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THE SPECTRAL EIGENMATRIX PROBLEMS OF PLANAR SELF-AFFINE MEASURES WITH FOUR DIGITS

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Abstract Given a Borel probability measure μ on $\mathbb{R}^n$ and a real matrix $R \in M_n(\mathbb{R})$. We call R a spectral eigenmatrix of the measure μ if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that the sets $E_\Lambda = \left\{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\right\}$ and $E_{R\Lambda} = \left\{e^{2\pi i\langle R\lambda, x\rangle} : \lambda \in \Lambda\right\}$ are both orthonormal bases for the Hilbert space $L^2(\mu)$. In this paper, we study the structure of spectral eigenmatrix of the planar self-affine measure $\mu_{M,D}$ generated by an expanding integer matrix $M \in M_2(2\mathbb{Z})$ and the four-elements digit set $D = \{(0,0)^t, (1,0)^t, (0,1)^t, (-1,-1)^t\}$. Some sufficient and/or necessary conditions for R to be a spectral eigenmatrix of $\mu_{M,D}$ are given.

Keywords: self-affine measure; spectral measure; spectrum; spectral eigenmatrix

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1. Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$. We call μ a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that the set $\left\{e^{2\pi i\langle \lambda, x\rangle}\right\}_{\lambda\in\Lambda}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a *spectrum* of μ, and we also say that (μ, Λ) forms a spectral pair. If $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set with finite positive Lebesgue measure and $d\mu = \chi_\Omega\, dx$ is a spectral measure, then we call Ω a *spectral set*. Classical spectral measures were first introduced by Fuglede [23], and he proposed his famous conjecture stating that a bounded measurable set $\Omega \subset \mathbb{R}^n$ is a spectral set if and only if Ω is a translational tile. The conjecture was proven to be false on $n \geq 3$ [29, 44], but it is still open in one and two dimensions, and it is related to the construction of Gabor and wavelet bases [13, 38, 46].

The studies entered into the realm of fractals when Jorgensen and Pedersen discovered that some singular fractal measures can also be spectral [27]. Since then, singular spectral measures have become an active research topic, which involves classifying classes of measures which are spectral [1, 3, 4, 6, 9, 14–17, 30], finding their possible spectra [8, 10, 11, 21, 24, 33, 34] and studying all kinds of convergence problems of the associated Mock Fourier series [19, 40, 41], etc. Many new exotic phenomena which are different from Lebesgue measures were discovered for singular fractal measures. Here we list some typical examples; Dai [7] found that the spectra of some singular spectral measures might have zero Beurling dimension, while the spectra for Lebesgue measures must have positive Beurling dimension [32]. Strichartz [40, 41] found a large class of singular spectral measures with uncountably many spectra such that their associated mock Fourier series of continuous functions converge uniformly, and the mock Fourier series of L^p-functions converge pointwise almost everywhere; nevertheless, there are also singular spectral measures and spectra such that the associated Fourier series of continuous function diverges at some point [19]. These surprising phenomena motivate researchers to find more singular spectral measures and study their related problems, and it is connected with a number of areas in mathematics such as number theory, dynamical system, harmonic analysis, etc. [18, 28, 39]. The following two types of problems are basic in the study of spectrality of singular measures.

I. **Spectral Problems:** What kind of measures are spectral measures?
II. **Spectral Eigenmatrix Problems:** Let μ be a singular spectral measure on $\mathbb{R}^n$. The spectral eigenmatrix problems contain two themes in general. (1) Fix a spectrum Λ, find all $R \in M_n(\mathbb{R})$ such that $R\Lambda$ is a spectrum of μ (we call it the first type of spectral eigenmatrix problems); (2) Find all matrices $R \in M_n(\mathbb{R})$ such that $R\Lambda$ is a spectrum of μ for some spectrum Λ (we call it the second type of spectral eigenmatrix problems). In the two cases, R is called a *spectral eigenmatrix* of μ and Λ is called a *eigenspectrum* of μ corresponding to R.

Let $M \in M_n(\mathbb{R})$ be an $n \times n$ expanding real matrix (i.e., all the eigenvalues of M have modulus strictly greater than one), and $D \subset \mathbb{R}^n$ be a finite digit set. Then there exists a unique Borel probability measure $\mu_{M,D}$, which satisfies that

$$\mu_{M,D}(E) = \frac{1}{\#D} \sum_{d \in D} \mu_{M,D}(ME - d) \quad \text{for any Borel set } E, \tag{1.1}$$

where $\#D$ denotes the cardinality of D [26]. Moreover, the measure is supported on

$$T(M, D) = \left\{ \sum_{k=1}^{\infty} M^{-k} d_k : d_k \in D \right\} := \sum_{k=1}^{\infty} M^{-k} D.$$

We call $\mu_{M,D}$ a *self-affine measure* and $T(M, D)$ a *self-affine set*. In particular, if M is a multiple of an orthonormal matrix, then $\mu_{M,D}$ and $T(M, D)$ are called *self-similar* *measure* and *self-similar set*, respectively. It is known that a self-affine measure $\mu_{M,D}$ can be expressed by the infinite convolution of Dirac measures with equal weights as follows:

$$\mu_{M,D} = \delta_{M^{-1}D} * \delta_{M^{-2}D} * \delta_{M^{-3}D} * \cdots ,$$

where $*$ is the convolution sign, $\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$ for a finite set E and δ_e is the Dirac measure at the point e, and the convergence is in the weak sense.

The spectral eigenmatrix problems are also called the scaling matrix problems. The origin of spectral eigenmatrix problems goes back to Łaba and Wang [30], who first discovered a countable set Λ such that Λ and 2Λ are spectra of a measure μ. This surprise discovery attracted many researchers, and a lot of new results about the spectral eigenmatrix have been obtained. For examples, Dutkay and Jorgensen [17] proved that $R = 5^n (n \in \mathbb{N}^+)$ is a spectral eigenmatrix of $\mu_{4,\{0,2\}}$ for the spectrum $\Lambda_1 := \sum_{k=1}^{\infty}\{0,1\}4^{k-1}$ and $R = 3n(n \in \mathbb{N}^+)$ is not a spectral eigenmatrix of $\mu_{4,\{0,2\}}$ for the spectrum Λ_1. Dutkay and Haussermann [18] proved that if p is a prime number greater than 3, then $p^n\Lambda_1$ is also a spectrum of $\mu_{4,\{0,2\}}$ for any $n \geq 1$. He *et al.* [25] gave an answer to the spectral eigenmatrix problems (1) about the measure $\mu_{q,\{0,ar,br\}}$ for the spectrum $\Lambda := \sum_{k=1}^{\infty}\{0,\pm1\}q^{k-1}$. Fu *et al.* [22] gave a complete characterization on the spectral eigenmatrix problems (2) of the Bernoulli convolution μ_{2k}. Wang and Wu [45] studied the spectral eigenmatrix problems (2) for a class of self-similar spectral measures with consecutive digits. The known results on spectral eigenvalue problems mainly focus on self-similar spectral measures in one-dimensional case; however, there are not much discussions about the spectral eigenmatrix problems in high-dimensional case since the methods in one dimension are difficult to apply to higher dimensions even for the simple cases. As far as we know, the only high-dimensional example was by An *et al.* [2], who discussed the spectral eigenmatrix problems of the Sierpinski-type measure $\mu_{M,D}$ generated by an expanding matrix $M = \mathrm{diag}[3q, 3q]$ and $D = \{(0,0)^t, (0,1)^t, (1,0)^t\}$.

The planar Cantor-dust measure $\mu_{M,D'}$ is the most typical self-affine measure except for the planar Sierpinski-type measure, and it is generated by an expanding matrix $M \in M_2(\mathbb{Z})$ and the integer digit set $D' = \{(0,0)^t, (\alpha_1,\alpha_2)^t, (\beta_1,\beta_2)^t, (-\alpha_1-\beta_1, -\alpha_2-\beta_2)^t\}$. There are many researches about its spectrality or non-spectrality [35, 37, 42, 43]. Recently, Chen *et al.*[5] gave the following complete characterization on the spectrality of $\mu_{M,D'}$.

Theorem 1.1. *[5] Let $D' = \{(0,0)^t, (\alpha_1,\alpha_2)^t, (\beta_1,\beta_2)^t, (-\alpha_1-\beta_1, -\alpha_2-\beta_2)^t\}$ be an integer digit set and $M \in M_2(\mathbb{Z})$ be an expanding matrix. If $\alpha_1\beta_2 - \alpha_2\beta_1 \notin 2\mathbb{Z}$, then $\mu_{M,D'}$ is a spectral measure if and only if $M \in M_2(2\mathbb{Z})$.*

Let

$$M \in M_2(2\mathbb{Z}) \quad \text{and} \quad D = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix} \right\}. \tag{1.2}$$

The corresponding self-affine measure $\mu_{M,D}$ is the simplest planar Cantor-dust measure. Motivated by the above works, our goal in the present paper is to investigate the spectral structure and spectral eigenvalue problems of $\mu_{M,D}$, and we hope that the methods used in this paper can shed some light on the study of high-dimensional spectral eigenvalue problems.

