Appendix F

Feynman integrals

F.1 Feynman parametrization

The Feynman parametrization is needed to recombine the product of denominators appearing in the momentum integral. We shall discuss the most usual ways of parametrization.

F.1.1 Schwinger representation

The first one consists of an exponentiation of the propagator denominators and leads to:

$$\frac{1}{a_1 \cdots a_n} = \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \exp\left(-\sum_{i=1}^n a_i z_i\right) . \tag{F.1}$$

In connection to this, the following Gaussian integral is useful:

$$\int \frac{d^n k}{(2\pi)^n} e^{-\alpha k^2} = \frac{1}{(4\pi\alpha)^{n/2}} . \tag{F.2}$$

F.1.2 Original Feynman parametrization

The second alternative is obtained from the original Feynman parametrization:

$$\frac{1}{a_1 \cdots a_n} = (n-1)! \int_0^1 dz_1 \cdots \int_0^1 dz_n \, \delta\left(1 - \sum_i z_i\right) \frac{1}{\left(\sum_i a_i z_i\right)^n} . \tag{F.3}$$

After a suitable change of variables, one can eliminate the δ-function and one obtains:

$$\begin{aligned} \frac{1}{a_1 \cdots a_n} = \; & (n-1)! \int_0^1 u_1^{n-2} du_1 \int_0^1 u_2^{n-3} du_2 \cdots \int_0^1 du_{n-1} \\ & \times \left[(a_1 - a_2) u_1 \cdots u_{n-1} + (a_2 - a_3) u_1 \cdots u_{n-2} + \cdots + a_n\right]^{-n} . \end{aligned} \tag{F.4}$$

This parametrization is quite convenient as it allows possible cancellations among terms of two propagators, and has the advantage to provide finite bounds of integration, which is convenient in various numerical integration calculations encountered e.g. in QED

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calculations (g-2, ...). A particularly useful case of Eq. (F.4) are:

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[(a-b)x+b]^{\alpha+\beta}} , \tag{F.5}$$

and:

$$\frac{1}{a^n b^m c^r} = \frac{\Gamma(n+m+r)}{\Gamma(n)\Gamma(m)\Gamma(r)} \int_0^1 dx\; x^{m+n-1}(1-x)^{r-1} \times \int_0^1 dy \frac{(1-y)^{m-1} y^{n-1}}{[(a-b)xy+(b-c)x+c]^{m+n+r}} , \tag{F.6}$$

entering in a one-loop calculation. In the case where a_1 is $\ln k^2$, the following representation integral is useful:

$$\frac{1}{(\ln k^2)^{n+1}} = \frac{1}{\Gamma(n+1)} \int_0^\infty dx\; x^n (k^2)^{-x} . \tag{F.7}$$

F.2 The Γ function

It is defined for complex z by the Euler integral:

$$\Gamma(z) = \int_0^\infty dt\; t^{z-1} e^{-t} , \tag{F.8}$$

If the previous integral does not exist, it can be defined, using an analytic continuation, by:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^\infty dt\; t^{z-1} e^{-t} , \tag{F.9}$$

which expresses that $\Gamma(z)$ is analytic in the entire z-plane but contains simple poles at $z = 0, -1, -2, \ldots$. It has the properties:

$$\Gamma(1+z) = z\Gamma(z) , \tag{F.10}$$

and:

$$\Gamma(1+z) = \exp\left\{ -z\gamma_E + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n) \right\} , \tag{F.11}$$

with:

$$\gamma_E \equiv \gamma = \lim_{n\to\infty} \left\{ 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right\} = 0.577\,215\,664\,9\ldots \tag{F.12}$$

and:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} , \tag{F.13}$$

is the Riemann function: with:

$$\zeta(2) = \frac{\pi^2}{6} , \qquad \zeta(3) = 1.202\,056\,903\,1\ldots , \qquad \zeta(4) = \frac{\pi^4}{90} . \tag{F.14}$$

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The following expansion is particularly useful in dimensional regularization:

$$\lim_{\epsilon\to 0}\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \frac{\epsilon^2}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) - \frac{\epsilon^3}{3}\left(\frac{\gamma_E^3}{2} + \frac{\pi^2}{4}\gamma_E + \zeta(3)\right) + \mathcal{O}(\epsilon^4)\,, \tag{F.15}$$

from which one can deduce $\Gamma(\epsilon)$ with the help of Eq. (F.10). For integer n, one has:

$$\Gamma(n) = (n-1)!\,, \tag{F.16}$$

while one also has the following properties:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}\,,\\ \Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin\pi x}\end{aligned} \tag{F.17}$$

F.3 The beta function $B(x, y)$

It is defined as:

$$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \tag{F.18}$$

and has the useful properties:

$$\begin{aligned}B(x+1,y) &= \left(\frac{x}{x+y}\right)B(x,y)\,,\\ B(x,1+y) &= \left(\frac{y}{x+y}\right)B(x,y)\,.\end{aligned} \tag{F.19}$$

