6.3.1}$$

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Table 7. The restrictions of W-types.

	Nilpotent					
	E_6	E_6A_1	$E_7(a_2)$	$E_8(b_5)$	$E_8(b_5)$	$E_8(a_5)$
W-type	$525'_x$	$448'_z$	$1344'_x$	$1008'_z$	$1400'_z$	$700'_x$
Multiplicity	1	1	2	1	2	1
$E_6 \subset E_7$	21_b	21_b	$27'_a,\ 21_b$	$27'_a$	$27'_a,\ 21_b$	$27'_a$
A_1	(2)	(11)	(2), (11)	(2)	(2), (11)	(11)
$W(G_2)$	1_1	1_4	2_2	1_3	2_1	1_2

The intertwining operator $A(E_6, \mathsf{St}, \nu)$ decomposes according to the decomposition $w_m = (\bar{s}_1 \cdot \bar{s}_2)^3$.

The restrictions of W-types are given in Table 7.

On the factor corresponding to $\bar{s}_2$, the root α_8 takes values $3\nu_1 + 2\nu_2,\ 3\nu_1 + \nu_2$, and ν_2.

The factor corresponding to $\bar{s}_1$ is induced from an intertwining operator for the Hecke algebra of type E_7, with the nilpotent orbit E_6 in E_7, and infinitesimal character $(0, 1, 2, 3, 4, -4, -4, 4) + \bar{\nu}(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$, where $\bar{\nu}$ takes the values $\nu_1,\ 2\nu_1 + \nu_2$ and $\nu_1 + \nu_2$.

The reducibility hyperplanes for $X(E_6, \mathsf{St}, \nu)$ are $\nu_1, \nu_2 + 2\nu_1, \nu_1 + \nu_2 = 1$ and $2\nu_2 + 3\nu_1, \nu_2 + 3\nu_1, \nu_2 = 1$ (as in the centralizer G_2), and $\nu_1, \nu_2 + 2\nu_1, \nu_1 + \nu_2 = 5, 9$.

The operators match as follows:

$\widehat{W}$	$525'_x$	$448'_z$	$1344'_x$	$1008'_z$	$1400'_z$	$700'_x$
$\widehat{W(G_2)}$	1_1	1_4	2_2	1_3	2_1	1_2

and all of the relevant $W(G_2)$-types are matched.

6.3.2 *Nilpotent* $\mathbf{D_4 + A_1}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = C_3$, the lowest W-type is $\mu_0 = 700_{xx}$, and the infinitesimal character is $(0, 1, 2, 3, -\frac{1}{2}, \frac{1}{2}, 0, 0) + (0, 0, 0, 0, \nu_1, \nu_1, -\nu_2 + \nu_3, \nu_2 + \nu_3)$. The standard module is $X(D_4 + A_1, \mathsf{St}, \nu)$, where $\nu = (\nu_1, \nu_2, \nu_3)$.

The hyperplanes of reducibility are $\nu_i = \frac{1}{2},\ \nu_i \pm \nu_j = 1$, as for the centralizer C_3, and $\nu_i = \frac{3}{2}, \frac{7}{2}, \frac{9}{2}, \pm\nu_i \pm \nu_j = 4$ and $\pm\nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2}$.

The operators match as follows:

$\widehat{W}$	700_{xx}	2800_z	6075_x	5600_z
$\widehat{W(C_3)}$	3×0	0×3	1×2	0×12

The W-types that match operators from C_3 are not sufficient for concluding that the generic complementary series is included in that for C_3. They are positive in the unitary region for C_3: $\{0 \leqslant \nu_3 \leqslant \nu_2 \leqslant \nu_1 < \frac{1}{2}\}$, but also in the region $\mathcal{R} = \{\nu_1 + \nu_3 > 1,\ \nu_2 > \frac{1}{2},\ \nu_1 - \nu_3 < 1,\ \nu_2 + \nu_3 < 1\}$.

We also need to use the signature of the operator on the W-type 4200_z, which has multiplicity four.

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Among the extra hyperplanes of reducibility, $\nu_1+\nu_2-\nu_3=\frac{3}{2}$ cuts the region $\mathcal{R}$ into two open subregions as follows.

- Region $\mathcal{R}_1$: $\nu_1+\nu_2-\nu_3<\frac{3}{2}$, sample point $(1,\frac{5}{8},\frac{3}{16})$. The determinant of 4200_z is negative in this region.
- Region $\mathcal{R}_2$: $\nu_1+\nu_2-\nu_3>\frac{3}{2}$, and the determinant of 4200_z is positive. We choose a point on the boundary of the region, which is unitarily induced: $(\frac{7}{8},\frac{7}{8},\frac{1}{8})$. The corresponding parameter is induced from D_7, with $(0,1,2,3,-\frac{3}{8},\frac{5}{8},\frac{7}{4})$ in D_7. Furthermore, this can be deformed irreducibly to $(0,1,2,3,-\frac{1}{2},\frac{1}{2},\frac{7}{4})$, which unitarily induced from $(0,1,2,3,\frac{7}{4})$ in D_5 and the signatures are induced from D_5. In D_5, for this parameter, $21^3\times 0$ and $1^4\times 1$ have opposite signs.

Since 4200_z contains in this restriction $21^3\times 0$ and $1^4\times 1$, it follows that the form is indefinite on it, so the lowest W-type factor is not unitary at this boundary point. However, then the entire region $\mathcal{R}_2$ must be nonunitary.

6.3.3 We list the matching of W-types for the other nilpotent orbits in E_8 of similar kind. The infinitesimal characters $\chi=h/2+\nu$ are in the tables in § 7.

6.4 Exceptions

6.4.1 *Nilpotent* $\mathbf{4A_1}$. We present the case of the complementary series for the nilpotent orbit $4A_1$ in detail. This is the only case in which the complementary series is larger than the 0-complementary series of the centralizer, which is of type C_4. The standard module is $X(4A_1,\mathsf{St},\nu)$, $\nu=(\nu_1,\nu_2,\nu_3,\nu_4)$, and it has lowest W-type $\mu_0=50_x$. The infinitesimal character is $(0,1,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(0,0,\nu_1,\nu_1,\nu_2,\nu_2,-\nu_3+\nu_4,\nu_3+\nu_4)$.

The operators match as follows.

$\widehat{W}$	50_x	210_x	560_z	567_x	300_x
$\widehat{W(C_4)}$	4×0	0×4	1×3	0×13	0×22

These W-types only change sign when passing a hyperplane as in C_4: $\nu_i=\frac{1}{2}$ and $\pm\nu_j+\nu_i=1$. We know that the region $\nu_4<\frac{1}{2}$ is the only unitary 0-complementary series in C_4. The W-types above are not sufficient however to rule out all other (four-dimensional) open regions in C_4. They are all positive semidefinite also in the region $\mathcal{R}=\{\nu_1+\nu_4<1,\nu_2+\nu_3<1,\nu_2+\nu_4>1,\nu_3>\frac{1}{2}\}$.

The hyperplanes of reducibility $-\nu_2+\nu_3+\nu_4=\frac{3}{2}$, $-\nu_1+\nu_3+\nu_4=\frac{3}{2}$ and $\nu_1+\nu_3+\nu_4=\frac{3}{2}$ cut the region $\mathcal{R}$ into the following open regions:

(i) $\mathcal{R}_1=\{\nu_1+\nu_4<1,\nu_2+\nu_3<1,\nu_2+\nu_4>1,\nu_3>\frac{1}{2},-\nu_2+\nu_3+\nu_4>\frac{3}{2}\}$;

(ii) $\mathcal{R}_2=\{\nu_1+\nu_4<1,\nu_2+\nu_3<1,\nu_2+\nu_4>1,\nu_3>\frac{1}{2},-\nu_2+\nu_3+\nu_4<\frac{3}{2}<-\nu_1+\nu_3+\nu_4\}$;

(iii) $\mathcal{R}_3=\{\nu_1+\nu_4<1,\nu_2+\nu_3<1,\nu_2+\nu_4>1,\nu_3>\frac{1}{2},-\nu_1+\nu_3+\nu_4<\frac{3}{2}<\nu_1+\nu_3+\nu_4\}$;

(iv) $\mathcal{R}_4=\{\nu_1+\nu_4<1,\nu_2+\nu_3<1,\nu_2+\nu_4>1,\nu_3>\frac{1}{2},\nu_1+\nu_3+\nu_4>\frac{3}{2}\}$.

In $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_4$, one can deform the parameter to $\nu_1=0$, where the module is unitarily induced irreducible from a nonunitary module attached to $4A_1$ in E_7.

PROPOSITION 6.1. *The open region* $\mathcal{R}_3$ *is unitary:*

$$\{\nu_1+\nu_4<1,\ \nu_2+\nu_3<1,\nu_2+\nu_4>1,\ -\nu_1+\nu_3+\nu_4<\tfrac{3}{2}<\nu_1+\nu_3+\nu_4\}. \tag{6.4.1}$$

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TABLE 8. Nilpotent orbits $\mathcal{O}$ with single lowest W-type.