In order to characterize the structure of the spectra of $\mu_{M,D}$, we need to introduce some notation from symbolic dynamical system. Let $\Sigma = \{0, 1, 2, 3\}$, $\Sigma^0 = \emptyset$ and $\Sigma^n = \{i_1 i_2 \cdots i_n : i_j \in \Sigma, 1 \leq j \leq n\}$ to be the set of all words $i_1 \cdots i_n$ with length $n \geq 1$. For any $I \in \Sigma^n$ and $J \in \Sigma^m$, the word IJ is their natural conjunction. In particular, $\emptyset i_1 \cdots i_n = i_1 \cdots i_n$, $Ij = i_1 \cdots i_n j$ for all $I = i_1 \cdots i_n \in \Sigma^n$, $n \geq 1$, and $0^k = \underbrace{0 \cdots 0}_{k}$. We define that $I|_k = i_1 i_2 \cdots i_k$ for $I = i_1 i_2 \cdots i_n \in \Sigma^n$ and $k \leq n$.

Let

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}. \tag{1.3}$$

Then we can state our first result as follows.

Theorem 1.2. *Let $M \in M_2(2\mathbb{Z})$ be an expanding matrix, D be given by Equation (1.2). If Λ is a spectrum of $\mu_{M,D}$ with $0 \in \Lambda$, then for each $n \geq 1$, Λ has a decomposition*

$$\Lambda = \bigcup_{I \in \Sigma^n} \left(M^* \left(\frac{1}{2} s_{I|_1} + z_{I|_1} \right) + M^{*2} \left(\frac{1}{2} s_{I|_2} + z_{I|_2} \right) + \cdots + M^{*n} \left(\frac{1}{2} s_I + \Gamma_I \right) \right), \tag{1.4}$$

where $z_{0^k} = s_{0^k} = s_{0^n} = 0$, $\cup_{i \in \Sigma} s_{Ji} = \mathcal{S}$, $z_{Ji} \subset \mathbb{Z}^2$ for every $J \in \Sigma^k$, $0 \leq k \leq n-1$ and each $\Gamma_I \subset \mathbb{Z}^2$ is a spectrum of $\mu_{M,D}$ with $0 \in \Gamma_{0^n}$. Conversely, if Λ can be decomposed into the form of Equation (1.4) for some positive integer n, then Λ is a spectrum of $\mu_{M,D}$ with $0 \in \Lambda$.

Theorem 1.2 characterizes the structure of the spectra of $\mu_{M,D}$, which is useful for us to find the necessary conditions for a matrix to be a spectral eigenmatrix of $\mu_{M,D}$. Our main results about spectral eigenmatrices are the following three theorems.

Theorem 1.3. *Let $M \in M_2(2\mathbb{Z})$ be an expanding matrix, D be given by Equation (1.2) and let $R \in M_2(\mathbb{R})$ and $R_k = M^{*-k} R M^{*k}$ for any $k \in \mathbb{Z}$. Suppose R is a spectral eigenmatrix of $\mu_{M,D}$, then the following statements hold.*

(i) *$R = \frac{1}{l} M^* R' M^{*-1}$ for some $l \in 2\mathbb{Z}+1$ and $R' \in M_2(\mathbb{Z})$ with $\det(R') \in 2\mathbb{Z}+1$;*
(ii) *For each $k \geq 0$, R_k is a spectral eigenmatrix of $\mu_{M,D}$;*
(iii) If $M = \operatorname{diag}[2p, 2q]$ with $\frac{pq}{\gcd(p^2,q^2)} \in 2\mathbb{Z}+1$, then for each $k \geq 0$, R_{-k} is a spectral eigenmatrix of $\mu_{M,D}$.

Theorem 1.3 just gives some necessary conditions for a matrix to be a spectral eigenmatrix of $\mu_{M,D}$. For some special case, the sufficient and necessary conditions for the spectral eigenmatrices of $\mu_{M,D}$ can be characterized.

Theorem 1.4. *Let $M = \operatorname{diag}[2p, 2q] \in M_2(2\mathbb{Z})$ with $|p|, |q| > 1$, D be given by Equation (1.2). Suppose $R \in M_2(\mathbb{R})$ satisfies $RM = MR$, then R is a spectral eigenmatrix of $\mu_{M,D}$ if and only if $R \in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1})$ and $\det(R) \in \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.*

We remark that the condition $|p|, |q| > 1$ can be removed in the proof of the necessity of Theorem 1.4, but this condition plays a key role in constructing the spectrum in the proof of the sufficiency of Theorem 1.4. For the condition '$RM = MR$', we do not know whether it is superfluous. But the following theorem tells us that this condition may be necessary in Theorem 1.4.

Theorem 1.5. *Let $M = \mathrm{diag}[2p, 2q] \in M_2(2\mathbb{Z})$ be an expanding matrix, D be given by Equation (1.2). Suppose $R \in M_2(\mathbb{R})$ is a spectral eigenmatrix of $\mu_{M,D}$, then the following statements hold.*

(i) *If $\frac{q}{\gcd(p,q)} \in 2\mathbb{Z}$, then $R = \frac{1}{l}\begin{bmatrix} a & b(\frac{p}{q}) \\ 0 & d \end{bmatrix}$ for some integers $l, a, d \in 2\mathbb{Z}+1$ and $b \in \mathbb{Z}$;*

(ii) *If $\frac{p}{\gcd(p,q)} \in 2\mathbb{Z}$, then $R = \frac{1}{l}\begin{bmatrix} a & 0 \\ c(\frac{q}{p}) & d \end{bmatrix}$ for some integers $l, a, d \in 2\mathbb{Z}+1$ and $c \in \mathbb{Z}$.*

This paper is organized as follows. In § 2, we introduce some basic definitions and results that will be used in the proof of our main results. In § 3, we study the structure of the spectra of $\mu_{M,D}$ and prove Theorem 1.2. In § 4, we will discuss the structure of the spectral eigenmatrices of $\mu_{M,D}$ and prove Theorems 1.3 and 1.5. In § 5, we will prove Theorem 1.4. In § 6, some remarks and open questions related to our main results will be given.

2. Preliminaries

For a probability measure μ on $\mathbb{R}^n$, the Fourier transform of μ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi i \langle x, \xi \rangle} \, \mathrm{d}\mu(x).$$

It follows from Equations (1.1) and (1.2) that the Fourier transform $\hat{\mu}_{M,D}$ of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi) = \int_{\mathbb{R}^2} \mathrm{e}^{2\pi i \langle x, \xi \rangle} \, \mathrm{d}\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi), \quad \xi \in \mathbb{R}^2, \tag{2.1}$$

where M^* denotes the transposed conjugate of M and

$$m_D(\xi) = \frac{1}{\#D} \sum_{d \in D} \mathrm{e}^{2\pi i \langle d, \xi \rangle} = \frac{1}{4}\left(1 + \mathrm{e}^{2\pi i \xi_1} + \mathrm{e}^{2\pi i \xi_2} + \mathrm{e}^{-2\pi i (\xi_1 + \xi_2)}\right), \quad \xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2.$$

Denote $\mathcal{Z}(\hat{\mu}) = \{x \in \mathbb{R}^2 : \hat{\mu}(x) = 0\}$ to be the zero set of $\hat{\mu}$. For convenience, we let

$$\mathcal{F}_2^2 = \frac{1}{2}\big\{(\ell_1, \ell_2)^t : \ell_i \in \{0, 1\}, i = 1, 2\big\} \quad \text{and} \quad \mathring{\mathcal{F}}_2^2 := \mathcal{F}_2^2 \setminus \{0\}. \tag{2.2}$$

It follows from Equations (2.2) and (2.1) that

$$\mathcal{Z}(m_D) = \mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2, \tag{2.3}$$

$$\mathcal{Z}(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)) = \bigcup_{j=1}^{\infty} M^{*j}(\mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2). \tag{2.4}$$

For a probability measure μ on $\mathbb{R}^n$ and a countable set $\Lambda \subset \mathbb{R}^n$, it is easy to check that the family $E_\Lambda := \{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal set for $L^2(\mu)$, which is equivalent to the condition that

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}). \tag{2.5}$$

We call Λ an *orthogonal set* (respectively, *spectrum*) of μ if E_Λ forms an orthogonal system (respectively, Fourier basis) for $L^2(\mu)$. Since the properties of orthogonal set (or spectrum) are invariant under a translation, it will be convenient to assume that $0 \in \Lambda$, and hence $\Lambda \subset (\Lambda - \Lambda)$.

Let

$$Q_{\mu,\Lambda}(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2, \quad \xi \in \mathbb{R}^n. \tag{2.6}$$

The well-known result of Jorgensen and Pedersen [27, Lemma 4.2] shows that $Q_{\mu,\Lambda}(\xi)$ is an entire function if Λ is an orthogonal set of μ. The following provides a universal test, which allows us to decide whether an orthogonal set Λ is a spectrum of the measure μ.

Theorem 2.1. *[27] Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$, and let $\Lambda \subset \mathbb{R}^n$ be a countable set. Then*

(i) Λ is an orthogonal set of μ if and only if $Q_{\mu,\Lambda}(\xi) \le 1$ for $\xi \in \mathbb{R}^n$.
(ii) Λ is a spectrum of μ if and only if $Q_{\mu,\Lambda}(\xi) \equiv 1$ for $\xi \in \mathbb{R}^n$.

In the study of the spectrality of self-affine measures, the concept of Hadamard triple plays an important role. To the best of our knowledge, almost all spectra of self-affine measures are generated by Hadamard triples.