Therefore, one can deduce:

$$\begin{aligned}&B(1+az,1+bz)\\ &\quad = \frac{1}{1+(a+b)z}\exp\left\{-z\gamma_E + \sum_{n=2}^{\infty}(-1)^n\frac{z^n}{n}\zeta(n)[a^n+b^n-(a+b)^n]\right\}\,.\end{aligned} \tag{F.20}$$

In the limit $\epsilon\to 0$, it has the Taylor expansion:

$$\begin{aligned}\lim_{\epsilon\to 0} B(1-a\epsilon, 1-b\epsilon) &= 1 - \epsilon(a+b) + \epsilon^2\left[(a+b)^2 - ab\frac{\pi^2}{6}\right]\\ &\quad + \epsilon^3(a+b)[-(a+b)^2 + ab\zeta(2) + ab\zeta(3)] + \cdots\,,\\ \lim_{\epsilon\to 0} B\left(n-\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}\right) &= \frac{1}{n}\left\{1 + \frac{\epsilon}{2}\left[\frac{2}{n} + \sum_{j=1}^{n-1}\frac{1}{j}\right]\right\} + \mathcal{O}(\epsilon^2)\,,\\ \lim_{\epsilon\to 0} B\left(n-\frac{\epsilon}{2}, 2-\frac{\epsilon}{2}\right) &= \frac{1}{n(n+1)}\left\{1 - \frac{\epsilon}{2}\left[1 - \frac{2}{n} - \frac{2}{n+1} - \sum_{j=1}^{n-1}\frac{1}{j}\right]\right\} + \mathcal{O}(\epsilon^2)\,,\end{aligned} \tag{F.21}$$

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F.4 The incomplete beta function $B_a(x, y)$

It is defined as:

$$B_a(x, y) = \int_0^a dt\, t^{x-1}(1-t)^{y-1} \,. \tag{F.22}$$

Defining the function:

$$I_a(x, y) = \frac{B_a(x, y)}{B_1(x, y)} \,, \tag{F.23}$$

one has the properties:

$$\begin{aligned}
&aI_a(x, y) - I_a(x+1, y) + (1-a)I_a(x+1, y-1) = 0 \,, \\
&(x+y-xa)I_a(x, y) - yI_a(x, y+1) - x(1-a)I_a(x+1, y-1) = 0 \,, \\
&yI_a(x, y+1) + xI_a(x+1, y) - (x+y)I_a(x, y) = 0 \,.
\end{aligned} \tag{F.24}$$

F.5 The hypergeometric function ${}_2F_1(a, b, c; z)$

It is defined as:

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \tag{F.25}$$

for Re c and Re $b > 0$, and arg$(1-\text{z}) < \pi$. It has the properties:

$$\begin{aligned}
{}_2F_1(a, b, c; z) &= 1 + \frac{ab}{1.c}z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2 + \cdots \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \,.
\end{aligned} \tag{F.26}$$

The hypergeometric function enters frequently in the calculation of multiloop Feynman integrals when the Gegenbauer polynomial techniques are used.

F.6 One-loop massless integrals

The most useful integral is:

$$\begin{aligned}
I(\alpha, \beta) &\equiv \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + i\epsilon')^\alpha} \frac{1}{[(k-q)^2 + i\epsilon']^\beta} \\
&= \frac{i}{(16\pi^2)^{n/4}} (-1)^{-\alpha-\beta} (-q^2)^{-\alpha-\beta+n/2} \frac{\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)} B\left(\frac{n}{2} - \beta, \frac{n}{2} - \alpha\right) \,.
\end{aligned} \tag{F.27}$$

Combining this result with the one in Eq. (8.24), one can derive in $n = 4 - \epsilon$ dimensions:

$$I^\mu(\alpha, \beta) \equiv \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu}{(k^2 + i\epsilon')^\alpha((k-q)^2 + i\epsilon')^\beta}$$

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Table F.1. *Some values of* $I(\alpha, \beta)$

α	β	$I(\alpha,\beta)\nu^{\epsilon}\left(\frac{16\pi^2}{i}\right)(q^2)^{\alpha+\beta-2}$
1	1	$\frac{2}{\tilde{\epsilon}}+2$
2	1	$-\frac{2}{\tilde{\epsilon}}+0$
3	1	-1
2	2	$-\frac{4}{\tilde{\epsilon}}-2$
4	1	$-1/3$
3	2	$-\frac{4}{\tilde{\epsilon}}-5$