$\mathcal{O}$	$\mathfrak{z}(\mathcal{O})$	Matching
$\mathbf{E_6}$	G_2	$525'_x$ $448'_z$ $1344'_x$ $1008'_z$ $1400'_z$ $700'_x$ 1_1 1_4 2_2 1_3 2_1 1_2
$\mathbf{D_6}$	B_2	$972'_x$ $2268'_x$ $3240'_z$ $1050'_x$ 2×0 11×0 1×1 0×2
$\mathbf{A_6}$	$2A_1$	$4200'_z$ $6075'_x$ $2835'_x$ $(2)\otimes(2)$ $(11)\otimes(2)$ $(2)\otimes(11)$
$\mathbf{D_5+A_1}$	$2A_1$	$3200'_x$ $5600'_z$ $6075'_x$ $(2)\otimes(2)$ $(2)\otimes(11)$ $(11)\otimes(11)$
$\mathbf{A_5+A_1}$	$2A_1$	2016_w 3150_y 4200_y $(2)\otimes(2)$ $(2)\otimes(11)$ $(11)\otimes(2)$
$\mathbf{D_5}$	B_3	2100_y $3200'_x$ $5600'_z$ $2400'_z$ $6075'_x$ 3×0 12×0 2×1 0×3 1×2
$\mathbf{D_5(a_1)+A_1}$	$2A_1$	6075_x 4200_z 2400_z $(2)\otimes(2)$ $(2)\otimes(11)$ $(11)\otimes(2)$
$\mathbf{E_6(a_3)}$	G_2	5600_z 3150_y 7168_w 5600_w 4480_y 1680_y 1_1 1_4 2_2 2_2 2_1 1_3
$\mathbf{A_5}$	G_2+A_1	3200_x 2016_w 5600_z 4200_y 3150_y 4480_y 1680_y $(1_1),(2)$ $(1_4),(2)$ $(1_1),(11)$ $(2_2),(2)$ $(1_4),(11)$ $(2_1),(11)$ $(1_3),(11)$
$\mathbf{A_4+A_2}$	$2A_1$	4536_z 2835_x 6075_x $(2)\otimes(2)$ $(2)\otimes(11)$ $(11)\otimes(2)$
$\mathbf{2A_3}$	B_2	840_x 4200_x 4536_z 2×0 11×0 1×1
$\mathbf{D_4+A_1}$	C_3	700_{xx} 2800_z 6075_x 5600_z 3×0 0×3 1×2 0×12
$\mathbf{A_3+A_2+A_1}$	$2A_1$	$1400'_{zz}$ $4096'_x$ $2240'_x$ $(2)\otimes(2)$ $(11)\otimes(2)$ $(2)\otimes(11)$
$\mathbf{A_3+2A_1}$	B_2+A_1	1050_x 1400_x 972_x 3240_z $(2\times 0)\otimes(2)$ $(0\times 2)\otimes(2)$ $(11\times 0)\otimes(2)$ $(0\times 2)\otimes(11)+(1\times 1)\otimes(2)$
$\mathbf{2A_2+2A_1}$	B_2	175_x 1050_x 972_x 3240_z 2×0 11×0 0×11 1×1[1]
$\mathbf{D_4}$	F_4	525_x 700_{xx} 2800_z 2100_x 6075_x 4200_z 5600_z 1_1 2_3 4_2 2_1 9_1 8_3 8_1
$\mathbf{A_3+A_1}$	B_3+A_1	1344_x 1400_z 1050_x 1400_x 350_x $(3\times 0)\otimes(2)$ $(3\times 0)\otimes(11)$ $(21\times 0)\otimes(2)$ $(2\times 1)\otimes(11)$ $(0\times 3)\otimes(11)$
$\mathbf{2A_2+A_1}$	G_2+A_1	448_z 1344_x 175_x 1050_x $1_1\otimes(2)$ $1_1\otimes(11)$ $1_4\otimes(2)$ $2_2\otimes(11)$
$\mathbf{A_3}$	B_5	567_x 1344_x 1400_z 56_z 1050_x 1400_x 350_x 5×0 41×0 4×1 0×5 32×0 3×2 1×4
$\mathbf{A_2+3A_1}$	G_2+A_1	400_z 700_x 448_z 1344_x 1008_z 1400_z $1_1\otimes(2)$ $1_1\otimes(11)$ $1_4\otimes(11)$ $2_2\otimes(11)$ $1_3\otimes(11)$ $2_1\otimes(11)$
$\mathbf{A_2+2A_1}$	B_3+A_1	560_z 567_x 400_z 700_x 300_x $(3\times 0)\otimes(2)$ $(3\times 0)\otimes(11)$ $(12\times 0)\otimes(2)$ $(2\times 1)\otimes(2)$ $(0\times 3)\otimes(2)$
$\mathbf{4A_1}$	C_4	50_x 210_x 560_z 567_x 300_x 4×0 0×4 1×3 0×13 0×22

TABLE 8. (Continued.)

$\mathcal{O}$	$\mathfrak{z}(\mathcal{O})$	Matching				
$\mathbf{3A_1}$	F_4+A_1	84_x $1_1\otimes(2)$	112_z $1_1\otimes(11)$	50_x $2_3\otimes(2)$	210_x $4_2\otimes(11)$	160_z $2_1\otimes(11)$
$\mathbf{2A_1}$	B_6	35_x 6×0	84_x 15×0	112_z 5×1	28_x 0×6	50_x 33×0
$\mathbf{A_1}$	E_7	8_z 1_a	35_x 7_a	84_x $21'_b$	50_x $15'_a$	

$\mathcal{O}=2A_2+2A_1$: although not identical with 1×1, the operator on 3240_z has the same signature as 1×1 in the open regions of B_2.

Proof. We divide the proof into four parts.

Step 1. The generic modules are unitary on the walls of $\mathcal{R}_3$.

For each wall, we find the nilpotent orbit $\mathcal{O}'$ parameterizing the generic module: $\nu_1+\nu_4=1$, $\nu_2+\nu_3=1$, $\nu_2+\nu_4=1$ correspond to A_2+2A_1, and $\nu_1+\nu_3+\nu_4=\frac{3}{2}$, $-\nu_1+\nu_3+\nu_4=\frac{3}{2}$ correspond to A_2+3A_1. The claim follows then by comparison with the complementary series attached to the nilpotent orbits A_2+2A_1 and A_2+3A_1.

We deform the parameter to a particular point on the walls: $p=(\frac{1}{12},\frac{3}{12},\frac{9}{12},\frac{9}{12})$. The point p lies at the intersection of the walls $\nu_2+\nu_3=1$ and $\nu_2+\nu_4=1$. The corresponding point $\bar{p}=\frac{1}{2}h+p$, in E_8-coordinates is $\bar{p}=(0,1,-\frac{5}{12},\frac{7}{12},-\frac{3}{12},\frac{9}{12},\frac{18}{12},0)$.

Step 2. The standard module $X(4A_1,\mathsf{St},p)$ has two composition factors: $\overline{X}(4A_1,\mathsf{St},p)$ and $X(A_2+2A_1,\mathsf{St},p)$.

The standard module $X(4A_1,\mathsf{St},p)$ is reducible. A necessary condition for a nilpotent $\mathcal{O}'>\mathcal{O}$ to parameterize a composition factor is that $w\bar{p}=\frac{1}{2}h'+\nu'$, for h' the middle element of a Lie triple $\{e',h',f'\}$ of $\mathcal{O}'$, $w\in W$ and $\nu'\in\mathfrak{z}(\mathcal{O}')$. We check that the nilpotent $\mathcal{O}'$ satisfying the condition are A_2+A_1 and A_2+2A_1, so potentially there are three factors. Here A_2+2A_1 parameterizes the generic factor. The lowest W-type of A_2+A_1 is 210_x, and the operator on 210_x matches 0×4 in C_4, so it is invertible at p.

Step 3. The non-generic factor $\overline{X}(4A_1,\mathsf{St},p)$ is unitary.

The point $\bar{p}$ is unitarily induced reducible from D_7. The corresponding nilpotent in D_7 is (1^32^43) and the infinitesimal character is of the form $(0,1,-\frac{1}{2}+\bar{\nu}_1,\frac{1}{2}+\bar{\nu}_1,-\frac{1}{2}+\bar{\nu}_2,\frac{1}{2}+\bar{\nu}_2,\bar{\nu}_3)$, with $(\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3)=(\frac{1}{12},\frac{3}{12},\frac{18}{12})$. Moreover, the parameter in D_7 can be deformed irreducibly to a unitarily induced from $D_3\times\mathrm{GL}(4)$, where the parameter on D_3 is $(0,1,\frac{18}{12})$ (nilpotent (1^33) in D_3) and on $\mathrm{GL}(4)$, it is $(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})+(\frac{1}{12},-\frac{1}{12},-\frac{1}{12},\frac{1}{12})$ (nilpotent (22)). Therefore, the signature of the form on E_8 can be computed from the signatures on D_3 and $\mathrm{GL}(4)$ as follows.

D_3	$(0,1,\nu)$	$\nu=\frac{18}{12}$	111×0 $+$	11×1 $+$	12×0 $-$
$\mathrm{GL}(4)$	$(-\frac{1}{2}+\nu,\frac{1}{2}+\nu,-\frac{1}{2}-\nu,\frac{1}{2}-\nu)$	$\nu=\frac{1}{12}$	(22) $+$	(211) $+$	(1^4) $+$

The signature of the hermitian form on $D_3\times\mathrm{GL}(4)$ will therefore be $(24,12)$. The unitarily induced form on D_7 will have signature $(13\,440,6\,720)$ and the unitarily induced form in E_8 has signature $(29\,030\,400,14\,515\,200)$. Since the W-dimension of $X(A_2+2A_1)$ is

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$|W|/|W(A_2)W(A_1)^2| = 29\,030\,400$ and $X(A_2+2A_1)$ is unitary, it follows that the induced form on the factor $\overline{X}(4A_1)$ is (negative) definite, so after the appropriate normalization, the factor $\overline{X}(4A_1)$ is also unitary.

Step 4. In the interior of $\mathcal{R}_3$, the intertwining operator $A_\mu(4A_1, \mathsf{St}, \nu)$ is positive definite for all $\mu \in \widehat{W}$.