Definition 2.2. *Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, and let $D, L \subset \mathbb{Z}^n$ be two finite digit sets with the same cardinality. We say that the pair (M, D) is admissible if the matrix*

$$H_{M^{-1}D,L} := \frac{1}{\sqrt{\#D}} \left[e^{2\pi i\langle M^{-1}d,l\rangle}\right]_{d\in D,l\in L}$$

is unitary, that is, $H_{M^{-1}D,L}H^*_{M^{-1}D,L} = I_{\#D}$. In this case, we call the triple (M, D, L) a Hadamard triple and also call $(M^{-1}D, L)$ a compatible pair.

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Lemma 2.3. *[12, 20] Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, and let $D, L \subset \mathbb{Z}^n$ be two finite digit sets with the same cardinality. Then the following statements are equivalent.*

(i) *(M, D, L) is a Hadamard triple.*
(ii) *$m_D(M^{*-1}(l_1 - l_2)) = 0$ for any $l_1 \neq l_2 \in L$.*
(iii) *$(\delta_{M^{-1}D}, L)$ is a spectral pair.*

Moreover, if (M, D, L) is a Hadamard triple, then for any $n \geq 1$, (M^n, D_n, L_n) is a Hadamard triple where $D_n := D + MD + M^2D + \cdots + M^{n-1}D$ and $L_n := L + M^*L + M^{*2}L + \cdots + M^{*(n-1)}L$.

Lemma 2.4. *[40] Let $\mu_{M,D}$ be defined by Equation (1.1), and $(M^{-1}D, L_k)$ be compatible pairs with $0 \in L_k$, $k \geq 1$. Suppose that the set of zeros of the Fourier transform $\widehat{\mu}_{M,D}$ is uniformly disjoint from the sets $M^{*-n}L_1 + M^{*-(n-1)}L_2 + \cdots + M^{*-1}L_n$ for all large n. Then $\mu_{M,D}$ is a spectral measure and has a spectrum $\Lambda = L_1 + M^*L_2 + M^{*2}L_3 + \cdots$.*

Recall that $\mathcal{S}$ is defined by Equation (1.3), that is,

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

In the end of this section, we will discuss the relationship between D and $\mathcal{S}$, which is an elementary but useful fact in our investigation.

Lemma 2.5. *With the above notation, the following statements hold.*

(i) *$\mathcal{S} = 2\mathcal{F}_2^2$.*
(ii) *$\mathcal{S}$ is a complete residue system modulo $2\mathbb{Z}^2$.*
(iii) *$(\delta_D, \frac{1}{2}\mathcal{S})$ is a spectral pair.*

Proof. By the definition of $\mathcal{S}$ and $\mathcal{F}_2^2$, (i) and (ii) are obvious. We now prove (iii). For any $s_1 \neq s_2 \in \mathcal{S}$, it is easy to check that $\frac{1}{2}(s_1 - s_2) \in \mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2$. Using Equation (2.3), we get $m_D(\frac{1}{2}(s_1 - s_2)) = 0$. Therefore, (iii) holds by Lemma 2.3. □

3. Structure of the spectra of $\mu_{M,D}$

In this section, we first investigate the structure of the spectra of $\mu_{M,D}$ under the hypothesis that $\mu_{M,D}$ is a spectral measure, and then prove Theorem 1.2. According to Theorem 1.1, we may assume the expanding matrix $M \in M_2(2\mathbb{Z})$.

Let Λ be a spectrum of $\mu_{M,D}$ with $0 \in \Lambda$, according to Equations (2.4) and (2.5), we know

$$2M^{*-1}\Lambda \subset \{0\} \cup 2\bigcup_{j=0}^{\infty} M^{*j}(\mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2) \subset \bigcup_{j=0}^{\infty} M^{*j}(\mathcal{S} + 2\mathbb{Z}^2) \subset \mathbb{Z}^2. \tag{3.1}$$

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For any $v \in \mathbb{Z}^2$, it follows from Lemma 2.5(*ii*) that there exist $s \in \mathcal{S}$ and $v' \in \mathbb{Z}^2$ such that

$$v = s + 2v'. \tag{3.2}$$

It is clear that the expression is unique. By Equations (3.1) and (3.2), we can get that, for any $\lambda \in \Lambda$, there exists a unique $s \in \mathcal{S}$ such that $\lambda = M^*(\frac{1}{2}s + \omega)$ for some $\omega \in \mathbb{Z}^2$. Define

$$\Gamma_s = \left\{\omega \in \mathbb{Z}^2 : M^*\left(\frac{1}{2}s + \omega\right) \in \Lambda\right\}. \tag{3.3}$$

Then we have the following decomposition:

$$\Lambda = \bigcup_{s\in\mathcal{S}} M^*\left(\frac{1}{2}s + \Gamma_s\right), \tag{3.4}$$

where $M^*(\frac{1}{2}s + \Gamma_s) = \emptyset$ if $\Gamma_s = \emptyset$. Moreover, the union is disjoint. As $0 \in \Lambda$, it follows that

$$\Gamma_0 \neq \emptyset. \tag{3.5}$$

Lemma 3.1. *Let Λ be a spectrum of $\mu_{M,D}$ with $0 \in \Lambda$. If Γ_s is a non-empty set, then Γ_s is an orthogonal set of $\mu_{M,D}$.*

Proof. Suppose that Γ_s is a non-empty set for some $s \in \mathcal{S}$. For any two distinct elements $\lambda_1, \lambda_2 \in \Gamma_s \subset \mathbb{Z}^2$, it follows from Equation (3.4) that $M^*(\frac{1}{2}s + \lambda_1)$, $M^*(\frac{1}{2}s + \lambda_2) \in \Lambda$. By Equation (2.5), we have $M^*(\lambda_1 - \lambda_2) \in \mathcal{Z}(\hat{\mu}_{M,D})$. Together with Equation (2.1) and $m_D(\xi) = 1$ for any $\xi \in \mathbb{Z}^2$, it is easy to get

$$0 = \hat{\mu}_{M,D}(M^*(\lambda_1 - \lambda_2)) = m_D(\lambda_1 - \lambda_2)\hat{\mu}_{M,D}(\lambda_1 - \lambda_2) = \hat{\mu}_{M,D}(\lambda_1 - \lambda_2).$$

Thus, $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\mu}_{M,D})$, which implies that Γ_s is an orthogonal set of $\mu_{M,D}$. □

The following lemma gives the structure of the spectra of $\mu_{M,D}$.

Lemma 3.2. *Let Λ be a spectrum of $\mu_{M,D}$ with $0 \in \Lambda$. Then Λ has a decomposition*

$$\Lambda = \bigcup_{s\in\mathcal{S}} \left(\frac{1}{2}M^*s + M^*\Gamma_s\right), \tag{3.6}$$

where Γ_s are also spectra of $\mu_{M,D}$ for all $s \in \mathcal{S}$.

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Proof. We first prove that $\Gamma_s \neq \emptyset$ for all $s \in \mathcal{S}$. Let $\mathcal{S}' = \{s \in \mathcal{S} : \Gamma_s \neq \emptyset\}$. In view of Equation (3.4), Λ can be written as

$$\Lambda = \bigcup_{s\in\mathcal{S}'} M^*\left(\frac{1}{2}s + \Gamma_s\right).$$

Then for any $\xi \in \mathbb{R}^2$, using Equation (2.1), Theorem 2.1, Lemma 3.1 and the fact that $\Gamma_s \subset \mathbb{Z}^2$, we get

$$\begin{aligned} 1 &\equiv \sum_{\lambda'\in\Lambda} |\hat{\mu}_{M,D}(\xi+\lambda')|^2 \\ &= \sum_{s\in\mathcal{S}'}\sum_{\lambda\in\Gamma_s}\left|\hat{\mu}_{M,D}\left(\xi + M^*\left(\frac{1}{2}s+\lambda\right)\right)\right|^2 \\ &= \sum_{s\in\mathcal{S}'}\sum_{\lambda\in\Gamma_s}\left|m_D\left(M^{*-1}\xi+\frac{1}{2}s+\lambda\right)\right|^2\left|\hat{\mu}_{M,D}\left(M^{*-1}\xi+\frac{1}{2}s+\lambda\right)\right|^2 \\ &= \sum_{s\in\mathcal{S}'}\left|m_D\left(M^{*-1}\xi+\frac{1}{2}s\right)\right|^2\sum_{\lambda\in\Gamma_s}\left|\hat{\mu}_{M,D}\left(M^{*-1}\xi+\frac{1}{2}s+\lambda\right)\right|^2 \\ &\leq \sum_{s\in\mathcal{S}'}\left|m_D\left(M^{*-1}\xi+\frac{1}{2}s\right)\right|^2. \end{aligned} \tag{3.7}$$

On the other hand, choose $\xi \in M^*(\mathbb{R}^2 \setminus \mathbb{Q}^2)$. As $\mathcal{Z}(m_D) = \mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2$, it follows that

$$\left|m_D\left(M^{*-1}\xi+\frac{1}{2}s\right)\right|^2 > 0 \quad \text{for all } s \in \mathcal{S}. \tag{3.8}$$

Following from Lemma 2.5(iii) and Theorem 2.1 , one may get that

$$\sum_{s\in\mathcal{S}}\left|m_D\left(M^{*-1}\xi+\frac{1}{2}s\right)\right|^2 = 1. \tag{3.9}$$

If $\mathcal{S}' \neq \mathcal{S}$, according to Equations (3.8) and (3.9), we deduce that $\sum_{s\in\mathcal{S}'}|m_D(M^{*-1}\xi + \frac{1}{2}s)|^2 < 1$. This contradicts Equation (3.7); therefore, $\mathcal{S}' = \mathcal{S}$, that is, $\Gamma_s \neq \emptyset$ for all $s \in \mathcal{S}$.