Table F.2. *Some values of* $I^{\mu}(\alpha, \beta)$

α	β	$I^{\mu}(\alpha,\beta)\nu^{\epsilon}\left(\frac{16\pi^2}{i}\right)(q^{\mu})^{-1}(q^2)^{\alpha+\beta-2}$
1	1	$\frac{1}{\tilde{\epsilon}}+1$
2	1	$+1$
1	2	$-\frac{2}{\tilde{\epsilon}}-1$
3	1	$-\frac{1}{\tilde{\epsilon}}-\frac{1}{2}$
2	2	$-\frac{2}{\tilde{\epsilon}}-1$
1	3	$\frac{1}{\tilde{\epsilon}}-\frac{1}{2}$

$$= \nu^{-\epsilon}\frac{i}{16\pi^2}\left(\frac{-q^2}{4\pi\nu^2}\right)^{-\epsilon/2}(q^2)^{2-\alpha-\beta}q^{\mu} \times \frac{\Gamma(3-\alpha-\epsilon/2)\Gamma(2-\beta-\epsilon/2)\Gamma(\alpha+\beta-2+\epsilon/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(5-\alpha-\beta-\epsilon)}\,, \tag{F.28}$$

$$\begin{aligned} I^{\mu\nu}(\alpha,\beta) &\equiv \int \frac{d^n k}{(2\pi)^n}\frac{k^{\mu}k^{\nu}}{(k^2+i\epsilon')^{\alpha}(k-q)^2+i\epsilon')^{\beta}} \\ &= \nu^{-\epsilon}\frac{i}{16\pi^2}\left(\frac{-q^2}{4\pi\nu^2}\right)^{-\epsilon/2}(q^2)^{2-\alpha-\beta} \\ &\quad\times\left\{g^{\mu\nu}q^2\frac{\Gamma(3-\alpha-\epsilon/2)\Gamma(3-\beta-\epsilon/2)\Gamma(\alpha+\beta+\epsilon/2)}{2\Gamma(\alpha)\Gamma(\beta)\Gamma(6-\alpha-\beta-\epsilon)}\right. \\ &\quad\left.+q^{\mu}q^{\nu}\frac{\Gamma(4-\alpha-\epsilon/2)\Gamma(2-\beta-\epsilon/2)\Gamma(\alpha+\beta-2+\epsilon/2)}{2\Gamma(\alpha)\Gamma(\beta)\Gamma(6-\alpha-\beta-\epsilon)}\right\}. \end{aligned} \tag{F.29}$$

We give (Tables F.1–F.3 in values of the integrals for some values of α and β, where:

$$\frac{2}{\tilde{\epsilon}} \equiv \frac{2}{\epsilon} - \gamma_E - \ln\frac{-q^2}{4\pi\nu^2}\,. \tag{F.30}$$

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Table F.3. *Some values of*
$I^{\mu\nu}(\alpha,\beta) \equiv \nu^{-\epsilon}\left(\frac{i}{16\pi^2}\right)(q^2)^{2-\alpha-\beta}[Aq^2 g^{\mu\nu} + Bq^\mu q^\nu]$

α	β	A	B
1	1	$-\frac{1}{6\bar{\epsilon}} - \frac{2}{9}$	$\frac{2}{3\bar{\epsilon}} + \frac{13}{18}$
2	1	$\frac{1}{2\bar{\epsilon}} + \frac{1}{2}$	$\frac{1}{2}$
1	2	$\frac{1}{2\bar{\epsilon}} + \frac{1}{2}$	$-\frac{2}{\bar{\epsilon}} - \frac{3}{2}$
3	1	$-\frac{1}{2\bar{\epsilon}} - \frac{1}{4}$	$\frac{1}{2}$
2	2	$\frac{1}{2}$	$-\frac{2}{\bar{\epsilon}} - 2$
1	3	$-\frac{1}{2\bar{\epsilon}} - \frac{1}{4}$	$\frac{2}{\bar{\epsilon}} + \frac{1}{2}$

F.7 Two- and three-loop massless integrals

Most of the integral encountered in the evaluation of loop diagrams can be reduced to the following integrals by means of the formula:

$$kq = \frac{1}{2}[k^2 + q^2 - (k-q)^2]\ . \tag{F.31}$$

These integral come from [2] and reads:

$$\begin{aligned}
I_2 &\equiv \left\{\int \frac{d^n k_1}{(2\pi)^n}\frac{1}{k_1^2(k_1-q)^2}\right\}^2 ,\\
&= \frac{(-1)}{(16\pi^2)^{n/2}}(-1)^{-4}(-q^2)^{-4+n}\left\{\Gamma\left(2-\frac{n}{2}\right)B\left(\frac{n}{2}-1,\frac{n}{2}-1\right)\right\}^2\\
&= \frac{(-1)}{(4\pi)^{4-\epsilon}}(-q^2)^{-\epsilon}4\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}(2-\gamma)+3-2\gamma+\frac{\gamma^2}{2}-\frac{\pi^2}{24}\right\}\quad \text{for } n=4-\epsilon\ ,\\
I_3 &\equiv \int\frac{d^n k_1}{(2\pi)^n}\int\frac{d^n k_2}{(2\pi)^n}\frac{1}{k_1^2(k_1-k_2)^2(k_1-q)^2}\\
&= \frac{(-1)}{(16\pi^2)^{n/2}}(-1)^{-3+n}(-q^2)^{-3+n}\Gamma(3-n)\,B\left(\frac{n}{2}-1,\frac{n}{2}-1\right)B\left(\frac{n}{2}-1,n-2\right)\\
&= \frac{(-1)^{-\epsilon}}{(4\pi)^{4-\epsilon}}(q^2)(-q^2)^{-\epsilon}\frac{1}{2}\left\{\frac{1}{\epsilon}+\frac{13}{4}-\gamma\right\}\quad \text{for } n=4-\epsilon\ ,\\
I_4 &\equiv \int\frac{d^n k_1}{(2\pi)^n}\int\frac{d^n k_2}{(2\pi)^n}\frac{1}{k_1^2(k_1-q)^2(k_1-k_2)^2(k_2-q)^2}\\
&= \frac{(-1)}{(16\pi^2)^{n/2}}(-1)^{-4+n}(-q^2)^{-4+n}\frac{\Gamma(2-n/2)\,\Gamma(4-n)}{\Gamma(3-n/2)}\\
&\quad\times B\left(\frac{n}{2}-1,\frac{n}{2}-1\right)B\left(\frac{n}{2}-1,n-3\right)\\
&= \frac{(-1)^{1-\epsilon}}{(4\pi)^{4-\epsilon}}(-q^2)^{-\epsilon}\left\{\frac{2}{\epsilon^2}+\frac{5-2\gamma}{\epsilon}+\frac{19}{2}-5\gamma\gamma^2-\frac{\pi^2}{12}\right\}\quad \text{for } n=4-\epsilon\ ,\\
I_5 &\equiv \int\frac{d^n k_1}{(2\pi)^n}\int\frac{d^n k_2}{(2\pi)^n}\frac{k_1^2}{k_1^2(k_1-q)^2(k_1-k_2)^2(k_2-q)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{(4\pi)^{4-\epsilon}}(-q^2)^{1-\epsilon}\frac{\Gamma(\epsilon)}{\epsilon}B\left(1-\frac{\epsilon}{2},2-\frac{\epsilon}{2}\right)B\left(2-\frac{\epsilon}{2},1-\epsilon\right)\\
&= \frac{1}{(4\pi)^{4-\epsilon}}(-q^2)^{1-\epsilon}\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{4}-\gamma\right)\right.\\
&\left.+\frac{1}{2}\left(\frac{89}{8}-\frac{11}{2}\gamma+\gamma^2-\frac{\pi^2}{12}\right)\right\} \quad \text{for } n=4-\epsilon\,. \qquad \text{(F.32)}
\end{aligned}
$$

Some more complicated integrals entering in the three-loop calculation has been done analytically in [875] using Gegenbauer techniques. Within our notations and conventions, these integrals are generally of the form:

$$
\begin{aligned}
I(\alpha,\beta,\lambda,\delta) &\equiv \int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n}\frac{1}{k_1^{2\alpha}(k_1-q)^{2\beta}(k_1-k_2)^{2\gamma}(k_2-q)^{2\delta}}\\
&= \frac{(-1)}{(4\pi)^{4-\epsilon}}(-q^2)^{-\epsilon}(q^2)^{4-\alpha-\beta-\gamma-\delta}\frac{\Gamma(\gamma+\delta-2+\epsilon/2)}{\Gamma(\gamma)\Gamma(\delta)}\\
&\times\frac{\Gamma(\alpha+\beta+\gamma+\delta-4+\epsilon)}{\Gamma(\alpha)\Gamma(\beta+\gamma+\delta-2+\epsilon/2)}B\left(2-\gamma-\frac{\epsilon}{2},2-\delta-\frac{\epsilon}{2}\right)\\
&\times B\left(2-\alpha-\frac{\epsilon}{2},4-\beta-\gamma-\delta-\epsilon\right) \quad \text{for } n=4-\epsilon\,. \qquad \text{(F.33)}
\end{aligned}
$$

One also has:

$$
\begin{aligned}
I_\lambda(\alpha,\beta) &\equiv \int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n}\frac{1}{k_1^{2\alpha}(k_1-q)^{2\beta}(k_1-k_2)^2k_2^2(k_2-q)^2}\\
&= \frac{(-1)}{(16\pi^2)^{\lambda+1}}(-1)^{2\lambda-1-\alpha-\beta}(-q^2)^{2\lambda-1-\alpha-\beta}F_\lambda(\alpha,\beta) \qquad \text{(F.34)}
\end{aligned}
$$

for $\alpha, \beta \geq 1$ and where:

$$
\begin{aligned}
\lambda &\equiv n/2-1\,,\\
F_1(1,1) &= 6\sum_{k=o}^{\infty}\frac{1}{(k+1)^3} = 6\zeta(3)\,,\\
F_\lambda(\alpha>1,\beta>1) &= \frac{\Gamma(1-2\lambda)\Gamma(\lambda-\alpha)\Gamma(\lambda-\beta)\Gamma(\lambda)\Gamma(\alpha+\beta-2\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(3\lambda-\alpha-\beta)}\\
&\times\left\{\frac{\Gamma(3\lambda-\alpha-\beta)}{\Gamma(1+\lambda-\alpha-\beta)}-\frac{\Gamma(\alpha+\beta-\lambda)}{\Gamma(\alpha+\beta+1-3\lambda)}+\frac{\Gamma(\alpha)}{\Gamma(1+\alpha-2\lambda)}\right.\\
&\left.-\frac{\Gamma(2\lambda-\alpha)}{\Gamma(1-\alpha)}+\frac{\Gamma(\beta)}{\Gamma(1+\beta-2\lambda)}-\frac{\Gamma(2\lambda-\beta)}{\Gamma(1-\beta)}\right\}. \qquad \text{(F.35)}
\end{aligned}
$$