It is sufficient to calculate the intertwining operator on a single W-type which appears in both factors $\overline{X}(4A_1, \mathsf{St}, p)$ and $X(A_2+2A_1, \mathsf{St}, p)$. The W-type 560_z has this property, and $A_{560_z}(4A_1, \mathsf{St}, \nu) = a_{1\times 3}(\nu)$ which is positive definite inside $\mathcal{R}_3$. This concludes the proof. □

6.4.2 *Nilpotent* $\mathbf{D_5(a_1)+A_1}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = 2A_1$, the lowest W-type is $\mu_0 = 6075_x$, and the infinitesimal character is $(0, 1, 1, 2, 3, -\frac{1}{2}+\nu_2, \frac{1}{2}+\nu_2, 2\nu_1)$. The standard module is $X(D_5+A_1, \sigma\otimes\mathsf{St}, \nu)$, where $\nu = (\nu_1, \nu_2)$ and σ is the discrete series parameterized by the nilpotent (73) in D_5.

The matching of operators in Table 8 imply that the complementary series is included in $\{0 \leqslant \nu_1 < 1, 0 \leqslant \nu_2 < 1/2\}$, the complementary series of the centralizer. There are hyperplanes of reducibility $2\nu_1 \pm \nu_2 = \frac{3}{2}$ which cut this region. We need to use the scalar operator on 1344_w (a W-type with multiplicity one). This is negative in the region $\{2\nu_1 - \nu_2 < \frac{3}{2} < 2\nu_1 + \nu_2, \nu_2 < \frac{1}{2}\}$. It follows that the complementary series is $\{0 \leqslant \nu_2 < \frac{1}{2},\ 2\nu_1 + \nu_2 < \frac{3}{2}\}$ and $\{0 \leqslant \nu_1 < 1, 2\nu_1 - \nu_2 > \frac{3}{2}\}$.

6.4.3 *Nilpotent* $\mathbf{A_4+A_2}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = 2A_1$, the lowest W-type is 4536_z, and the infinitesimal character is $s = (-\frac{1}{2}, \frac{1}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}) + \nu_2(1, 1, 0, 0, 0, 0, 0, 0) + \nu_1(0, 0, 1, 1, 1, 1, 1, 5)$, with $\nu_1 \geqslant 0$ and $\nu_2 \geqslant 0$. The standard module is $X(A_4+A_2, \mathsf{St}, \nu)$, $\nu = (\nu_1, \nu_2)$.

The matching of operators in Table 8 imply that the complementary series is included in $\{0 \leqslant \nu_1 < 1/2, 0 \leqslant \nu_2 < 1/2\}$, the complementary series of the centralizer. There are hyperplanes of reducibility $5\nu_1 \pm \nu_2 = 2$ which cut this region. We need to use the scalar operator on 420_y (a W-type with multiplicity one). This is negative in the region $\{5\nu_1 - \nu_2 < 2 < 5\nu_1 + \nu_2, \nu_2 < \frac{1}{2}\}$. It follows that the complementary series is $\{0 \leqslant \nu_2 < \frac{1}{2}, 5\nu_1 + \nu_2 < 2\} \cup \{0 \leqslant \nu_1 < \frac{1}{2}, 5\nu_1 - \nu_2 > 2\}$.

6.4.4 *Nilpotent* $\mathbf{A_2+3A_1}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = G_2 + A_1$, the lowest W-type is $\mu_0 = 400_z$, and the infinitesimal character is $(0, 1, -1, 0, -1, 0, -\frac{1}{2}, \frac{1}{2}) + \nu_1(0, 0, 1, 1, 1, 1, -2, 2) + \nu_2(0, 0, 0, 0, 1, 1, -1, 1) + \nu_3(0, 0, 0, 0, 0, 0, 1, 1)$. The standard module is $X(A_2+3A_1, \mathsf{St}, \nu)$, $\nu = (\nu_1, \nu_2, \nu_3)$.

The matching of operators in Table 8 imply that the complementary series is included in $\{3\nu_1 + 2\nu_2 < 1, \nu_3 < \frac{1}{2}\}$ and $\{3\nu_1 + \nu_2 > 1 > 2\nu_1 + \nu_2, \nu_3 < \frac{1}{2}\}$, the complementary series of the centralizer. There are hyperplanes of reducibility $3\nu_1 + 2\nu_2 + \nu_3 = \frac{3}{2}$, $3\nu_1 + 2\nu_2 - \nu_3 = \frac{3}{2}$ and $3\nu_1 + \nu_2 + \nu_3 = \frac{3}{2}$ which cut the second region into five (open) subregions. We need to use the determinant of the operator on 175_x (a W-type with multiplicity two). This is negative in two of the five subregions. It follows that the complementary series is the union of four regions (see § 7 for the explicit description).

6.4.5 *Nilpotent* $\mathbf{A_2+2A_1}$. The centralizer is $\mathfrak{z}(\mathcal{O}) = B_3 + A_1$, the lowest W-type is $\mu_0 = 560_z$, and the infinitesimal character is $(0, 1, -1, 0, 1, 0, 0, 0)+$ $(0, 0, \nu_1, \nu_1, \nu_1, \nu_2, \nu_3, \nu_4)$. The standard module is $X(A_2+A_1, \mathsf{St}, \nu)$, $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$.

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Table 9. The restrictions of W-types.

	Nilpotent				
	$D_4(a_1)$	$D_4(a_1)$	$D_4(a_1)$	$D_4(a_1)A_1$	A_3A_2
W-type	$1400'_z$	$1008'_z$	$56'_z$	$1400'_x$	$3240'_z$
Multiplicity	$1+0+0$	$0+1+0$	$0+0+1$	$2+1+0$	$3+3+0$
D_5	211×1	211×1	$1^3\times 2$	$2\cdot 211\times 1$	$6\cdot 211\times 1$
		$1^3\times 2$		$2\cdot 1^3\times 2$	$3\cdot 1^3\times 2$
A_1	(2)	$2\cdot(2)$	(2)	$3\cdot(2),(11)$	$6\cdot(2),3\cdot(11)$
$W(F_4)$	1_1	2_1	1_2	4_2	9_1
D_4	4×0	$2\cdot 4\times 0$	4×0	3×1	$2\times 2, 31\times 0$

The matching of operators in Table 8 imply that the complementary series is included in the complementary series for the centralizer $B_3+A_1: \mathcal{R}_1=\{0\leqslant\nu_1<1,\ \nu_3+\nu_4<1\}$ and $\mathcal{R}_2=\{0\leqslant\nu_1<1,\nu_2+\nu_4>1,\nu_2+\nu_3<1,\nu_4<1\}$. There are hyperplanes of reducibility $3\nu_1+\nu_2+\nu_3-\nu_4=3,\ 3\nu_1-\nu_2-\nu_3+\nu_4=3,\ 3\nu_1+\nu_2-\nu_3+\nu_4=3,\ 3\nu_1-\nu_2+\nu_3+\nu_4=3,\ 3\nu_1+\nu_2+\nu_3+\nu_4=3$, which cut $\mathcal{R}_1$ and $\mathcal{R}_2$ into 12 open subregions. We need to use the determinant of the operator on 448_z (a W-type with multiplicity four). This is negative in five subregions, the other seven forming the complementary series (see § 7 for the explicit description).

6.5 Multiple lowest W-types orbits

We begin with two typical examples.

6.5.1 *Nilpotent* $\mathbf{D_4(a_1)}$. The centralizer is $\mathfrak{z}(\mathcal{O})=D_4$, with component group $A(\mathcal{O})=S_3$. The infinitesimal character is $(0,1,1,2,0,0,0,0)+$ $(0,0,0,0,\nu_4,\nu_3,\nu_2,\nu_1)$.

The standard module $X(D_4,\sigma,\nu)$, $\nu=(\nu_1,\nu_2,\nu_3,\nu_4)$, with σ the discrete series parameterized by the nilpotent (53) in D_4, has three lowest W-types, $\mu_0=1400_z$, $\mu_0'=1008_z$, and $\mu_0''=56_z$. Note that μ_0' has multiplicity two. They have the same signature at infinity, and stay in the same factor unless the parameter satisfies $\nu_4=0$ or $\nu_1-\nu_2-\nu_3-\nu_4=0$.

If, for example, $\nu_4=0$, the standard module corresponding to the generic case is $X(D_5,\sigma',\nu')$, $\nu'=(\nu_1,\nu_2,\nu_3)$, where σ' is the generic limit of discrete series parameterized by the nilpotent (5311) in D_5, and it contains two lowest W-types, μ_0 and μ_0'.

If, $\nu_4=0$ and $\nu_1-\nu_2-\nu_3-\nu_4=0$, the standard module corresponding to the generic case is $X(E_6,\sigma'',\nu'')$, $\nu''=(\nu_1,\nu_2)$, where σ'' is the generic limit of discrete series module parameterized by the nilpotent orbit $D_4(a_1)$ in E_6, and it contains a single lowest W-type, μ_0.

The subgroup $C(\mathfrak{a},M)\cong W(F_4)$ is generated by

$$\bar{s}_1=s_8,\quad \bar{s}_2=s_7,\quad \bar{s}_3=w_m(D_5(2),D_4),\quad \bar{s}_4=w_m(D_5(1),D_4), \tag{6.5.1}$$

and the subgroup $W(\mathfrak{z})\cong W(D_4)$ by $\{\bar{s}_3\cdot\bar{s}_2\cdot\bar{s}_3,\ \bar{s}_2,\ \bar{s}_1,\ \bar{s}_4\cdot\bar{s}_3\cdot\bar{s}_2\cdot\bar{s}_3\cdot\bar{s}_4\}$.

The restrictions of W-types are as in Table 9.

In addition to the hyperplanes of reducibility as in D_4, there are the following reducibility hyperplanes: $\nu_i=2,3,\ i=1,4,\ \pm\nu_1\pm\nu_2\pm\nu_3\pm\nu_4=4,6$.