Now we show that Γ_s is a spectrum of $\mu_{M,D}$ for each $s \in \mathcal{S}$. Suppose, on the contrary, that there exists a $s_0 \in \mathcal{S}$ such that Γ_{s_0} is not a spectrum of $\mu_{M,D}$. Then, by Theorem 2.1, there must exist $\xi_0 \in M^*(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ such that

$$\sum_{\lambda\in\Gamma_{s_0}}\left|\hat{\mu}_{M,D}\left(M^{*-1}\xi_0+\frac{1}{2}s_0+\lambda\right)\right|^2 < 1.$$

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Hence, using $\mathcal{S}' = \mathcal{S}$, Lemma 3.1, Equation (3.9) and the similar argument as Equation (3.7), we conclude that

$$\begin{aligned} 1 &= \sum_{\lambda' \in \Lambda} |\hat{\mu}_{M,D}(\xi_0 + \lambda')|^2 \\ &= \sum_{s \in \mathcal{S}} \left| m_D \left(M^{*-1}\xi_0 + \frac{1}{2}s \right) \right|^2 \sum_{\lambda \in \Gamma_s} \left| \hat{\mu}_{M,D} \left(M^{*-1}\xi_0 + \frac{1}{2}s + \lambda \right) \right|^2 \\ &\leq \sum_{s \in \mathcal{S} \setminus \{s_0\}} \left| m_D \left(M^{*-1}\xi_0 + \frac{1}{2}s \right) \right|^2 + \left| m_D \left(M^{*-1}\xi_0 + \frac{1}{2}s_0 \right) \right|^2 \\ &\quad \times \sum_{\lambda \in \Gamma_{s_0}} \left| \hat{\mu}_{M,D} \left(M^{*-1}\xi_0 + \frac{1}{2}s_0 + \lambda \right) \right|^2 \\ &< \sum_{s \in \mathcal{S}} \left| m_D \left(M^{*-1}\xi_0 + \frac{1}{2}s \right) \right|^2 = 1. \end{aligned}$$

This is a contradiction, and hence Γ_s is a spectrum of $\mu_{M,D}$ for each $s \in \mathcal{S}$. □

Now we are in a position to prove Theorem 1.2, which can be seen as an extension of Lemma 3.2.

Proof of Theorem 1.2. We prove the first result Equation (1.4) by induction. It is obvious that the assertion holds for $n = 1$ by Lemma 3.2. Now we suppose it holds for $n = m - 1$, that is, Λ has a decomposition

$$\Lambda = \bigcup_{I \in \Sigma^{m-1}} \left(M^* \left(\frac{1}{2}s_{I|_1} + z_{I|_1} \right) + M^{*2} \left(\frac{1}{2}s_{I|_2} + z_{I|_2} \right) + \cdots + M^{*m-1} \left(\frac{1}{2}s_I + \Gamma_I \right) \right), \tag{3.10}$$

where $z_{0^k} = s_{0^k} = 0$, $z_{Ji} \subset \mathbb{Z}^2$, $\cup_{i \in \Sigma} s_{Ji} = \mathcal{S}$ for $J \in \Sigma^k$, $0 \leq k \leq m-2$, and every Γ_I is a spectrum of $\mu_{M,D}$ with $\Gamma_I \subset \mathbb{Z}^2$, $0 \in \Gamma_{0^{m-1}}$ and $s_{0^{m-1}} = 0$. For each Γ_I with $I \in \Sigma^{m-1}$, we choose a $z_I \in \Gamma_I$ (choose $z_{0^{m-1}} = 0$), then all $\Gamma_I - z_I$ are also spectra with $0 \in \Gamma_I - z_I$. According to Equation (3.4) and Lemma 3.2, one may conclude that $\Gamma_I - z_I$ has a decomposition

$$\Gamma_I - z_I = \bigcup_{i \in \Sigma} M^* \left(\frac{1}{2}s_{Ii} + \Gamma_{Ii} \right), \tag{3.11}$$

where $\bigcup_{i \in \Sigma} s_{Ii} = \mathcal{S}$, $\Gamma_{Ii} \subset \mathbb{Z}^2$ is a spectrum of $\mu_{M,D}$ for any $i \in \Sigma$, $s_{I0} = 0$ and $0 \in \Gamma_{I0}$. Combining Equations (3.10) and (3.11), we can get

$$\begin{aligned} \Lambda = \bigcup_{I \in \Sigma^{m-1}, i \in \Sigma} & \left(M^* \left(\frac{1}{2}s_{I|_1} + z_{I|_1} \right) + M^{*2} \left(\frac{1}{2}s_{I|_2} + z_{I|_2} \right) + \cdots + M^{*m-1} \left(\frac{1}{2}s_I + z_I \right) \right. \\ & \left. + M^{*m} \left(\frac{1}{2}s_{Ii} + \Gamma_{Ii} \right) \right). \end{aligned}$$

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This shows that Equation (1.4) holds for the case $n=m$.

Now we prove the second result. Suppose there exists n such that Λ can be decomposed as the following

$$\Lambda = \bigcup_{i_1 i_2\cdots i_n\in\Sigma^n}\left(M^*\left(\frac{1}{2}s_{i_1}+z_{i_1}\right)+M^{*2}\left(\frac{1}{2}s_{i_1i_2}+z_{i_1i_2}\right)+\cdots\right.$$
$$\left.+M^{*n}\left(\frac{1}{2}s_{i_1\cdots i_n}+\Gamma_{i_1\cdots i_n}\right)\right),$$

where $z_{0^k}=s_{0^k}=s_{0^n}=0$, $z_{Ji}\subset\mathbb{Z}^2$, $\cup_{i\in\Sigma}s_{Ji}=\mathcal{S}$ for any $J\in\Sigma^k$ $(0\le k\le n-1)$ and each $\Gamma_{i_1\cdots i_n}\subset\mathbb{Z}^2$ is a spectrum of $\mu_{M,D}$ with $0\in\Gamma_{0^n}$. One has

$$\begin{aligned}
Q_{\mu_{M,D},\Lambda}(\xi) &= \sum_{\lambda'\in\Lambda}\left|\hat{\mu}_{M,D}(\xi+\lambda')\right|^2\\
&=\sum_{s_{i_1}\in\mathcal{S}}\cdots\sum_{s_{i_1\cdots i_n}\in\mathcal{S}}\sum_{\lambda\in\Gamma_{i_1\cdots i_n}}\left|m_D\left(M^{*-1}\xi+\frac{s_{i_1}}{2}\right)\right|^2\cdots\\
&\quad\times\left|m_D\left(\xi_{i_1\cdots i_{n-1}}+\frac{s_{i_1\cdots i_n}}{2}\right)\right|^2\left|\hat{\mu}_{M,D}\left(\xi_{i_1\cdots i_{n-1}}+\frac{s_{i_1\cdots i_n}}{2}+\lambda\right)\right|^2\\
&=\sum_{s_{i_1}\in\mathcal{S}}\left|m_D\left(M^{*-1}\xi+\frac{s_{i_1}}{2}\right)\right|^2\cdots\sum_{s_{i_1\cdots i_n}\in\mathcal{S}}\left|m_D\left(\xi_{i_1\cdots i_{n-1}}+\frac{s_{i_1\cdots i_n}}{2}\right)\right|^2\\
&\quad\times\sum_{\lambda\in\Gamma_{i_1\cdots i_n}}\left|\hat{\mu}_{M,D}\left(\xi_{i_1\cdots i_{n-1}}+\frac{s_{i_1\cdots i_n}}{2}+\lambda\right)\right|^2\\
&=\sum_{s_{i_1}\in\mathcal{S}}\left|m_D\left(M^{*-1}\xi+\frac{s_{i_1}}{2}\right)\right|^2\cdots\sum_{s_{i_1\cdots i_n}\in\mathcal{S}}\left|m_D\left(\xi_{i_1\cdots i_{n-1}}+\frac{s_{i_1\cdots i_n}}{2}\right)\right|^2\\
&=\cdots\\
&=1,
\end{aligned}$$

where $\xi_{i_1\cdots i_{n-1}}=M^{*-n}\big(\xi+M^*(\frac{1}{2}s_{i_1}+z_{i_1})+\cdots+M^{*n-1}(\frac{1}{2}s_{i_1\cdots i_{n-1}}+z_{i_1\cdots i_{n-1}})\big)$. Also $0\in\Lambda$ follows from $s_{0^n}=0$, $0\in\Gamma_{0^n}$ and $s_{0^k}=z_{0^k}=0$ for all $1\le k\le n-1$. Hence, Λ is a spectrum of $\mu_{M,D}$ with $0\in\Lambda$. This completes the proof of Theorem 1.2. $\square$

4. Structure of spectral eigenmatrix of $\mu_{M,D}$

In the present section, we first study the structure of spectral eigenmatrices of $\mu_{M,D}$ and then find the conditions for a matrix to be a spectral eigenmatrix of $\mu_{M,D}$. Based on these preparations, we will complete the proofs of Theorems 1.3 and 1.5.