A final type of integral is:

$$
\begin{aligned}
I_\lambda(\alpha,\beta,\gamma) &\equiv \int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n}\frac{1}{k_1^{2\alpha}(k_1-q)^2(k_1-k_2)^{2\beta}k_2^{2\gamma}(k_2-q)^2}\\
&= \frac{(-1)}{(4\pi^{4+\epsilon}}(q^2)^{2-\alpha-\beta-\gamma}2\lambda-1-\alpha-\beta F_\lambda(\alpha,\beta,\gamma)\,. \qquad \text{(F.36)}
\end{aligned}
$$

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where:

$$F_\lambda(\alpha,\beta,\gamma) = \frac{\Gamma(\lambda+1-\alpha)\Gamma(\lambda+1-\beta)\Gamma(\lambda+1-\gamma)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\lambda)}$$
$$\times \sum_{m,n=0}^{\infty} \frac{(-1)^m}{m!n!(n+\lambda)} \frac{\Gamma(n+2\lambda)\Gamma(m+n+\alpha+\beta+\gamma-2\lambda)}{\Gamma(1+3\lambda-m-\alpha-\beta-\gamma)\Gamma(m+n+\lambda+1)}$$
$$\times \left\{ \frac{1}{(n+\beta)(m+n+\alpha+\beta-\lambda)} + \frac{1}{(m+n+\alpha)(m+n+\alpha+\beta-\lambda)} \right.$$
$$\left. + \frac{1}{(m+n+\alpha)(n+2\lambda-\beta)} + (\alpha \leftrightarrow \gamma) \right\}, \tag{F.37}$$

where the series is convergent for ($A \equiv \alpha+\beta+\gamma$):

$$A < 3\lambda+1\,, \qquad A < 2\lambda+2\,, \qquad A < \lambda+4\,. \tag{F.38}$$

F.8 One-loop massive integrals

$$I(\alpha,\beta,m^2) \equiv \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2+i\epsilon')^\alpha} \frac{1}{[k^2-m^2+i\epsilon']^\beta}$$
$$= \nu^{-\epsilon} \frac{i}{16\pi^2} (-m^2)^{2-\alpha-\beta} \left(\frac{m^2}{4\pi\nu^2} \right)^{-\epsilon/2}$$
$$\times \frac{\Gamma(2-\alpha-\epsilon/2)\,\Gamma(\alpha+\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\Gamma(\beta)}$$

$$I(\alpha,\beta,q^2,m^2) \equiv \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k-q)^2-m^2+i\epsilon']^\alpha} \frac{1}{(k^2+i\epsilon')^\beta}$$
$$= \nu^{-\epsilon} \frac{i}{16\pi^2} (q^2)^{2-\alpha-\beta} \left(\frac{-q^2}{4\pi\nu^2} \right)^{-\epsilon/2}$$
$$\times \frac{\Gamma(2-\beta-\epsilon/2)\,\Gamma(\alpha+\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\Gamma(\alpha)} \left(1-\frac{m^2}{q^2}\right)^{2-\alpha-\beta-\epsilon/2}$$
$$\times\, {}_2F_1\left(\alpha+\beta-2+\epsilon/2,\, 2-\beta-\epsilon/2,\, 2-\epsilon/2;\, \frac{1}{1-\frac{m^2}{q^2}}\right)$$

$$I^\mu(\alpha,\beta,q^2,m^2) \equiv \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu}{[(k-q)^2-m^2+i\epsilon']^\alpha} \frac{1}{(k^2+i\epsilon')^\beta}$$
$$= \nu^{-\epsilon} \frac{i}{16\pi^2} (q^2)^{2-\alpha-\beta} \left(\frac{-q^2}{4\pi\nu^2} \right)^{-\epsilon/2}$$
$$\times\, q^\mu \frac{\Gamma(2-\beta-\epsilon/2)\,\Gamma(\alpha+\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\Gamma(\alpha)} \left(1-\frac{m^2}{q^2}\right)^{2-\alpha-\beta-\epsilon/2}$$
$$\times\, {}_2F_1\left(\alpha+\beta-2+\epsilon/2,\, 3-\beta-\epsilon/2,\, 3-\epsilon/2;\, \frac{1}{1-\frac{m^2}{q^2}}\right) \tag{F.39}$$