The operators (normalized by the scalar on μ_0) match operators for the Hecke algebra of type F_4 with parameter 0 on the long roots, or equivalently operators for the Hecke algebra of type D_4 (see § 4.6):

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$\widehat{W}$	1400_z	1008_z	56_z	1400_x	3240_z
$\widehat{W(F_4)}$	1_1	2_1	1_2	4_2	9_1
$\widehat{W(D_4)}$	4×0	$2\cdot 4\times 0$	4×0	3×1	$2\times 2+31\times 0$

6.5.2 *Nilpotent* $\mathbf{A_4}$. We realize the Bala–Carter Levi subalgebra $\mathfrak{m}=\{\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$. The centralizer of the nilpotent orbit is $\mathfrak{z}(\mathcal{O})=A_4$, and it is realized by $\{\alpha_3,\alpha_1,(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),\alpha_2\}$. The infinitesimal character is $\chi=(0,0,-2,-1,\ 0,1,2,0,0)+(\nu_4,-\nu_1+\nu_2,\nu_3,\nu_3,\nu_3,\nu_3,\nu_3,\nu_1+\nu_2)$.

The standard module $X(A_4,\mathsf{St},\nu)$, $\nu=(\nu_1,\nu_2,\nu_3,\nu_4)$ has two lowest W-types, $\mu_0=2268_x$ and $\mu_0'=1296_z$. They have opposite signs at infinity, and they are separate if and only if $\nu_3=\nu_4=0$. We assume that this is the case; therefore, $\chi=(0,-\nu_1+\nu_2,-2,-1,0,1,2,\nu_1+\nu_2)$. The standard module corresponding to the generic case is $X(D_6,\sigma,\nu)$, $\nu=(\nu_1,\nu_2)$, where σ is the generic limit of discrete series parameterized by the nilpotent (5511) in D_6.

The subgroup $W(\mathfrak{z})\cong W(A_4)\subset W$ is generated by

$$\bar{s}_1=s_3,\quad \bar{s}_2=s_1,\quad \bar{s}_3=w_m(A_5,A_4)\cdot w_m(D_6,A_4)\cdot s_1\cdot w_m(D_6,A_4)\cdot w_m(A_5,A_4),\quad \bar{s}_4=s_2. \tag{6.5.2}$$

The intertwining operator decomposes according to the decomposition $w_m=\bar{s}_1\cdot\bar{s}_2\cdot\bar{s}_3\cdot\bar{s}_4\cdot\bar{s}_1\cdot\bar{s}_2\cdot\bar{s}_3\cdot\bar{s}_1\cdot\bar{s}_2\cdot\bar{s}_1$.

We compute the restrictions of W-types as in § 6.5.1. The operators on W-types in the generic factor of $X(A_4,\mathsf{St},\nu)$ match hermitian spherical operators in A_4 as follows:

W-type	2268_x	4096_x	4096_z	4200_x	3360_z
$W(A_4)$-type	(5)	(41)	(41)	(32)	(32)
eigenspace of $w_0(A_4)$	+1-eig.	+1-eig.	−1-eig.	+1-eig.	−1-eig.

6.5.3 We list the matching of W-types for the other nilpotent orbits of similar kind in Table 10. The infinitesimal characters are in the tables in § 7.

7. Tables of generic unitary parameters

7.1 Parameters for $\mathcal{O}\neq 0$

Tables 11, 12 and 13 contain the nilpotent orbits (see [Car85]), the hermitian infinitesimal character, and the coordinates and type of the centralizer.

The nilpotent orbits which are exceptions are marked with $*$ in the tables. The description of the complementary series for them is recorded after the tables. For the rest of the nilpotents, an infinitesimal character χ is in the complementary series if and only if the corresponding parameter ν is in the 0-complementary series for $\mathfrak{z}(\mathcal{O})$. The parameter ν is given by a string $(\nu_1,\dots,\nu_k)$, and the order agrees with the way the centralizer $\mathfrak{z}(\mathcal{O})$ is written in the tables. The parts of ν corresponding to a torus T_1 or T_2 in $\mathfrak{z}(\mathcal{O})$ must be zero, in order for χ to be unitary. In addition, if ν corresponds to A_1, the complementary series is $0\leqslant\nu<\frac{1}{2}$, while the notation A_1^ℓ means that it is $0\leqslant\nu<1$. If a string $(\nu_1,\dots,\nu_k)$ of ν corresponds to type A_k, the last $k-[k/2]$ entries must be zero in order for χ to be unitary. For example, in the table for E_8, for the nilpotent A_4+A_1, the ν-string is (ν_1,ν_2,ν_3) and the centralizer is A_2+T_1. This means that the

TABLE 10. Nilpotent orbits $\mathcal{O}$ with multiple lowest W-type.

$\mathcal{O}$	$\mathfrak{z}(\mathcal{O}), \mathbf{A}(\mathcal{O})$	Matching						
$\mathbf{E_6(a_1)}$	$A_2, \mathbb{Z}_2$	$2800'_z$ (3) $+1$-eig.	4096_z (21) $+1$-eig.	4096_x (21) -1-eig.				
$\mathbf{D_6(a_1)}$	$2A_1, \mathbb{Z}_2$	$5600'_z$ $(2) \otimes (2)$	$2400'_z$ $(2) \otimes (2)$	$6075'_x$ $(2) \otimes (11) + (11) \otimes (2)$				
$\mathbf{D_6(a_2)}$	$2A_1, \mathbb{Z}_2$	4200_y $(2) \otimes (2)$	2688_y $(2) \otimes (2)$	7168_w $(2) \otimes (11) + (11) \otimes (2)$				
$\mathbf{D_4 + A_2}$	$A_2, \mathbb{Z}_2$	4200_z (3) $+1$-eig.	1344_w (21) $+1$-eig.	3150_y (111) $+1$-eig.				
$\mathbf{A_4 + 2A_1}$	$A_1 + T_1, \mathbb{Z}_2$	4200_x (2)	4536_z (11)					
$\mathbf{D_5(a_1)}$	$A_3, \mathbb{Z}_2$	2800_z (4) $+1$-eig.	6075_x (31) $+1$-eig.	4200_z (22) $+1$-eig.				
$\mathbf{A_4 + A_1}$	$A_2 + T_1, \mathbb{Z}_2$	4096_x (3) $+1$-eig.	4200_x (21) $+1$-eig.	3360_z (21) -1-eig.				
$\mathbf{A_4}$	$A_4, \mathbb{Z}_2$	2268_x (5) $+1$-eig.	4096_x (41) $+1$-eig.	4096_z (41) -1-eig.	4200_x (32) $+1$-eig.	3360_z (32) -1-eig.		
$\mathbf{D_4(a_1) + A_2}$	$A_2, \mathbb{Z}_2$	2240_x (3) $+1$-eig.	4096_x (21) $+1$-eig.	4096_z (21) -1-eig.				
$\mathbf{A_3 + A_2}$	$B_2 + T_1, \mathbb{Z}_2$	3240_z 2×0	1400_{zz} 11×0	2240_x 1×1	840_z 0×2			
$\mathbf{D_4(a_1) + A_1}$	$3A_1, S_3$	1400_x $(2) \otimes (2) \otimes (2)$	3240_z $(11) \otimes (2) \otimes (2) + (2) \otimes (11) \otimes (2) + (2) \otimes (2) \otimes (11)$					
$\mathbf{D_4(a_1)}$	D_4, S_3	1400_z 4×0	1008_z $2 \cdot 4 \times 0$	56_z 4×0	1400_x 3×1	3240_z $2 \times 2 + 31 \times 0$		
$\mathbf{2A_2}$	$2G_2, \mathbb{Z}_2$	700_x $1_1 \otimes 1_1$	300_x $1_1 \otimes 1_1$	448_z $1_4 \otimes 1_1$ $1_1 \otimes 1_4$	1344_x $2_2 \otimes 1_1$ $1_1 \otimes 2_2$	1400_z $2_1 \otimes 1_1$ $1_1 \otimes 2_1$	1008_z $1_3 \otimes 1_1$ $1_1 \otimes 1_3$	
$\mathbf{A_2 + A_1}$	A_5	210_x (6) $+1$-eig.	560_z (51) $+1$-eig.	400_z (33) $+1$-eig.				
$\mathbf{A_2}$	E_6	112_z 1_p $+1$-eig.	210_x 6_p $+1$-eig.	160_z 6_p -1-eig.	560_z 20_p $+1$-eig.	400_z 15_q $+1$-eig.	700_x 30_p $+1$-eig.	300_x 15_p -1-eig.

unitary parameters are those for which $\nu_3 = 0$ (this is the T_1-piece), $\nu_2 = 0$ and $0 \leqslant \nu_1 < \frac{1}{2}$ (this is the 0-complementary series of A_2).

There is one difference in E_6 due to the fact that we only consider hermitian spherical infinitesimal characters χ. In Table 11, the ν-string already refers to the semisimple and hermitian spherical parameter of the centralizer. For example, the nilpotent $A_2 + A_1$ in E_6 has centralizer $A_2 + T_1$, and the corresponding χ has a single ν. This ν corresponds to the hermitian parameter in the A_2 part of $\mathfrak{z}(\mathcal{O})$, so it must satisfy $0 \leqslant \nu < \frac{1}{2}$.

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Table 11. Table of hermitian parameters $(\mathcal{O}, \nu)$ for E_6.