The following lemma is useful for us to study the structure of spectral eigenmatrices.

Lemma 4.1. *Let $R\in M_2(\mathbb{R})$ and $\mathcal{S}$ be defined by Equation (1.3). Suppose $\bigcup_{s\in\mathcal{S}}R(s+2\Gamma_s)\equiv\mathcal{S}(\bmod 2\mathbb{Z}^2)$ for some $\Gamma_s\subset\mathbb{Z}^2$, then the following statements hold.*

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(i) $R \in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1})$.
(ii) $\det(R) \in \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(iii) $2R\Gamma \equiv 0 (\bmod 2\mathbb{Z}^2)$ *if* $2R\Gamma \subset \mathbb{Z}^2$ *with* $\Gamma \subset \mathbb{Z}^2$.
(iv) *For any* $s \in \mathcal{S}$, *there exists a unique* $s' \in \mathcal{S}$ *such that* $R(s + 2\Gamma_s) \equiv s' (\bmod 2\mathbb{Z}^2)$.

Proof. (i) Write $s_0 = (0,0)^t$, $s_1 = (1,0)^t$, $s_2 = (0,1)^t$ and $s_3 = (1,1)^t$. Since $\bigcup_{s\in\mathcal{S}} R(s + 2\Gamma_s) \equiv \mathcal{S} (\bmod 2\mathbb{Z}^2)$, we have $\bigcup_{i=0}^{3} R\big(s_i + 2\Gamma_{s_i}\big) \subset \mathbb{Z}^2$. For $i = 1, 2$, choosing $\mathbf{z}_i = (k_{1i}, k_{2i})^t \in \Gamma_{s_i} \subset \mathbb{Z}^2$, there exist $\mathbf{v}_i = (l_{1i}, l_{2i})^t \in \mathbb{Z}^2$ such that $R(s_i + 2\mathbf{z}_i) = \mathbf{v}_i$. This is equivalent to

$$R\begin{bmatrix} 2k_{11}+1 & 2k_{12} \\ 2k_{21} & 2k_{22}+1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}. \tag{4.1}$$

Let $l = (1+2k_{11})(1+2k_{22}) - 4k_{21}k_{12}$. It follows from Equation (4.1), $\mathbf{v}_i \in \mathbb{Z}^2$ and $\mathbf{z}_i \in \mathbb{Z}^2$ that

$$R = \frac{1}{l}\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}\begin{bmatrix} 2k_{22}+1 & -2k_{12} \\ -2k_{21} & 2k_{11}+1 \end{bmatrix} \in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1}).$$

This proves (i).

(ii) According to (i), we can let $R = \frac{1}{l}R'$, where $l \in 2\mathbb{Z}+1$ and $R' \in M_2(\mathbb{Z})$. As $\bigcup_{s\in\mathcal{S}} R(s + 2\Gamma_s) \equiv \mathcal{S} (\bmod 2\mathbb{Z}^2)$, there must exist $\mathcal{V}_i = (a_{1i}, a_{2i})^t \in \bigcup_{s\in\mathcal{S}} (s + 2\Gamma_s)$ and $\mathcal{Z}_i = (h_{1i}, h_{2i})^t \in \mathbb{Z}^2$ such that $R\mathcal{V}_i = s_i + 2\mathcal{Z}_i$ for $i = 1, 2$. Thus, one has

$$R'\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = l\begin{bmatrix} 2h_{11}+1 & 2h_{12} \\ 2h_{21} & 2h_{22}+1 \end{bmatrix}.$$

Consequently,

$$\det(R')\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = l\det\left(\begin{bmatrix} 2h_{11}+1 & 2h_{12} \\ 2h_{21} & 2h_{22}+1 \end{bmatrix}\right) \in 2\mathbb{Z}+1.$$

This concludes that $\det(R') \in 2\mathbb{Z}+1$. Combining $R = \frac{1}{l}R'$ and $l \in 2\mathbb{Z}+1$, we derive $\det(R) \in \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$. So the assertion follows.

(iii) By (i) and (ii), we let $R = \frac{1}{l}R'$ with $l, \det(R') \in 2\mathbb{Z}+1$ and $R' \in M_2(\mathbb{Z})$. Suppose, on the contrary, that $2R\Gamma \not\equiv 0 (\bmod 2\mathbb{Z}^2)$. Then there exists a vector $(l_1, l_2)^t \in \Gamma \subset \mathbb{Z}^2$ such that

$$2R\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} h_1 + 2k_1 \\ h_2 + 2k_2 \end{pmatrix} \quad \text{for some } (h_1, h_2)^t \in \mathcal{S} \setminus \{0\} \text{ and } (k_1, k_2)^t \in \mathbb{Z}^2. \tag{4.2}$$

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Multiplying both sides of Equation (4.2) by l, we have

$$2R'\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = l\begin{pmatrix} h_1+2k_1 \\ h_2+2k_2 \end{pmatrix}.$$

It is clear that $2R'(l_1, l_2)^t \in 2\mathbb{Z}^2$ since $R' \in M_2(\mathbb{Z})$. However, the right side of the above equation belongs to $\mathbb{Z}^2\backslash 2\mathbb{Z}^2$. This leads to a contradiction, and hence $2R\Gamma \equiv 0 (\mathrm{mod}\ 2\mathbb{Z}^2)$.

(iv) Suppose there are $s_0' \in \mathcal{S}$ and two distinct $(k_1, k_2)^t, (k_3, k_4)^t \in \Gamma_{s_0'}$ such that $R(s_0' + 2(k_1, k_2)^t) = s_1' (\mathrm{mod}\ 2\mathbb{Z}^2)$ and $R(s_0' + 2(k_3, k_4)^t) = s_2' (\mathrm{mod}\ 2\mathbb{Z}^2)$ for two distinct $s_1', s_2' \in \mathcal{S}$. By a simple calculation, we have

$$2R\left(\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} - \begin{pmatrix} k_3 \\ k_4 \end{pmatrix}\right) \equiv (s_1' - s_2')(\mathrm{mod}\ 2\mathbb{Z}^2) \subset \mathcal{S}\setminus\{0\} + 2\mathbb{Z}^2 \subset \mathbb{Z}^2,$$

which contradicts (iii). This ends the proof. □

Recall that $\mathcal{F}_2^2 = \frac{1}{2}\{(\ell_1, \ell_2)^t : \ell_i \in \{0,1\}, i = 1,2\}$. For an integer matrix R with $\det(R) \in 2\mathbb{Z}+1$, the next lemma shows that $R\mathcal{F}_2^2$ is invariant in the sense of modulo $\mathbb{Z}^2$.

Lemma 4.2. *Let $R \in M_2(\mathbb{Z})$, then $R\mathcal{F}_2^2 = \mathcal{F}_2^2 (\mathrm{mod}\ \mathbb{Z}^2)$ if and only if $\det(R) \in 2\mathbb{Z}+1$.*

Proof. The sufficiency follows immediately from [36, Proposition 2.2]. For the necessity, we suppose $R\mathcal{F}_2^2 = \mathcal{F}_2^2\ (\mathrm{mod}\ \mathbb{Z}^2)$ and take $\Gamma_s = \{0\}$ in Lemma 4.1. In view of Lemma 2.5(i) and Lemma 4.1(ii), we deduce that $\det(R) \in 2\mathbb{Z}+1$. □

Now, by using Lemmas 3.2 and 4.1, we first prove the results (i) and (ii) of Theorem 1.3.

Theorem 4.3. *Let $\mu_{M,D}$ be defined by Equation (1.1), where $M \in M_2(2\mathbb{Z})$ is an expanding matrix and D is given by Equation (1.2). Suppose Λ and $R\Lambda$ are spectra of $\mu_{M,D}$ with $R \in M_2(\mathbb{R})$ and $0 \in \Lambda$, then the following statements hold.*

(i) $R = \frac{1}{l}M^*R'M^{*-1}$ for some $l \in 2\mathbb{Z}+1$ and $R' \in M_2(\mathbb{Z})$ with $\det(R') \in 2\mathbb{Z}+1$.
(ii) For any $n \geq 1$, $R_n = M^{*-n}RM^{*n}$ is a spectral eigenmatrix of $\mu_{M,D}$.