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One also has:

$$\tilde{I}(\alpha, \beta, q^2, m^2) \equiv \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k-q)^2 - m^2 + i\epsilon']^\alpha} \frac{1}{(k^2 - m^2 + i\epsilon')^\beta}$$
$$= \nu^{-\epsilon} \frac{i}{16\pi^2} (q^2)^{2-\alpha-\beta} \left(\frac{-q^2}{4\pi\nu^2}\right)^{-\epsilon/2} \frac{\Gamma(\alpha+\beta-2+\epsilon/2)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\times \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \left[x(1-x) - \frac{m^2}{q^2}\right]^{2-\alpha-\beta-\epsilon/2}, \tag{F.40}$$

with:

$$\tilde{I}(\alpha, \beta, q^2, m^2) = \tilde{I}(\beta, \alpha, q^2, m^2)\,. \tag{F.41}$$

For $\alpha + \beta > 2$, one can rewrite the x-integral by letting $\epsilon \to 0$. For some particular values of α and β, one has, by using the definition of $\tilde{\epsilon}$ in Eq. (F.30):

$$I(1,1,q^2,m^2) = \nu^{-\epsilon} \frac{i}{16\pi^2} \left\{ \frac{2}{\tilde{\epsilon}} - \frac{m^2}{q^2} \ln \frac{m^2}{-q^2} - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{m^2}{q^2}\right) + 2 \right\},$$
$$I(1,2,q^2,m^2) = \nu^{-\epsilon} \frac{i}{16\pi^2} \frac{1}{q^2 - m^2} \left\{ -\frac{2}{\tilde{\epsilon}} - \frac{m^2}{q^2} \ln \frac{m^2}{-q^2} + \left(1 + \frac{m^2}{q^2}\right) \ln\left(1 - \frac{m^2}{q^2}\right) \right\},$$
$$I^\mu(1,1,q^2,m^2) = \nu^{-\epsilon} \frac{i}{16\pi^2} \frac{q^\mu}{2} \left\{ \frac{2}{\tilde{\epsilon}} - \frac{m^2}{q^2} \left(2 - \frac{m^2}{q^2}\right) \ln \frac{m^2}{q^2} - \left(1 - \frac{m^2}{q^2}\right)^2 \right.$$
$$\left. \times \ln\left(1 - \frac{m^2}{q^2}\right) - \frac{m^2}{q^2} + 2 \right\},$$
$$I^\mu(1,2,q^2,m^2) = \nu^{-\epsilon} \frac{i}{16\pi^2} \frac{q^\mu}{q^2} \left\{ -\frac{m^2}{q^2} \ln \frac{m^2}{-q^2} + \frac{m^2}{q^2} \ln\left(1 - \frac{m^2}{q^2}\right) + 1 \right\},$$
$$\tilde{I}(1,1,q^2,m^2) = \nu^{-\epsilon} \frac{i}{16\pi^2} \left\{ \frac{2}{\tilde{\epsilon}} - \sqrt{1 - 4m^2/q^2} \ln \frac{\sqrt{1-4m^2/q^2}+1}{\sqrt{1-4m^2/q^2}-1} + 2 \right\},$$
$$\tilde{I}(1,2,q^2,m^2) = \nu^{-\epsilon} \frac{i}{16\pi^2} \left\{ \frac{1}{q^2\sqrt{1-4m^2/q^2}} \ln \frac{\sqrt{1-4m^2/q^2}+1}{\sqrt{1-4m^2/q^2}-1} \right\}, \tag{F.42}$$

F.9 A two-loop massive integral

$$I_\lambda(\alpha, \beta, m^2) \equiv \int \frac{d^n k_1}{(2\pi)^n} \int \frac{d^n k_2}{(2\pi)^n} \frac{1}{\left(k_1^2 + m^2\right)^\alpha \left(k_2^2 + m^2\right)^\beta (k_1 - k_2)^2}$$
$$= \frac{(-1)}{(4\pi)^{4+\epsilon}} (m^2)^{3-\alpha-\beta+\epsilon} \frac{B(\alpha-\lambda, \beta-\lambda)\Gamma(\alpha+\beta-2\lambda-1)}{(1+\epsilon/2)\Gamma(\alpha)\Gamma(\beta)}\,. \tag{F.43}$$

F.10 The dilogarithm function

For complex z, it is defined as:

$$\mathrm{Li}_2(z) \equiv -\int_0^z \frac{dt}{t} \ln(1-t). \tag{F.44}$$

When z is real and bigger than one, the log. is complex, and there is a branch cut from $z = 1$ to ∞. Therefore, the function develops an imaginary part:

$$\mathrm{Li}_2(x + i0) \to -i\pi \ln x\ . \tag{F.45}$$

For $z \le 1$, one can write as:

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\ , \tag{F.46}$$

which is convenient for numerical calculations. In particular, its values are:

$$\begin{aligned} \mathrm{Li}_2(1) &= \frac{\pi^2}{6}\ , \qquad \mathrm{Li}_2(-1) = -\frac{\pi^2}{12}\ , \\ \mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\ , \qquad \mathrm{Li}_2(2 - i0) = \frac{\pi^2}{4} - i\pi \ln 2\ . \end{aligned} \tag{F.47}$$