$\mathcal{O}$	χ	$\mathfrak{z}(\mathcal{O})$
E_6	$(0, 1, 2, 3, 4, -4, -4, 4)$	1
$E_6(a_1)$	$(0, 1, 1, 2, 3, -3, -3, 3)$	1
D_5	$(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, \frac{5}{2})$	T_1
$E_6(a_3)$	$(0, 0, 1, 1, 2, -2, -2, 2)$	1
$D_5(a_1)$	$(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, -\frac{7}{4}, -\frac{7}{4}, \frac{7}{4})$	T_1
A_5	$(-\frac{11}{4}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{5}{4}, -\frac{5}{4}, \frac{5}{4}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	A_1
$A_4 + A_1$	$(0, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$	T_1
D_4	$(0, 1, 2, 3, \nu, -\nu, -\nu, \nu)$	A_2
A_4	$(-2, -1, 0, 1, 2, 0, 0, 0) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$A_1 + T_1$
$D_4(a_1)$	$(0, 0, 1, 1, 1, -1, -1, 1)$	T_2
$A_3 + A_1$	$(-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{5}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$A_1 + T_1$
$2A_2 + A_1$	$(0, 1, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu(0, 0, 1, 1, 1, -1, -1, 1)$	A_1
A_3	$(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 0, 0, 0, 0) + (\frac{\nu_1}{2}, \frac{\nu_1}{2}, \frac{\nu_1}{2}, \frac{\nu_1}{2}, \frac{\nu_2}{2}, -\frac{\nu_2}{2}, -\frac{\nu_2}{2}, \frac{\nu_2}{2})$	$B_2 + T_1$
$A_2 + 2A_1$	$(\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}) + \nu(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$	$A_1 + T_1$
$2A_2$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})+$ $(\frac{\nu_2}{2}, \frac{\nu_2}{2}, \frac{2\nu_1+\nu_2}{2}, \frac{2\nu_1+\nu_2}{2}, \frac{2\nu_1+\nu_2}{2}, -\frac{2\nu_1+\nu_2}{2}, -\frac{2\nu_1+\nu_2}{2}, \frac{2\nu_1+\nu_2}{2})$	G_2
$A_2 + A_1$	$(-\frac{1}{2}, \frac{1}{2}, -1, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$A_2 + T_1$
A_2	$(0, -1, 0, 1, 0, 0, 0, 0) + (\frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2},$ $\frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2})$	$2A_2$
$3A_1$	$(0, 1, -\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0) + (0, 0, \nu_1, \nu_2, \nu_1, -\nu_1, -\nu_1, \nu_1)$	$A_2 + A_1$
$2A_1$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)+$ $(\frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, -\frac{\nu_1}{2}, -\frac{\nu_1}{2}, \frac{\nu_1}{2})$	$B_3 + T_1$
A_1	$(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0) + (\frac{-\nu_1+\nu_2}{2}, \frac{\nu_1-\nu_2}{2},$ $\frac{-\nu_1+\nu_2}{2} + \nu_3, \frac{\nu_1-\nu_2}{2} + \nu_3, \frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2})$	A_5

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TABLE 12. Table of parameters $(\mathcal{O}, \nu)$ for E_7.

$\mathcal{O}$	χ	$\mathfrak{z}(\mathcal{O})$
E_7	$(0, 1, 2, 3, 4, 5, -\frac{17}{2}, \frac{17}{2})$	1
$E_7(a_1)$	$(0, 1, 1, 2, 3, 4, -\frac{13}{2}, \frac{13}{2})$	1
$E_7(a_2)$	$(0, 1, 1, 2, 2, 3, -\frac{11}{2}, \frac{11}{2})$	1
$E_7(a_3)$	$(0, 0, 1, 1, 2, 3, -\frac{9}{2}, \frac{9}{2})$	1
E_6	$(0, 1, 2, 3, 4, -4, -4, 4) + \nu(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	A_1^ℓ
D_6	$(0, 1, 2, 3, 4, 5, 0, 0) + \nu(0, 0, 0, 0, 0, 0, -1, 1)$	A_1
$E_6(a_1)$	$(0, 1, 1, 2, 3, -3, -3, 3) + \nu(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	T_1
$E_7(a_4)$	$(0, 0, 1, 1, 1, 2, -\frac{7}{2}, \frac{7}{2})$	1
$D_6(a_1)$	$(0, 1, 1, 2, 3, 4, 0, 0) + \nu(0, 0, 0, 0, 0, 0, -1, 1)$	A_1
A_6	$(-\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, 1)$	A_1^ℓ
$D_5 + A_1$	$(0, 1, 2, 3, -\frac{5}{2}, -\frac{3}{2}, -2, 2) + \nu(0, 0, 0, 0, 1, 1, -1, 1)$	A_1
$E_7(a_5)$	$(0, 0, 1, 1, 1, 2, -\frac{5}{2}, \frac{5}{2})$	1
$D_6(a_2)$	$(0, 1, 1, 2, 2, 3, 0, 0) + \nu(0, 0, 0, 0, 0, 0, -1, 1)$	A_1
$A_5 + A_1$	$(\frac{11}{4}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, -\frac{1}{4}, \frac{1}{4}) + \nu(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{3}{2})$	A_1
D_5	$(0, 1, 2, 3, -2, -2, -2, 2) + \nu_1(0, 0, 0, 0, 1, 1, -1, 1)$ $+\nu_2(0, 0, 0, 0, -1, 1, 0, 0)$	$2A_1$
$E_6(a_3)$	$(0, 0, 1, 1, 2, -2, -2, 2) + \nu(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	A_1^ℓ
$D_5(a_1)A_1$	$(0, 1, 1, 2, -2, -1, -\frac{3}{2}, \frac{3}{2}) + \nu(0, 0, 0, 0, 1, 1, -1, 1)$	A_1^ℓ
$(A_5)'$	$(-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 0, 0) + \nu_1(0, 0, 0, 0, 0, 0, -1, 1)$ $+\nu_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$	$A_1 + A_1^\ell$
$A_4 + A_2$	$(0, 1, 2, -2, -1, 0, -1, 1) + \nu(0, 0, 0, 1, 1, 1, -\frac{3}{2}, \frac{3}{2})$	A_1^ℓ
$(A_5)''$	$(\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 0, 0) + \nu_2(0, 0, 0, 0, 0, 0, -1, 1)$ $+\nu_1(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{3}{2})$	G_2
$D_5(a_1)$	$(0, 1, 1, 2, 3, 0, 0, 0) + \nu_1(0, 0, 0, 0, 0, 0, -1, 1)$ $+\nu_2(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	$A_1 + T_1$
$A_4 + A_1$	$(\frac{9}{4}, -\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, -\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}) + \nu_1(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, 1)$ $+\nu_2(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	T_2
$D_4 + A_1$	$(0, 1, 2, 3, -\frac{1}{2}, \frac{1}{2}, 0, 0) + \nu_1(0, 0, 0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ $+\nu_2(0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	B_2
$A_3 + A_2 + A_1$	$(0, 1, -2, -1, 0, 1, -\frac{1}{2}, \frac{1}{2}) + \nu(0, 0, 1, 1, 1, 1, -2, 2)$	A_1
A_4	$(0, -2, -1, 0, 1, 2, 0, 0) + \nu_1(0, 0, 0, 0, 0, 0, -1, 1)$ $+\nu_2(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{3}{2}) + \nu_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, 1)$	$A_2 + T_1$
$A_3 + A_2$	$(0, 1, 2, -1, 0, 1, 0, 0) + \nu_1(0, 0, 0, 0, 0, 0, -1, 1)$ $+\nu_2(0, 0, 0, 1, 1, 1, 0, 0)$	$A_1 + T_1$
D_4	$(0, 1, 2, 3, \nu_2 - \nu_1, \nu_2 + \nu_1, -\nu_3, \nu_3)$	C_3
$D_4(a_1) + A_1$	$(0, 1, 1, 2, -\frac{1}{2}, \frac{1}{2}, 0, 0) + (0, 0, 0, 0, \nu_2, \nu_2, -\nu_1, \nu_1)$	$2A_1$

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TABLE 12. (Continued.)

$\mathcal{O}$	χ	$\mathfrak{z}(\mathcal{O})$
A_3+2A_1	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0)+(0,0,\nu_2,\nu_2,\nu_2,\nu_2,-\nu_1,\nu_1)$	$2A_1$
$D_4(a_1)$	$(0,1,1,2,\nu_2-\nu_3,\nu_2+\nu_3,-\nu_1,\nu_1)$	$3A_1$
$(A_3+A_1)'$	$(0,1,2,0,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,2\nu_2,\nu_3,\nu_3,-\nu_1,\nu_1)$	$3A_1$
$2A_2+A_1$	$(\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{1}{4},\frac{1}{4})+\nu_1(1,-1,-1,1,1,1,0,0)$ $+\nu_2(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$	$2A_1$
$(A_3+A_1)''$	$(\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},-\frac{1}{2},\frac{1}{2},0,0)$ $+(-\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_3-\nu_2}{2},\frac{\nu_3-\nu_2}{2},-\frac{\nu_3+\nu_2}{2},\frac{\nu_3+\nu_2}{2})$	B_3
A_2+3A_1	$(0,1,-1,0,-1,0,-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,1,-2,2)$ $+\nu_2(0,0,0,0,1,1,-1,1)$	G_2
$2A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,-1,-1,1)$ $+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_3(0,0,0,0,0,1,-\frac{1}{2},\frac{1}{2})$	G_2+A_1
A_3	$(0,1,2,\nu_1,\nu_2,\nu_3,\nu_4)$	B_3+A_1
$*A_2+2A_1$	$(0,1,-1,0,1,0,0,0)+(0,0,\nu_2,\nu_2,\nu_2,\nu_3,-\nu_1,\nu_1)$	$A_1+2A_1^{\ell}$
A_2+A_1	$(1,0,1,0,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,0,\nu_2,\nu_2,-\nu_1,\nu_1)$ $+\nu_3(0,0,0,1,1,1,-\frac{3}{2},\frac{3}{2})+\nu_4(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$	A_3+T_1
$4A_1$	$(0,1,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(0,0,\nu_3,\nu_3,\nu_2,\nu_2,-\nu_1,\nu_1)$	C_3
A_2	$(1,0,1,0,0,0,0,0)+(0,0,0,0,\nu_2-\nu_3,\nu_2+\nu_3,-\nu_1,\nu_1)$ $+\nu_4(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})+\nu_5(0,0,0,1,1,1,-\frac{3}{2},\frac{3}{2})$	A_5
$(3A_1)'$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(\nu_1,\nu_1,\nu_2,\nu_2,\nu_3,\nu_3,-\nu_4,\nu_4)$	C_3+A_1
$(3A_1)''$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(-\nu_4,\nu_4,\nu_3,\nu_3,\nu_2,\nu_2,-\nu_1,\nu_1)$	F_4
$2A_1$	$(0,1,\nu_1,\nu_2,\nu_3,\nu_4,-\nu_5,\nu_5)$	B_4+A_1
A_1	$(\frac{\nu_1+\nu_2+\nu_3-\nu_4}{2},\frac{\nu_1+\nu_2-\nu_3+\nu_4}{2},\frac{\nu_1-\nu_2+\nu_3+\nu_4}{2},\frac{-\nu_1+\nu_2+\nu_3+\nu_4}{2},$ $-\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},-\frac{\nu_5+\nu_6}{2},\frac{\nu_5+\nu_6}{2})$	D_6