Proof. (i). By Lemma 3.2, Λ can be written as

$$\Lambda = \bigcup_{s\in\mathcal{S}}\left(\frac{1}{2}M^*s + M^*\Gamma_s\right),$$

where $\Gamma_s \subset \mathbb{Z}^2$ are also spectra of $\mu_{M,D}$ for all $s \in \mathcal{S}$. By a simple calculation, we obtain $RM^* = M^*R_1$ and

$$R\Lambda = \bigcup_{s\in\mathcal{S}}\left(\frac{1}{2}M^*R_1 s + M^*R_1\Gamma_s\right) = \bigcup_{s\in\mathcal{S}} M^*\left(\frac{1}{2}R_1 s + R_1\Gamma_s\right). \tag{4.3}$$

Similarly, by Lemma 3.2, the spectrum $R\Lambda$ can also be expressed as follows.

$$R\Lambda = \bigcup M^* \left(\frac{1}{2}s + \Gamma'_s \right) \tag{4.4}$$

where $\Gamma'_s \subset \mathbb{Z}^2$ are also spectra of $\mu_{M,D}$ for all $s \in \mathcal{S}$. It follows from Equations (4.3) and (4.4) that

$$R\Lambda = \bigcup_{s\in\mathcal{S}} M^* \left(\frac{1}{2}R_1 s + R_1\Gamma_s \right) = \bigcup_{s\in\mathcal{S}} M^* \left(\frac{1}{2}s + \Gamma'_s \right). \tag{4.5}$$

This implies that $\bigcup_{s\in\mathcal{S}} R_1\big(s+2\Gamma_s\big) = \bigcup_{s\in\mathcal{S}}\big(s+2\Gamma'_s\big)$. Using Lemma 4.1, one may easily get that $R_1 \in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1})$ and $\det(R_1) = \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$. So we can further find $l \in 2\mathbb{Z}+1$ such that $lR_1 \in M_2(\mathbb{Z})$ and $\det(lR_1) \in 2\mathbb{Z}+1$. By the definition of R_1, we have $R = \frac{1}{l}M^*R'M^{*-1}$ with $R' = lR_1$.

(*ii*). According to Equation (4.5) and Lemma 4.1(*iii*), we get $R_1\Gamma_0 = \Gamma'_0$, where Γ_0 and Γ'_0 are spectra for $\mu_{M,D}$ with $0 \in \Gamma'_0$. This shows that $R_1 = M^{*-1}RM^*$ is a spectral eigenmatrix of $\mu_{M,D}$. Similarly, we can prove that $R_2 = M^{*-1}R_1M^* = M^{*-2}RM^{*2}$ is a spectral eigenmatrix of $\mu_{M,D}$. Hence, by repeating this process many times, $R_n = M^{*-n}RM^{*n}$ is a spectral eigenmatrix of $\mu_{M,D}$ for any $n \geq 0$.

The proof of Theorem 4.3 is completed. □

Having established the above preparation, now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The results (*i*) and (*ii*) can be obtained directly from Theorem 4.3. We only need to prove (*iii*). Suppose Λ and $R\Lambda$ are spectra of $\mu_{M,D}$ with $0 \in \Lambda$. We first prove R_{-1} is also a spectral eigenmatrix of $\mu_{M,D}$. According to $M = \mathrm{diag}[2p, 2q]$ and Theorem 4.3, we can let

$$R = \frac{1}{l}M^*R'M^{*-1} = \frac{1}{l}\begin{bmatrix} a & b(\frac{p}{q}) \\ c(\frac{q}{p}) & d \end{bmatrix},$$

where $l \in 2\mathbb{Z}+1$ and $R' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$ with $\det(R') \in 2\mathbb{Z}+1$. As $\frac{pq}{\gcd(p^2,q^2)} \in 2\mathbb{Z}+1$, we can let $\frac{q}{p} = \frac{\alpha_1}{\beta_1}$ with $\alpha_1, \beta_1 \in 2\mathbb{Z}+1$ and $\gcd(\alpha_1,\beta_1) = 1$. Define

$$\Lambda' = \frac{1}{2}M^* \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \bigcup \begin{pmatrix} l\beta_1 \\ 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 \\ l\alpha_1 \end{pmatrix} \bigcup \begin{pmatrix} l\beta_1 \\ l\alpha_1 \end{pmatrix} + 2\Lambda \right).$$

By Lemma 4.2, we have $\Lambda' = \bigcup_{s\in\mathcal{S}} \left(\frac{1}{2}M^*s + M^*(v_s+\Lambda)\right)$ for some $v_s \in \mathbb{Z}^2$. Then Theorem 1.2 shows that Λ' is a spectrum of $\mu_{M,D}$ and

$$R_{-1}\Lambda' = M^*RM^{*-1}\Lambda' = \frac{1}{2}M^* \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \bigcup \begin{pmatrix} a\beta_1 \\ c\alpha_1 \end{pmatrix} \bigcup \begin{pmatrix} b\beta_1 \\ d\alpha_1 \end{pmatrix} \bigcup \begin{pmatrix} a\beta_1 + b\beta_1 \\ c\alpha_1 + d\alpha_1 \end{pmatrix} + 2R\Lambda \right)$$

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$$= \frac{1}{2} M^* \left(\begin{bmatrix} a\beta_1 & b\beta_1 \\ c\alpha_1 & d\alpha_1 \end{bmatrix} \mathcal{S} + 2R\Lambda \right).$$

As $\alpha_1, \beta_1, ad - bc \in 2\mathbb{Z}+1$, it follows that $\alpha_1\beta_1(ad-bc) \in 2\mathbb{Z}+1$. With Lemma 4.2, we obtain

$$\begin{bmatrix} a\beta_1 & b\beta_1 \\ c\alpha_1 & d\alpha_1 \end{bmatrix} \mathcal{S} = \bigcup_{s\in\mathcal{S}} (s + 2z_s)$$

for some $z_s \in \mathbb{Z}^2$. Therefore, $R_{-1}\Lambda'$ can be rewritten as

$$R_{-1}\Lambda' = \bigcup_{s\in\mathcal{S}} \left(\frac{1}{2} M^* s + M^*(z_s + R\Lambda) \right). \tag{4.6}$$

This together with Theorem 1.2 shows that $R_{-1}\Lambda'$ is a spectrum of $\mu_{M,D}$ and R_{-1} is a spectral eigenmatrix of $\mu_{M,D}$. So we know that R_{-1} is a spectral eigenmatrix of $\mu_{M,D}$ if R is a spectral eigenmatrix. Thus, R_{-2} is also a spectral eigenmatrix of $\mu_{M,D}$. By induction, one may derive that R_{-k} is a spectral eigenmatrix for any $k \in \mathbb{N}$. This completes the proof of Theorem 1.3. □

At the end of this section, we will prove Theorem 1.5.

Proof of Theorem 1.5. Since R is a spectral eigenmatrix, it follows from Theorem 4.3(*ii*) that $R_n = M^{*-n} R M^{*n}$ is also a spectral eigenmatrix for any $n \geq 0$. By Theorem 4.3(*i*), we get $R_{n+1} \in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1})$ for any $n \geq 0$. Let $R_1 = \frac{1}{l}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $l, ad - bc \in 2\mathbb{Z}+1$. Then, for any $n \geq 0$, one has

$$R_{n+1} = M^{*-n} R_1 M^{*n} = \frac{1}{l} \begin{bmatrix} a & b(\frac{q}{p})^n \\ c(\frac{p}{q})^n & d \end{bmatrix} \in M_2\left(\frac{\mathbb{Z}}{2\mathbb{Z}+1} \right). \tag{4.7}$$

(*i*) If $\frac{q}{\gcd(p,q)} \in 2\mathbb{Z}$, then Equation (4.7) implies that $2^n | c$. According to the arbitrariness of n, we have $c = 0$.

(*ii*) If $\frac{p}{\gcd(p,q)} \in 2\mathbb{Z}$, similar to (*i*), we can derive that $b = 0$ by Equation (4.7) and the arbitrariness of n.

Hence, we complete the proof of Theorem 1.5. □

5. Exchangeable spectral eigenmatrix

In this section, we consider a special kind of spectral eigenmatrix of $\mu_{M,D}$, which can be exchanged with matrix M, and then complete the proof of Theorem 1.4. We first consider the sufficiency of Theorem 1.4 whose main idea comes from [22]. We extend it to the two-dimensional case.

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Throughout this section, we let $M = \mathrm{diag}[2p, 2q]$ with integers $|p|, |q| > 1$, $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ and the measure $\mu_{M,D}$ is defined by Equation (1.1).

Lemma 5.1. *For any infinite word $w = w_0 w_1 \cdots \in \{-1, 1\}^{\mathbb{N}}$, the set*

$$\Lambda_w(M, M\mathcal{F}_2^2) = \left\{ \sum_{j=0}^{m} w_j M^j M s_j : s_j \in \mathcal{F}_2^2, m \in \mathbb{N} \right\} \tag{5.1}$$

is a spectrum of the measure $\mu_{M,D}$.