Some of its useful properties are:

$$\begin{aligned} &\mathrm{Li}_2\,(x) + \mathrm{Li}_2(1-x) + \ln x \ln(1-x) = \frac{\pi^2}{6}\ , \\ &\mathrm{Li}_2\left(-\frac{1}{x}\right) + \mathrm{Li}_2(-x) + \frac{1}{2}\ln^2 x = -\frac{\pi^2}{6} \quad : x > 0\ , \\ &\mathrm{Li}_2\left(\frac{1}{x}\right) + \mathrm{Li}_2(x) + \frac{1}{2}\ln^2 x = \frac{\pi^2}{3} - i\pi \ln x \quad : x > 1\ , \\ &\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{1-x}\right) = \frac{\pi^2}{2} - 2i\pi \ln x + i\pi \ln(x-1) - \frac{1}{2}\ln^2(1-x) \quad : x > 1\ , \\ &\mathrm{Li}_2(x) + \mathrm{Li}_2\left(-\frac{x}{1-x}\right) = -\frac{1}{2}\ln^2(1-x) \quad : x < 1\ , \\ &\mathrm{Li}_2\left(\frac{1}{1+x}\right) - \mathrm{Li}_2(-x) = \frac{\pi^2}{6} - \frac{1}{2}\ln(1+x)\ln\left(\frac{1+x}{x^2}\right)\ , \\ &\mathrm{Li}_2(1-x) - \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{\pi^2}{6} + \frac{1}{2}\ln x \ln\frac{x}{(x-1)^2}\ , \\ &\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2)\ , \\ &\mathrm{Li}_2(z) = \frac{1}{1+z}\left[2(1-z)\ln(1-z) + z\left[3 + \sum_{n=1}^{\infty}\frac{z^n}{n^2(n+1)^2}\right]\right]. \end{aligned} \tag{F.48}$$

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F.11 Some useful logarithmic integrals

$$\int dx\, x^n \ \ln(ax-b) = \frac{1}{n+1}\left\{x^{n+1} - \left(\frac{b}{a}\right)^{n+1} \ln(ax-b) - \sum_{i=1}^{n+1}\left(\frac{b}{a}\right)^{n+1-i}\frac{x^i}{i}\right\}$$

$$\int_0^1 dx\, x^n \ \ln x = -\frac{1}{(n+1)^2} \tag{F.49}$$

Defining:

$$I_n = \int_0^1 dx\, x^n \ \ln[a - x(1-x)]\,, \tag{F.50}$$

one has:

$$\begin{aligned}
I_0 &= -2 + \ln a + \sqrt{1-4a}\ln\frac{\sqrt{1-4a}+1}{\sqrt{1-4a}-1}\,,\\
I_1 &= \frac{1}{2}I_0\,,\\
I_2 &= \frac{1}{3}\left[-\frac{13}{6} + 2a + \ln a + (1-a)\sqrt{1-4a}\ln\frac{\sqrt{1-4a}+1}{\sqrt{1-4a}-1}\right],\\
I_3 &= \frac{1}{2}[I_0 - 3I_1 + 3I_2]\,,
\end{aligned} \tag{F.51}$$

where one has exploited the invariance under the change $x \leftrightarrow (1-x)$ allowing to deduce the integrals odd in n from the even ones. In addition to the properties of dilogarithm functions [876], one also needs [877,45,3]:

$$\begin{aligned}
\int_0^x \frac{dt}{t}\ln(1+t^2) &= -\frac{1}{2}\mathrm{Li}_2(-x^2)\,,\\
\int_0^{x>0} \frac{dt}{t}\ln(1-t+t^2) &= -\frac{1}{3}\mathrm{Li}_2(-x^3) + \mathrm{Li}_2(-x)\,,\\
\int_0^{x>0} \frac{dt}{t}\ln(1+t+t^2) &= -\frac{1}{3}\mathrm{Li}_2(x^3) + \mathrm{Li}_2(x)\,,\\
\int_0^1 \frac{dt}{1+t}\ln(1+t)\ln^2 t &= -\frac{3}{2}\zeta^2(2) + 4S_4 + \frac{7}{2}\zeta(3)\ln 2 - \zeta(2)\ln^2 2 + \frac{1}{6}\ln^4 2\\
\int_0^1 \frac{dt}{1+t}\ln^2(1+t)\ln t &= -\frac{4}{5}\zeta^2(2) + 2S_4 + \frac{7}{4}\zeta(3)\ln 2 - \frac{1}{2}\zeta(2)\ln^2 2 + \frac{1}{12}\ln^4 2\,,\\
\int_0^1 \frac{dt}{t}\ln t\ln^2(1+t) &= \frac{7}{10}\zeta^2(2) = -\frac{1}{3}\int_0^1\frac{dt}{1+t}\ln^2 t\,,\\
\int_0^1 \frac{dt}{t}\ln(1+t)\mathrm{Li}_2(-t) &= \frac{1}{8}\zeta^2(2)\,,\\
\int_0^1 \frac{dt}{1+t}\mathrm{Li}_2(-t)\ln(1+t) &= -\frac{6}{5}\zeta^2(2) - 3S_4 - \frac{21}{8}\zeta(3)\ln 2 + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{8}\ln^4 2\,,\\
\int_0^1 \frac{dt}{1+t}\mathrm{Li}_2(-t)\ln t &= \frac{13}{8}\zeta^2(2) - 4S_4 - \frac{7}{2}\zeta(3)\ln 2 + \zeta(2)\ln^2 2 - \frac{1}{6}\ln^4 2\,,
\end{aligned}$$