$\mathbf{E_7}$ exception:

$\mathbf{A_2+2A_1}$. Three regions: $\{0\leqslant\nu_1<1/2, 0\leqslant\nu_2<1, 0\leqslant\nu_3<1, \nu_1+3\nu_2/2+\nu_3/2<3/2\}$, $\{0\leqslant\nu_1<1/2, 0\leqslant\nu_2<1, 0\leqslant\nu_3<1, -\nu_1+3\nu_2/2+\nu_3/2<3/2, \nu_1+3\nu_2/2-\nu_3/2>3/2\}$, and $\{0\leqslant\nu_1<1/2, 0\leqslant\nu_2<1, 0\leqslant\nu_3<1, 3\nu_2/2+\nu_3/2>3/2, \nu_1+3\nu_2/2-\nu_3/2<3/2\}$.

$\mathbf{E_8}$ exceptions:

$\mathbf{A_4+A_2+A_1}$. $\{0\leqslant\nu<\frac{3}{10}\}$.

$\mathbf{D_5(a_1)+A_1}$. Two regions: $\{0\leqslant\nu_2<\frac{1}{2}, 2\nu_1+\nu_2<\frac{3}{2}\}$, $\{0\leqslant\nu_1<1, 2\nu_1-\nu_2>\frac{3}{2}\}$.

$\mathbf{A_4+A_2}$. Two regions: $\{0\leqslant\nu_2<\frac{1}{2}, 5\nu_1+\nu_2<2\}$, and $\{0\leqslant\nu_1<\frac{1}{2}, 5\nu_1-\nu_2>2\}$.

$\mathbf{A_2+3A_1}$. Four regions: $\{3\nu_1+2\nu_2<1, 0\leqslant\nu_3<\frac{1}{2}\}$, $\{2\nu_1+\nu_2<1<3\nu_1+\nu_2, 0\leqslant\nu_3<\frac{1}{2}, 3\nu_1+2\nu_2+\nu_3<\frac{3}{2}\}$, $\{2\nu_1+\nu_2<1<3\nu_1+\nu_2, 0\leqslant\nu_3<\frac{1}{2}, 3\nu_1+\nu_2+\nu_3<\frac{3}{2}<3\nu_1+2\nu_2-\nu_3\}$, and $\{2\nu_1+\nu_2<1<3\nu_1+\nu_2, 0\leqslant\nu_3<\frac{1}{2}, 3\nu_1+2\nu_2-\nu_3<\frac{3}{2}<3\nu_1+\nu_2+\nu_3\}$.

$\mathbf{A_2+2A_1}$. Seven regions: $\{0\leqslant\nu_1<1, \nu_3+\nu_4<1, 3\nu_1+\nu_2+\nu_3+\nu_4<3\}$, $\{0\leqslant\nu_1<1, \nu_3+\nu_4<1, 3\nu_1+\nu_2-\nu_3+\nu_4<3<3\nu_1-\nu_2+\nu_3+\nu_4\}$, $\{0\leqslant\nu_1<1, \nu_3+\nu_4<1, 3\nu_1-\nu_2-\nu_3+$

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TABLE 13. Table of parameters $(\mathcal{O}, \nu)$ for E_8.

$\mathcal{O}$	χ	$\mathfrak{z}(\mathcal{O})$
E_8	$(0,1,2,3,4,5,6,23)$	1
$E_8(a_1)$	$(0,1,1,2,3,4,5,18)$	1
$E_8(a_2)$	$(0,1,1,2,2,3,4,15)$	1
$E_8(a_3)$	$(0,0,1,1,2,3,4,13)$	1
$E_8(a_4)$	$(0,0,1,1,2,2,3,11)$	1
E_7	$(0,1,2,3,4,5,-\frac{17}{2},\frac{17}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_8(b_4)$	$(0,0,1,1,1,2,3,10)$	1
$E_8(a_5)$	$(0,0,1,1,1,2,2,9)$	1
$E_7(a_1)$	$(0,1,1,2,3,4,-\frac{13}{2},\frac{13}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_8(b_5)$	$(0,0,1,1,1,2,3,8)$	1
D_7	$(0,1,2,3,4,5,6,0)+\nu(0,0,0,0,0,0,0,2)$	A_1
$E_8(a_6)$	$(0,0,1,1,1,2,2,7)$	1
$E_7(a_2)$	$(0,1,1,2,2,3,-\frac{11}{2},\frac{11}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
E_6+A_1	$(0,1,2,3,4,-\frac{9}{2},-\frac{7}{2},4)+\nu(0,0,0,0,0,0,1,1,2)$	A_1
$D_7(a_1)$	$(0,1,1,2,3,4,5,0)+\nu(0,0,0,0,0,0,0,2)$	T_1
$E_8(b_6)$	$(0,0,1,1,1,1,2,6)$	1
$E_7(a_3)$	$(0,0,1,1,2,3,-\frac{9}{2},\frac{9}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_6(a_1)+A_1$	$(0,1,1,2,3,-\frac{7}{2},-\frac{5}{2},3)+\nu(0,0,0,0,0,1,1,2)$	T_1
A_7	$(-\frac{17}{4},-\frac{13}{4},-\frac{9}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{7}{4})+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{2})$	T_1
E_6	$(0,1,2,3,4,-4,-4,4)+\nu_1(0,0,0,0,0,1,1,2)$ $+\nu_2(0,0,0,0,0,0,1,1)$	G_2
D_6	$(0,1,2,3,4,5,\nu_1,\nu_2)$	B_2
D_5+A_2	$(0,1,2,3,-3,-2,-1,2)+\nu(0,0,0,0,1,1,1,3)$	T_1
$E_6(a_1)$	$(0,1,1,2,3,-3,-3,3)+\nu_2(0,0,0,0,0,1,1,2)$ $+\nu_1(0,0,0,0,0,0,1,1)$	A_2
$E_7(a_4)$	$(0,0,1,1,1,2,-\frac{7}{2},\frac{7}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
A_6+A_1	$(\frac{13}{4},-\frac{9}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{11}{4},\frac{1}{4})+\nu(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{2})$	A_1
$D_6(a_1)$	$(0,1,1,2,3,4,0,0)+\nu_1(0,0,0,0,0,0,-1,1)$ $+(0,0,0,0,0,0,1,1)$	$2A_1$
A_6	$(-3,-2,-1,0,1,2,3,0)+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $+\nu_1(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{7}{2})$	$2A_1$
$E_8(a_7)$	$(0,0,0,1,1,1,1,4)$	1
D_5+A_1	$(0,1,2,3,4,-\frac{1}{2},\frac{1}{2},0)+\nu_1(0,0,0,0,0,0,0,2)$ $+\nu_2(0,0,0,0,0,1,1,0)$	$2A_1$
$E_7(a_5)$	$(0,0,1,1,1,2,-\frac{5}{2},\frac{5}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1

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Table 13. (Continued.)