Proof. It is easy to know that $(M, D, M\mathcal{F}_2^2)$ and $(M, D, -M\mathcal{F}_2^2)$ are Hadamard triples. For any integer $m \geq 1, |b| > 1$, $c_i \in \{0, 1\}$, $1 \leq i \leq m$ and infinite word $w = w_0 w_1 \cdots \in \{-1, 1\}^{\mathbb{N}}$,

$$\left|(2b)^{-(m+1)} w_0 b c_1 + (2b)^{-m} w_1 b c_2 + \cdots + (2b)^{-1} w_m b c_m\right| \leq \sum_{k=1}^{\infty} \frac{|b|}{(2|b|)^k} \leq \frac{2}{3}.$$

Then the set

$$M^{-m} w_0 (M\mathcal{F}_2^2) + M^{-m+1} w_1 (M\mathcal{F}_2^2) + \cdots + M^{-1} w_m (M\mathcal{F}_2^2) \subset \left[-\frac{2}{3}, \frac{2}{3}\right] \times \left[-\frac{2}{3}, \frac{2}{3}\right]$$

is separated from the set $\mathcal{Z}(\hat{\mu}_{M,D})$ by a distance $\delta \geq \frac{1}{3}$, uniformly in m, for all $m \geq 1$. By Lemma 2.4, we know $\Lambda_w(M, M\mathcal{F}_2^2)$ is a spectrum of $\mu_{M,D}$. $\square$

Lemma 5.2. *Let $R \in M_2(\mathbb{Z})$ be a non-singular matrix that can be exchanged with M. Then each element $\boldsymbol{x} \in T(M, \pm MR\mathcal{F}_2^2)$ has a unique expansion in base M.*

Proof. Assume that there are two distinct sequence with infinite words $\{\boldsymbol{c}_j = (x_{j,1}, x_{j,2})^t\}_{j=1}^{\infty}$ and $\{\boldsymbol{c}'_j = (x'_{j,1}, x'_{j,2})^t\}_{j=1}^{\infty}$ in $\pm\mathcal{F}_2^2$ such that

$$\sum_{j=0}^{\infty} M^{-j} R \boldsymbol{c}_j = \sum_{j=0}^{\infty} M^{-j} R \boldsymbol{c}'_j. \tag{5.2}$$

Since $M = \mathrm{diag}[2p, 2q]$ and $MR = RM$, a simple calculation gives $M^{-j}R = RM^{-j}$ for each $j \in \mathbb{N}$. Let t be the smallest integer such that $\boldsymbol{c}_t \neq \boldsymbol{c}'_t$, without loss of generality we assume $x_{t,1} \neq x'_{t,1}$. From the first coordinate of Equation (5.2), we have

$$x_{t,1} - x'_{t,1} = \sum_{j=1}^{\infty} \frac{x'_{t+j,1} - x_{t+j,1}}{(2p)^j}. \tag{5.3}$$

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Observing Equation (5.3), the left hand must be chosen from $\{\pm\frac{1}{2}, \pm 1\}$, but

$$\left|\sum_{j=1}^{\infty} \frac{x'_{t+j,1} - x_{t+j,1}}{(2p)^j}\right| \leq \sum_{j=1}^{\infty} \frac{1}{(2|p|)^j} = \frac{1}{2|p|-1} \leq \frac{1}{3} < \frac{1}{2},$$

which is impossible. This ends the proof. □

Lemma 5.3. *Let $R \in M_2(\mathbb{Z})$ be a matrix with $\det(R) \in 2\mathbb{Z}+1$. Then for any element $\boldsymbol{x} \in T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D})$, the expansion of $\boldsymbol{x}$ cannot be finite.*

Proof. Suppose, on the contrary, that there exists a $\boldsymbol{x} \in T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D})$ whose expansion is finite, that is, $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = \sum_{j=0}^{n} M^{-j} R\boldsymbol{c}_j \tag{5.4}$$

for some $\boldsymbol{c}_j \in \pm\mathcal{F}_2^2$, $0 \leq j \leq n-1$, and $\boldsymbol{c}_n \in \pm\mathring{\mathcal{F}}_2^2$. Combining this with Equation (2.4), there exist $\boldsymbol{\nu} \in \mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2$ and positive integer k such that

$$M^k \boldsymbol{\nu} = \sum_{j=0}^{n} M^{-j} R\boldsymbol{c}_j.$$

Consequently,

$$M^{k+n}\boldsymbol{\nu} = R\boldsymbol{c}_n + MR\boldsymbol{c}_{n-1} + \cdots + M^n R\boldsymbol{c}_0.$$

Since $\det(R) \in 2\mathbb{Z}+1$, it follows from Lemma 4.2 that $R\boldsymbol{c}_n \in \mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2$. Then the right hand of the above equation belongs to $\mathring{\mathcal{F}}_2^2 + \mathbb{Z}^2$, but $M = \operatorname{diag}[2p, 2q]$ implies the left hand $M^{k+n}\boldsymbol{\nu} \in \mathbb{Z}^2$, which is a contradiction. Hence, any $\boldsymbol{x} \in T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D})$ has an infinite expansion. □

Fix a matrix $R \in M_2(\mathbb{Z})$ with $\det(R) \in 2\mathbb{Z}+1$. For any positive integers K and N, we define

$$S_{K,N}(R) = MR\mathcal{F}_2^2 + M^2R\mathcal{F}_2^2 + \cdots + M^K R\mathcal{F}_2^2 - \left(M^{K+1}R\mathcal{F}_2^2 + \cdots + M^{K+N}R\mathcal{F}_2^2\right) \tag{5.5}$$

and

$$D_{K,N} = D + MD + \cdots + M^{(K+N-1)}D. \tag{5.6}$$

Applying Lemma 2.3, one may get that $(M^{(K+N)}, D_{K,N}, S_{K,N}(R))$ form a Hadamard triple and the measure $\mu_{M,D} = \mu_{M^{K+N}, D_{K,N}}$. Based on these, we prove the following.

Theorem 5.4. *There exist two positive integers K_0 and N_0 such that $T(M^{K_0+N_0}, S_{K_0,N_0}(R))$ is separated from the set $\mathcal{Z}(\hat{\mu}_{M,D})$ by a distance $\delta > 0$. Moreover, the set $\Lambda\big(M^{K_0+N_0}, S_{K_0,N_0}(R)\big)$ is a spectrum of the measure $\mu_{M,D}$.*

Proof. First, we prove that there exist two positive integers K_0 and N_0 such that

$$T(M^{K_0+N_0}, S_{K_0,N_0}(R)) \cap \mathcal{Z}(\hat{\mu}_{M,D}) = \emptyset.$$

(a) If $T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D}) = \emptyset$, we take $K_0 = N_0 = 1$. Then it is easy to see that $T(M^{K_0+N_0}, S_{K_0,N_0}(R)) \cap \mathcal{Z}(\hat{\mu}_{M,D}) = \emptyset$.

(b) If $T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D}) \neq \emptyset$. Let $\mathcal{A}$ be the set of all point in $T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D})$, which has infinitely many positive terms in its expansion. By Lemma 5.3, we know that the expansion of the point in $\mathcal{B} := \big(T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D})\big) \setminus \mathcal{A}$ has infinitely many negative terms. Since the zero set $\mathcal{Z}(\hat{\mu}_{M,D})$ is uniformly discrete and $T(M, \pm MR\mathcal{F}_2^2)$ is a compact set, we know $T(M, \pm MR\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D}) \neq \emptyset$ contains at most finitely many points and $\mathcal{A}, \mathcal{B}$ are also finite sets. So, for every point $\boldsymbol{x} \in \mathcal{A}$, we can find an integer $N_{\boldsymbol{x}}$ such that there exists a positive term in the expansion of $\boldsymbol{x}$, with its index no more than $N_{\boldsymbol{x}}$. We let $N_0 = \max\{N_{\boldsymbol{x}} : x \in \mathcal{A}\}$ and $N_0 = 1$ when $\mathcal{A} = \emptyset$. We can also find a positive number $K_{\boldsymbol{x}}$ such that there exists a negative term in the expansion of $\boldsymbol{x} \in \mathcal{B}$ with its index in $[N_0 + 1, N_0 + K_{\boldsymbol{x}}]$. We let $K_0 = \max\{K_{\boldsymbol{x}} : x \in \mathcal{B}\}$ and $K_0 = 1$ when $\mathcal{B} = \emptyset$. By Lemma 5.2, one has $T(M^{K_0+N_0}, S_{K_0,N_0}(R)) \cap \mathcal{Z}(\hat{\mu}_{M,D}) = \emptyset$.

Second, note that $T(M^{K_0+N_0}, S_{K_0,N_0}(R))$ is a compact set and $\mathcal{Z}(\hat{\mu}_{M,D})$ is a uniform discrete set, then $T(M^{K_0+N_0}, S_{K_0,N_0}(R)) \cap \mathcal{Z}(\hat{\mu}_{M,D}) = \emptyset$ implies that $T(M^{K_0+N_0}, S_{K_0,N_0}(R))$ is separated from the sets $\mathcal{Z}(\hat{\mu}_{M,D})$ by a distance $\delta > 0$.

$\Lambda\big(M^{K_0+N_0}, S_{K_0,N_0}(R)\big)$ is a spectrum of the measure $\mu_{M,D}$ following from Lemma 2.4. □

More generally, one can similarly get the general version of Theorem 5.4.

Theorem 5.5. *Let $\{\mathcal{R}_k\}_{k=1}^n$ be a sequence of integer matrices which can be exchanged with M. Suppose $\det(\mathcal{R}_k) \in 2\mathbb{Z}+1$ for all $1 \le k \le n$, then there exists a spectrum Λ of $\mu_{M,D}$ such that $\mathcal{R}_1\Lambda, \mathcal{R}_2\Lambda, \ldots, \mathcal{R}_n\Lambda$ are spectra of $\mu_{M,D}$.*

Proof. Let $\mathcal{C} = \bigcup_{k=1}^n \big(T(M, \pm M\mathcal{R}_k\mathcal{F}_2^2) \cap \mathcal{Z}(\hat{\mu}_{M,D})\big)$, and let $\mathcal{A}$ be the set of all point in $\mathcal{C}$ which has infinitely many positive terms in its expansion. Then the expansion of the point in set $\mathcal{B} = \mathcal{C} \setminus \mathcal{A}$ has infinitely many negative terms by Lemma 5.3. Using the same argument as in the proof of Theorem 5.4, we can find two positive integers K_0 and N_0 such that all $\Lambda\big(M^{K_0+N_0}, S_{K_0,N_0}(\mathcal{R}_k)\big)$ are spectra of $\mu_{M,D}$, and $\Lambda\big(M^{K_0+N_0}, S_{K_0,N_0}(E)\big)$ with identity matrix E is also a spectrum of $\mu_{M,D}$ by Lemma 5.1.