$$\int_0^1 \frac{dt}{1+t}\ln^2(1+t)\ln(1-t) = -\frac{4}{5}\zeta^2(2) + 2S_4 + 2\zeta(3)\ln 2 - \zeta(2)\ln^2 2 + \frac{1}{3}\ln^4 2\,, \tag{F.52}$$

with:

$$S_4 \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^4} = 0.571\,479\,061\,6\ldots \tag{F.53}$$

Some other useful integrals are:

$$\begin{aligned}\int_0^1 dt\; t^{\alpha-1}(1-t)^{\beta-1}\ln t &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}[S_1(\alpha-1) - S_1(\alpha+\beta-1)]\,,\\ \int_0^1 dt\; t^{\alpha-1}(1-t)^{\beta-1}\ln^2 t &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}[[S_1(\alpha-1) - S_1(\alpha+\beta-1)]^2\\ &\quad + S_2(\alpha+\beta-1) - S_2(\alpha-1)]\,,\end{aligned} \tag{F.54}$$

where:

$$S_l(n) \equiv \sum_{k=1}^{n} \frac{1}{k^l} \quad : l = 1, 2, 3, \ldots \tag{F.55}$$

Differentiating with respect to β, one gets:

$$\begin{aligned}\int_0^1 dt\; t^{\alpha-1}(1-t)^{\beta-1}\ln t\ln(1-t) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}[S_2(\alpha+\beta-1) - \zeta(2)\\ &\quad + \left[S_1(\alpha-1) - S_1(\alpha+\beta-1)\right]\\ &\quad \times [S_1(\beta-1) - S_1(\alpha+\beta-1)]].\\ &\quad \times \int_0^1 dx\, \frac{1-x^\alpha}{1-x} = S_1(\alpha)\,.\end{aligned} \tag{F.56}$$

Integrals of the form:

$$\int_0^1 dt\; t^{\alpha-1}(1-t)^{\beta}\ln^m t, \tag{F.57}$$

for integer or half-integer values of α and β are also useful and are given in [876]. For some particular values, one has:

$$\begin{aligned}\int_0^1 \frac{dt}{(1-t)^2}\ln^2 t &= -4\zeta(3) + \frac{\pi^2}{3} + 2\,,\\ \int_0^1 \frac{dt}{(1-t)^2}t^{3/2}\ln^2 t &= -21\zeta(3) + \pi^2 + 16,\end{aligned} \tag{F.58}$$

and:

$$\int_0^1 \frac{dt}{(1-t)}t^{\alpha-1}\ln^m t = (-1)^m m![\zeta(m+1) - S_{m+1}(\alpha-1)]\,. \tag{F.59}$$

F.12 Further useful functions

The functions:

$$S_1(z) = \sum_{k=1}^{\infty} \frac{z}{k(z+k)} \,, \qquad S_l(z) = \zeta(l) - \sum_{k=1}^{\infty} \frac{1}{(z+k)^l} \,, \tag{F.60}$$

appear often in the evaluation of the parametric integrals. For $z = n$ integer, it can be reduced to the one in Eq. (F.55). They have the properties:

$$\begin{aligned} S_1(z+1) &= S_1(z) + \frac{1}{z+1} \,, \\ S_l(\infty) &= \zeta(l) \,, \\ \frac{dS_l(z)}{dz} &= l[\zeta(l+1) - S_{l+1}(z)], \end{aligned} \tag{F.61}$$

for $l = 1, 2, 3, \ldots$ They are related to the psi-function defined as:

$$\psi(z) \equiv \frac{d}{dz} \ln \Gamma(z) \,, \tag{F.62}$$

with the properties:

$$\begin{aligned} \psi(z) &= -\gamma_E - \frac{1}{z} + S_1(z) \,, \qquad \psi(1) = -\gamma_E \,, \\ \frac{d^l}{dz^l} \psi(z) &= l!(-1)^{l+1}[\zeta(l+1) - S_{l+1}(z-1)] \quad \text{for } l \geq 1, \end{aligned} \tag{F.63}$$

and therefore:

$$\psi'(1) = \zeta(2) \,, \quad \psi'(2) = \zeta(2) - 1 \,, \qquad \psi''(1) = -2\zeta(3) \,. \tag{F.64}$$

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