$\mathcal{O}$	χ	$\mathfrak{z}(\mathcal{O})$
$E_6(a_3)+A_1$	$(0,0,1,1,2,-\frac{5}{2},-\frac{3}{2},2)+\nu(0,0,0,0,0,1,1,2)$	A_1
$D_6(a_2)$	$(0,1,1,2,2,3,-\nu_1+\nu_2,\nu_1+\nu_2)$	$2A_1$
$D_5(a_1)+A_2$	$(0,1,1,2,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{3}{2})+\nu(0,0,0,0,1,1,1,3)$	A_1
A_5+A_1	$(\frac{1}{4},-\frac{11}{4},-\frac{7}{4},-\frac{3}{4},\frac{1}{4},\frac{5}{4},\frac{9}{4},\frac{1}{4})+\nu_2(-1,0,0,0,0,0,0,1)$ $+\nu_1(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$	$2A_1$
A_4+A_3	$(0,1,2,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},1)+\nu(0,0,0,1,1,1,1,4)$	A_1
D_5	$(0,1,2,3,4,\nu_1,\nu_2,\nu_3)$	B_3
$E_6(a_3)$	$(0,0,1,1,2,-2,-2,2)+\nu_1(0,0,0,0,0,1,1,2)$ $+\nu_2(0,0,0,0,0,0,1,1)$	G_2
D_4+A_2	$(0,1,2,3,-1,0,1,0)+\nu_2(0,0,0,0,1,1,1,3)$ $+\nu_1(0,0,0,0,0,0,0,2)$	A_2
$*A_4+A_2+A_1$	$(0,1,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2})+\nu(0,0,1,1,1,1,1,5)$	A_1
$*D_5(a_1)+A_1$	$(0,1,1,2,3,-\frac{1}{2}+\nu_2,\frac{1}{2}+\nu_2,2\nu_1)$	$A_1^\ell+A_1$
A_5	$(\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},0,0)+\nu_1(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$ $+\nu_2(0,0,0,0,0,0,-1,1)+\nu_3(0,0,0,0,0,0,1,1)$	G_2+A_1
$*A_4+A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2})+\nu_2(1,1,0,0,0,0,0,0)$ $+\nu_1(0,0,1,1,1,1,1,5)$	$2A_1$
A_4+2A_1	$(0,1,-2,-1,0,1,2,0)+\nu_1(0,0,0,0,0,0,0,2)$ $\nu_2(0,0,1,1,1,1,1,0)$	A_1+T_1
$D_5(a_1)$	$(0,1,1,2,3,\nu_3,\nu_2,\nu_1)$	A_3
$2A_3$	$(0,1,2,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0)+\nu_2(0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1)$	B_2
A_4+A_1	$(0,1,2,-\frac{3}{2},-\frac{1}{2},-1,-1,1)+\nu_2(0,0,0,0,0,1,1,2)$ $+\nu_1(0,0,0,0,0,0,1,1)+\nu_3(0,0,0,1,1,1,1,4)$	A_2+T_1
$D_4(a_1)+A_2$	$(0,1,1,2,-1,0,1,0)+\nu_1(0,0,0,0,1,1,1,3)$ $+\nu_2(0,0,0,0,0,0,0,2)$	A_2
D_4+A_1	$(0,1,2,3,-\frac{1}{2},\frac{1}{2},0,0)+$ $(0,0,0,0,\nu_1,\nu_1,-\nu_2+\nu_3,\nu_2+\nu_3)$	C_3
$A_3+A_2+A_1$	$(0,1,-2,-1,0,1,-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,1,-2,2)$ $+\nu_2(0,0,0,0,0,0,1,1)$	$2A_1$
A_4	$(0,-2,-1,0,1,2,0,0)+$ $(\nu_4,-\nu_1+\nu_2,\nu_3,\nu_3,\nu_3,\nu_3,\nu_3,\nu_1+\nu_2)$	A_4
A_3+A_2	$(0,1,2,-1,0,1,0,0)+(0,0,0,\nu_3,\nu_3,\nu_3,\nu_1,\nu_2)$	B_2+T_1
$D_4(a_1)+A_1$	$(0,1,1,2,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,0,\nu_1,\nu_1,-\nu_2+\nu_3,\nu_2+\nu_3)$	$3A_1$
A_3+2A_1	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0)+(0,0,\nu_1,\nu_1,\nu_1,\nu_1,\nu_2,\nu_3)$	A_1+B_2
$2A_2+2A_1$	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-1,0,\frac{1}{2})+\nu_1(0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},1,1,\frac{1}{2})$ $+\nu_2(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,\frac{3}{2})$	B_2
D_4	$(0,1,2,3,\nu_3-\nu_4,\nu_3+\nu_4,\nu_1-\nu_2,\nu_1+\nu_2)$	F_4

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TABLE 13. (Continued.)

$\mathcal{O}$	χ	$\mathfrak{z}(\mathcal{O})$
$D_4(a_1)$	$(0,1,1,2,\nu_4,\nu_3,\nu_2,\nu_1)$	D_4
A_3+A_1	$(0,1,2,-\frac{1}{2},\frac{1}{2},0,0,0)+(0,0,0,\nu_1,\nu_1,\nu_2,\nu_3,\nu_4)$	A_1+B_3
$2A_2+A_1$	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,-1,-1,1)$ $+\nu_2(0,0,0,0,0,1,1,2)+\nu_3(0,0,0,0,0,0,1,1)$	A_1+G_2
$2A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,-1,-1,1)$ $+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_3(0,0,0,0,0,1,1,2)$ $+\nu_4(0,0,0,0,0,0,1,1)$	$2G_2$
$*A_2+3A_1$	$(0,1,-1,0,-1,0,-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,1,-2,2)$ $+\nu_2(0,0,0,0,1,1,-1,1)+\nu_3(0,0,0,0,0,0,1,1)$	G_2+A_1
A_3	$(0,1,2,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$	B_5
$*A_2+2A_1$	$(0,1,-1,0,1,0,0,0)+(0,0,\nu_1,\nu_1,\nu_1,\nu_2,\nu_3,\nu_4)$	A_1+B_3
A_2+A_1	$(1,0,1,0,-\frac{1}{2},\frac{1}{2},0,0)+$ $(-\nu_5,\nu_5,\nu_5,\nu_4,\nu_3,\nu_3,-\nu_2+\nu_1,\nu_2+\nu_1)$	A_5
$*4A_1$	$(0,1,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+$ $(0,0,\nu_1,\nu_1,\nu_2,\nu_2,-\nu_3+\nu_4,\nu_3+\nu_4)$	C_4
A_2	$(\frac{\nu_1-\nu_2-\nu_3+\nu_4}{2},\frac{-\nu_1+\nu_2-\nu_3+\nu_4}{2},\frac{-\nu_1-\nu_2+\nu_3+\nu_4}{2},$ $\frac{\nu_1+\nu_2+\nu_3+\nu_4}{2},-1+\frac{\nu_5-\nu_6}{2},\frac{\nu_5-\nu_6}{2},1+\frac{\nu_5-\nu_6}{2},\frac{\nu_5+3\nu_6}{2})$	E_6
$3A_1$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(-\nu_4,\nu_4,\nu_3,\nu_3,\nu_2,\nu_2,-\nu_1,\nu_1)$ $+\nu_5(0,0,0,0,0,0,1,1)$	F_4+A_1
$2A_1$	$(0,1,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)$	B_6
A_1	$(\frac{\nu_1+\nu_2+\nu_3-\nu_4}{2},\frac{\nu_1+\nu_2-\nu_3+\nu_4}{2},\frac{\nu_1-\nu_2+\nu_3+\nu_4}{2},\frac{-\nu_1+\nu_2+\nu_3+\nu_4}{2},$ $\frac{-\nu_5-\nu_6+2\nu_7}{2},-\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},\frac{\nu_5+\nu_6+2\nu_7}{2})$	E_7

$\nu_4>3\}$, $\{0\leqslant\nu_1<1,\nu_3+\nu_4<1,3\nu_1+\nu_2+\nu_3-\nu_4>3\}$, $\{0\leqslant\nu_1<1,\nu_2+\nu_4>1,\nu_2+\nu_3<1,\nu_4<1,3\nu_1+\nu_2+\nu_3+\nu_4<3\}$, $\{0\leqslant\nu_1<1,\nu_2+\nu_4>1,\nu_2+\nu_3<1,\nu_4<1,3\nu_1-\nu_2-\nu_3+\nu_4>3\}$, and $\{0\leqslant\nu_1<1,\nu_2+\nu_4>1,\nu_2+\nu_3<1,\nu_4<1,3\nu_1+\nu_2+\nu_3-\nu_4>3\}$.

$\mathbf{4A_1}$. Two regions: $\{0\leqslant\nu_1\leqslant\nu-2\leqslant\nu_3\leqslant\nu_4<\frac{1}{2}\}$ and $\{\nu_1+\nu_4<1,\nu_2+\nu_3<1,\nu_2+\nu_4>1,-\nu_1+\nu_3+\nu_4<\frac{3}{2}<\nu_1+\nu_3+\nu_4\}$.

7.2 0-complementary series

We record next the precise description of the 0-complementary series (that is, the generic spherical unitary parameters) for types E_6, E_7, E_8. This answer is obtained inductively from Corollary 3.1.

7.2.1 *Type* E_6. In $W(E_6)$, the longest Weyl group element w_0 does not act by minus one. Therefore, we only consider dominant parameters χ such that $w_0\chi=-\chi$:

$$\left(\frac{\nu_1-\nu_2}{2}-\nu_3,\frac{\nu_1-\nu_2}{2}-\nu_4,\frac{\nu_1-\nu_2}{2}+\nu_4,\frac{\nu_1-\nu_2}{2}+\nu_3,\frac{\nu_1+\nu_2}{2},\right.$$
$$\left.-\frac{\nu_1+\nu_2}{2},-\frac{\nu_1+\nu_2}{2},\frac{\nu_1+\nu_2}{2}\right).$$

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The 0-complementary series is

(i) $\alpha_{36} < 1$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \geqslant 0$;

(ii) $\alpha_{34} < 1$, $\alpha_{35} > 1$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6 \geqslant 0$.