Let $\Lambda' = \Lambda\big(M^{K_0+N_0}, S_{K_0,N_0}(E)\big)$. Since $\mathcal{R}_k M = M\mathcal{R}_k$, the above argument shows that Λ' and $\mathcal{R}_k\Lambda' = \Lambda\big(M^{K_0+N_0}, S_{K_0,N_0}(\mathcal{R}_k)\big)$ are spectra of $\mu_{M,D}$ for all $1 \le k \le n$. □

In fact, Theorem 5.5 tells us that if integer matrices R_1, R_2 can be exchanged with M and $\det(R_1), \det(R_2) \in 2\mathbb{Z}+1$, then $R_1R_2^{-1}$ is a spectral eigenmatrix. This is the key to proving the sufficiency of Theorem 1.4. Next, we will consider the necessity of spectral eigenmatrix of $\mu_{M,D}$.

Let Λ and $R\Lambda$ be spectra of the measure $\mu_{M,D}$. By Lemma 3.2, we have

$$\Lambda=\bigcup_{s\in\mathcal{S}}\left(\frac{1}{2}M^{*}s+M^{*}\Gamma_{s}\right)\quad\text{and}\quad R\Lambda=\bigcup_{s\in\mathcal{S}}M^{*}\left(\frac{1}{2}s+\Gamma_{s}'\right),\tag{5.7}$$

with $\Gamma_s,\Gamma_s'\subset\mathbb{Z}^2$. Since $RM=MR$,

$$R\Lambda=\bigcup_{s\in\mathcal{S}}\left(\frac{1}{2}M^{*}Rs+M^{*}R\Gamma_{s}\right)=\bigcup_{s\in\mathcal{S}}M^{*}\left(\frac{1}{2}Rs+R\Gamma_{s}\right).\tag{5.8}$$

It follows from Equations (5.7) and (5.8) that

$$R\Lambda=\bigcup_{s\in\mathcal{S}}M^{*}\left(\frac{1}{2}Rs+R\Gamma_{s}\right)=\bigcup_{s\in\mathcal{S}}M^{*}\left(\frac{1}{2}s+\Gamma_{s}'\right).$$

This implies that

$$\bigcup_{s\in\mathcal{S}}R(s+2\Gamma_{s})=\bigcup_{s\in\mathcal{S}}(s+2\Gamma_{s}').\tag{5.9}$$

Then the following lemma follows immediately from Lemma 4.1.

Lemma 5.6. *Suppose $R\in M_2(\mathbb{R})$ is a spectral eigenmatrix of $\mu_{M,D}$ with $RM=MR$, then $R\in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1})$ and $\det(R)\in\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.*

We have all ingredients for the proof of Theorem 1.4.

Proof of Theorem 1.4. Sufficiency. Since $R\in M_2(\frac{\mathbb{Z}}{2\mathbb{Z}+1})$ and $\det(R)\in\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$, we can assume $R=\frac{1}{l}\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ with $a,b,c,d\in\mathbb{Z}$ and $l,ad-bc\in 2\mathbb{Z}+1$. Let $R_1=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $R_2=lE$, then $\det(R_1),\det(R_2)\in 2\mathbb{Z}+1$. By Theorem 5.5, there exists a discrete set Λ such that $\Lambda, R_1\Lambda$ and $R_2\Lambda$ are spectra of $\mu_{M,D}$. Let $\Lambda'=R_2\Lambda$. Then Λ' and $R\Lambda'=R_1\Lambda$ are spectra of $\mu_{M,D}$, which shows that R is a spectral eigenmatrix of $\mu_{M,D}$.

Necessity. The necessity of the theorem can be directly derived from Lemma 5.6. □

6. Concluding remarks

In the present section, we will give some remarks and open questions related to our main results.

For the diagonal matrix $M=\mathrm{diag}[2p,2q]$ with $|p|,|q|>1$, Theorem 1.4 characterizes the spectral eigenmatrix of $\mu_{M,D}$ that can be exchanged with M. In the case $\frac{pq}{\gcd(p^2,q^2)}\in 2\mathbb{Z}$, Theorem 1.5 tells us that the spectral eigenmatrix of $\mu_{M,D}$ must be a triangular matrix. Moreover, if we can further prove that the spectral eigenmatrix in Theorem 1.4 is a diagonal matrix, we can completely characterize the spectral eigenmatrix of $\mu_{M,D}$. It is natural for us to consider the following question.

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Q1: Let $M = \mathrm{diag}[2p, 2q] \in M_2(2\mathbb{Z})$ with two different integers $|p|, |q| > 1$ and $\frac{pq}{\gcd(p^2,q^2)} \in 2\mathbb{Z}$, D be given by Equation (1.2). If $R \in M_2(\mathbb{R})$ is a spectral eigenmatrix of $\mu_{M,D}$, whether R is a diagonal matrix? More general, if $|p| \neq |q|$, whether R is also a diagonal matrix?

Let $M = \mathrm{diag}[2p, 2q]$ with $|p|, |q| > 1$ and $\gcd(p, q) = 1$, and let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$, then $R_k = M^{*-k} R M^{*k} = \begin{bmatrix} a & b(\frac{q}{p})^k \\ c(\frac{p}{q})^k & d \end{bmatrix}$. If $0 \in \Lambda$ and $R\Lambda$ are spectra of $\mu_{M,D}$, then Equation (4.5) implies that

$$2M^{*-1}R\Lambda = \bigcup_{i\in\Sigma} R_1\big(s_i + 2\Gamma_i\big) = \bigcup_{i\in\Sigma} \big(s_i + 2\Gamma_i'\big) \subset \mathbb{Z}^2, \tag{6.1}$$

where $\cup_{i\in\Sigma} s_i = \mathcal{S}$, Γ_i, Γ_i' are spectra of $\mu_{M,D}$ for all $i \in \Sigma$. Let $\Lambda = \frac{1}{2}M^*\{(\lambda_{1n}, \lambda_{2n})^t\}_{n=1}^\infty$, by Equation (6.1), we have

$$2M^{*-1}R\Lambda = R_1\{(\lambda_{1n}, \lambda_{2n})^t\}_{n=1}^\infty = \left\{ \begin{bmatrix} a\lambda_{1n} + \frac{bq}{p}\lambda_{2n} \\ \frac{cp}{q}\lambda_{1n} + d\lambda_{2n} \end{bmatrix} \right\}_{n=1}^\infty \subset \mathbb{Z}^2.$$

This shows that $q \mid c\lambda_{1n}$ and $p \mid b\lambda_{2n}$ for all n.

Choose $z_i \in \mathbb{Z}^2$ such that $0 \in (\Gamma_i - z_i)$. From Lemmas 3.2 and 4.1 and Equation (6.1), it is easy to get $(\Gamma_i - z_i)$ and $R_1(\Gamma_i - z_i)$ are spectra of $\mu_{M,D}$. Using Equation (4.5) again, we have

$$2M^{*-1}R_1(\Gamma_i - z_i) = \bigcup_{j\in\Sigma} R_2\big(s_j + 2\Gamma_{ij}\big) = \bigcup_{j\in\Sigma} \big(s_j + 2\Gamma_{ij}'\big) \subset \mathbb{Z}^2. \tag{6.2}$$

Similarly, if we let $(\Gamma_i - z_i) = \frac{1}{2}M^*\{(\gamma_{1j}, \gamma_{2j})^t\}_{j=1}^\infty$, then Equation (6.2) implies $q^2 \mid c\gamma_{1j}$ and $p^2 \mid b\gamma_{2j}$ for all j. Continuing this process, we can see that the spectrum Λ increases rapidly if $bc \neq 0$. This is quite different from the spectrum that appeared in the previous references, although we cannot prove that $R\Lambda$ is not a spectrum of $\mu_{M,D}$.

Based on the above analysis, the following conjecture may be a reasonable conjecture to this end.

Conjecture. *Let $M = \mathrm{diag}[2p, 2q] \in M_2(2\mathbb{Z})$ with $|p| \neq |q| > 1$, D be given by Equation (1.2). Then $R \in M_2(\mathbb{R})$ is a spectral eigenmatrix of $\mu_{M,D}$ if and only if $R = \frac{1}{t}\mathrm{diag}[a, d]$ for some integers $t, a, d \in 2\mathbb{Z} + 1$.*

It is worth noting that the matrix $M = \mathrm{diag}[2p, 2q]$ given in Theorem 1.4 satisfies $|p|, |q| > 1$, which is only used in the proof of the sufficiency. If $|p| = |q| = 1$, then D is a complete set of coset representatives of $\mathbb{Z}^n/M\mathbb{Z}^n$. By the results of Lagarias and Wang [31], the attractor $T(M, D)$ is a self-affine tile and $\mu_{M,D}$ is the normalized Lebesgue measure supported on $T(M, D)$. It is interesting for us to answer the following question:

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Q2: Let $M \in M_2(2\mathbb{Z})$ with $\det(M) = 4$ and D be given by Equation (1.2). What are the sufficient and necessary conditions for $R \in M_2(\mathbb{R})$ to be a spectral eigenmatrix of $\mu_{M,D}$?

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