7.2.2 *Type E_7.* The parameters are $\chi = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, -\nu_7, \nu_7)$, assumed dominant. The 0-complementary series is:

(i) $\alpha_{63} < 1$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \geqslant 0$;

(ii) $\alpha_{61} < 1$, $\alpha_{62} > 1$ and $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \geqslant 0$;

(iii) $\alpha_{58} < 1$, $\alpha_{59} < 1$, $\alpha_{60} > 1$ and $\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7 \geqslant 0$;

(iv) $\alpha_{53} < 1$, $\alpha_{54} < 1$, $\alpha_{55} < 1$, $\alpha_{56} > 1$, $\alpha_{57} > 1$ and $\alpha_1, \alpha_3, \alpha_5 \geqslant 0.$;

(v) $\alpha_{46} < 1$, $\alpha_{47} < 1$, $\alpha_{48} < 1$, $\alpha_{49} < 1$, $\alpha_{50} > 1$, $\alpha_{51} > 1$, $\alpha_{52} > 1$ and $\alpha_2 \geqslant 0$;

(vi) $\alpha_{53} < 1$, $\alpha_{59} < 1$, $\alpha_{56} > 1$ and $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \geqslant 0$;

(vii) $\alpha_{49} < 1$, $\alpha_{53} < 1$, $\alpha_{54} < 1$, $\alpha_{52} > 1$, $\alpha_{56} > 1$ and $\alpha_3, \alpha_4, \alpha_5 \geqslant 0$;

(viii) $\alpha_{47} < 1$, $\alpha_{48} < 1$, $\alpha_{49} < 1$, $\alpha_{53} < 1$, $\alpha_{51} > 1$, $\alpha_{52} > 1$ and $\alpha_2, \alpha_4 \geqslant 0$.

7.2.3 *Type E_8.* The parameters are $\chi = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8)$, assumed dominant. The 0-complementary series is:

(i) $\alpha_{120} < 1$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \geqslant 0$;

(ii) $\alpha_{113} < 1, \alpha_{114} < 1$; $\alpha_{115} > 1$ and $\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \geqslant 0$;

(iii) $\alpha_{109} < 1, \alpha_{110} < 1$; $\alpha_{111} > 1, \alpha_{112} > 1$ and $\alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \geqslant 0$;

(iv) $\alpha_{91} < 1, \alpha_{92} < 1, \alpha_{97} < 1, \alpha_{98} < 1$; $\alpha_{95} > 1, \alpha_{96} > 1, \alpha_{101} > 1$ and $\alpha_3, \alpha_4 \geqslant 0$;

(v) $\alpha_{90} < 1, \alpha_{91} < 1, \alpha_{92} < 1, \alpha_{97} < 1$; $\alpha_{94} > 1, \alpha_{95} > 1, \alpha_{96} > 1$ and $\alpha_1, \alpha_3 \geqslant 0$;

(vi) $\alpha_{89} < 1, \alpha_{90} < 1, \alpha_{91} < 1, \alpha_{92} < 1$; $\alpha_{93} > 1, \alpha_{94} > 1, \alpha_{95} > 1, \alpha_{96} > 1$ and $\alpha_1 \geqslant 0$;

(vii) $\alpha_{104} < 1, \alpha_{110} < 1$; $\alpha_{107} > 1, \alpha_{112} > 1$ and $\alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8 \geqslant 0$;

(viii) $\alpha_{104} < 1, \alpha_{105} < 1, \alpha_{106} < 1$; $\alpha_{107} > 1, \alpha_{108} > 1$ and $\alpha_2, \alpha_4, \alpha_7, \alpha_8 \geqslant 0$;

(ix) $\alpha_{118} < 1$; $\alpha_{119} > 1$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8 \geqslant 0$;

(x) $\alpha_{97} < 1, \alpha_{110} < 1$; $\alpha_{101} > 1, \alpha_{112} > 1$ and $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \geqslant 0$;

(xi) $\alpha_{97} < 1, \alpha_{105} < 1, \alpha_{106} < 1$; $\alpha_{101} > 1, \alpha_{108} > 1$ and $\alpha_2, \alpha_4, \alpha_6, \alpha_7 \geqslant 0$;

(xii) $\alpha_{116} < 1$; $\alpha_{117} > 1$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8 \geqslant 0$;

(xiii) $\alpha_{97} < 1, \alpha_{98} < 1, \alpha_{106} < 1$; $\alpha_{101} > 1, \alpha_{102} > 1$ and $\alpha_2, \alpha_4, \alpha_5, \alpha_6 \geqslant 0$;

(xiv) $\alpha_{97} < 1, \alpha_{98} < 1, \alpha_{99} < 1$; $\alpha_{96} > 1, \alpha_{101} > 1, \alpha_{102} > 1$ and $\alpha_2, \alpha_4, \alpha_5 \geqslant 0$;

(xv) $\alpha_{97} < 1, \alpha_{98} < 1, \alpha_{99} < 1, \alpha_{100} < 1$; $\alpha_{101} > 1, \alpha_{102} > 1, \alpha_{103} > 1$ and $\alpha_2, \alpha_5 \geqslant 0$;

(xvi) $\alpha_{114} < 1$; $\alpha_{112} > 1$ and $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \geqslant 0$.

7.2.4 *Roots for type E.* The notation for the positive roots which appeared in the lists of 0-complementary series for E_6, E_7, E_8 is as follows.

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E6		
$\alpha_{34} = \frac{1}{2}(-1, 1, -1, 1, 1, -1, -1, 1)$	$\alpha_{35} = \frac{1}{2}(-1, -1, 1, 1, 1, -1, -1, 1)$	$\alpha_{36} = \frac{1}{2}(1, 1, 1, 1, 1, -1, -1, 1)$
E7		
$\alpha_{46} = \frac{1}{2}(-1, 1, -1, 1, 1, -1, -1, 1)$	$\alpha_{47} = \frac{1}{2}(-1, 1, 1, -1, -1, 1, -1, 1)$	$\alpha_{48} = \frac{1}{2}(1, -1, -1, 1, -1, 1, -1, 1)$
$\alpha_{49} = \epsilon_5 + \epsilon_6$	$\alpha_{50} = \frac{1}{2}(-1, -1, 1, 1, 1, -1, -1, 1)$	$\alpha_{51} = \frac{1}{2}(-1, 1, -1, 1, -1, 1, -1, 1)$
$\alpha_{52} = \frac{1}{2}(1, -1, -1, -1, 1, 1, -1, 1)$	$\alpha_{53} = \frac{1}{2}(1, 1, 1, 1, 1, -1, -1, 1)$	$\alpha_{54} = \frac{1}{2}(-1, -1, 1, 1, -1, 1, -1, 1)$
$\alpha_{55} = \frac{1}{2}(-1, 1, -1, -1, 1, 1, -1, 1)$	$\alpha_{56} = \frac{1}{2}(1, 1, 1, 1, -1, 1, -1, 1)$	$\alpha_{57} = \frac{1}{2}(-1, -1, 1, -1, 1, 1, -1, 1)$
$\alpha_{58} = \frac{1}{2}(1, 1, 1, -1, 1, 1, -1, 1)$	$\alpha_{59} = \frac{1}{2}(-1, -1, -1, 1, 1, 1, -1, 1)$	$\alpha_{60} = \frac{1}{2}(1, 1, -1, 1, 1, 1, -1, 1)$
$\alpha_{61} = \frac{1}{2}(1, -1, 1, 1, 1, 1, -1, 1)$	$\alpha_{62} = \frac{1}{2}(-1, 1, 1, 1, 1, 1, -1, 1)$	$\alpha_{63} = -\epsilon_7 + \epsilon_8$
E8		
$\alpha_{89} = \frac{1}{2}(1, -1, 1, 1, 1, 1, -1, 1)$	$\alpha_{90} = \frac{1}{2}(1, 1, -1, 1, 1, -1, 1, 1)$	$\alpha_{91} = \frac{1}{2}(1, 1, 1, -1, -1, 1, 1, 1)$
$\alpha_{92} = \frac{1}{2}(-1, -1, -1, 1, -1, 1, 1, 1)$	$\alpha_{93} = \frac{1}{2}(-1, 1, 1, 1, 1, 1, -1, 1)$	$\alpha_{94} = \frac{1}{2}(1, -1, 1, 1, 1, -1, 1, 1)$
$\alpha_{95} = \frac{1}{2}(1, 1, -1, 1, -1, 1, 1, 1)$	$\alpha_{96} = \frac{1}{2}(-1, -1, -1, -1, 1, 1, 1, 1)$	$\alpha_{97} = -\epsilon_7 + \epsilon_8$
$\alpha_{98} = \frac{1}{2}(-1, 1, 1, 1, 1, -1, 1, 1)$	$\alpha_{99} = \frac{1}{2}(1, -1, 1, 1, -1, 1, 1, 1)$	$\alpha_{100} = \frac{1}{2}(1, 1, -1, -1, 1, 1, 1, 1)$
$\alpha_{101} = -\epsilon_6 + \epsilon_8$	$\alpha_{102} = \frac{1}{2}(-1, 1, 1, 1, -1, 1, 1, 1)$	$\alpha_{103} = \frac{1}{2}(1, -1, 1, -1, 1, 1, 1, 1)$
$\alpha_{104} = -\epsilon_5 + \epsilon_8$	$\alpha_{105} = \frac{1}{2}(-1, 1, 1, -1, 1, 1, 1, 1)$	$\alpha_{106} = \frac{1}{2}(1, -1, -1, 1, 1, 1, 1, 1)$
$\alpha_{107} = -\epsilon_4 + \epsilon_8$	$\alpha_{108} = \frac{1}{2}(-1, 1, -1, 1, 1, 1, 1, 1)$	$\alpha_{109} = -\epsilon_3 + \epsilon_8$
$\alpha_{110} = \frac{1}{2}(-1, -1, 1, 1, 1, 1, 1, 1)$	$\alpha_{111} = -\epsilon_2 + \epsilon_8$	$\alpha_{112} = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1)$
$\alpha_{113} = \epsilon_1 + \epsilon_8$	$\alpha_{114} = -\epsilon_1 + \epsilon_8$	$\alpha_{115} = \epsilon_2 + \epsilon_8$
$\alpha_{116} = \epsilon_3 + \epsilon_8$	$\alpha_{117} = \epsilon_4 + \epsilon_8$	$\alpha_{118} = \epsilon_5 + \epsilon_8$
$\alpha_{119} = \epsilon_6 + \epsilon_8$	$\alpha_{120} = \epsilon_7 + \epsilon_8$	

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