Polynomial mean complexity and logarithmic Sarnak conjecture

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Abstract. In this paper, we reduce the logarithmic Sarnak conjecture to the {0, 1}-symbolic systems with polynomial mean complexity. By showing that the logarithmic Sarnak conjecture holds for any topologically dynamical system with sublinear complexity, we provide a variant of the 1-Fourier uniformity conjecture, where the frequencies are restricted to any subset of [0, 1] with packing dimension less than one.

Key words: Möbius function, topological dynamics, polynomial complexity, packing dimension
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1. *Introduction*

In this paper, a *topologically dynamical system* (t.d.s. for short) is a pair (X, T), where X is a compact metric space endowed with a metric d and $T : X \to X$ is a homeomorphism. Denote by $\mathcal{M}(X, T)$ the set of all T-invariant Borel probability measures on X, which is a non-empty convex and compact metric space with respect to the weak* topology. We say a sequence ξ is *realized* in (X, T) if there is an $f \in C(X)$ and an $x \in X$ such that $\xi(n) = f(T^n x)$ for any $n \in \mathbb{N}$. A sequence ξ is said to be *deterministic* if it is realized in a t.d.s. with zero topological entropy. The Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined by $\mu(1) = 1$ and

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In this paper, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{E}$ (respectively $\mathbb{E}^{\log}$) stands for a finite average (respectively a finite logarithmical average), that is,

$$\mathbb{E}_{n \leq N} A_n = \frac{1}{N} \sum_{n=1}^{N} A_n \quad \text{and} \quad \mathbb{E}_{n \leq N}^{\log} A_n = \frac{1}{\sum_{n=1}^{N} (1/n)} \sum_{n=1}^{N} \frac{A_n}{n}.$$

Here is the well-known conjecture by Sarnak [19].

Sarnak Conjecture. The Möbius function μ is linearly asymptotically disjoint from any deterministic sequence ξ. That is,

$$\lim_{N \to \infty} \mathbb{E}_{n \leq N} \mu(n) \xi(n) = 0. \tag{1.2}$$

The conjecture in the case when X is finite is equivalent to the prime number theorem in arithmetic progressions. The conjecture in the case when T is a rotation on the circle is equivalent to Davenport's theorem [2]. The conjecture in many other special cases has been established recently (see [8, 6, 12, 13] and references therein).

Tao introduced and investigated the following logarithmic version of the Sarnak conjecture [21, 22] (see also [7, 18, 23, 24]).

Logarithmic Sarnak Conjecture. For any topological dynamical system (X, T) with zero entropy, any continuous function $f : X \to \mathbb{C}$, and any point x in X,

$$\lim_{N \to \infty} \mathbb{E}_{n \leq N}^{\log} \mu(n) f(n) = 0. \tag{1.3}$$

Now we let (X, T) be a t.d.s. with a metric d. For any $n \in \mathbb{N}$, we consider the so-called *mean metric* induced by d:

$$\overline{d}_n(x, y) = \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y)$$

for any $x, y \in X$. For $\epsilon > 0$ and a subset K of X, we let

$$S_n(d, T, K, \epsilon) = \min \left\{ m \in \mathbb{N} : \text{there exists } x_1, x_2, \ldots, x_m \text{ such that } K \subset \bigcup_{i=1}^{m} B_{\overline{d}_n}(x_i, \epsilon) \right\},$$

where $B_{\overline{d}_n}(x, \epsilon) := \{y \in X : \overline{d}_n(x, y) < \epsilon\}$ for any $x \in X$. We say (X, T) has *polynomial mean complexity* if there exists a constant $k > 0$ such that $\liminf_{n \to +\infty} (S_n(d, T, X, \epsilon)/n^k) = 0$ for all $\epsilon > 0$. The following is our main result.

THEOREM 1.1. *The following statements are equivalent.*
(1) *The logarithmic Sarnak conjecture holds.*
(2) *The logarithmic Sarnak conjecture holds for any t.d.s. with polynomial mean complexity.*
(3) *The logarithmic Sarnak conjecture holds for any {0, 1}-symbolic system with polynomial mean complexity.*

We now briefly describe the main ingredients in the proof of Theorem 1.1. It is clear that statement (1) implies statement (2) which in turn implies statement (3). So it remains to prove statement (2) implies statement (1) and statement (3) implies statement (2). To show statement (2) implies statement (1), we use Tao's result as a starting point, which states

that the logarithmic Sarnak conjecture is equivalent to a conjecture involving the limit of averages on nilmanifolds, see Conjecture 2.1. By assuming that Conjecture 2.1 fails, we are able then to construct a system with polynomial mean complexity which does not satisfy the logarithmic Sarnak conjecture, and hence prove that statement (2) implies statement (1). To construct the system, we need to work on nilsystems and figure out the complexity of polynomial sequences, see Proposition 2.5. Precisely, we will show that for a given $\epsilon > 0$, for any $n \in \mathbb{N}$, the minimal number of ϵ-dense subsets of strings of lengths n of the set of all polynomial sequences on G/Γ is bounded by a polynomial which is only dependent on ϵ and G/Γ, where G/Γ is an s-step nilmanifold. With the help of this proposition, we finish the construction and thus show that statement (2) implies statement (1). To show statement (3) implies statement (2), we study a t.d.s. with the small boundary property which was introduced by Lindenstrauss when studying mean dimension. Proposition 2.10 plays a key role for the proof, which states that for a t.d.s. (X, T) with polynomial mean complexity and a subset U with small boundary, each $x \in X$ is associated with a point in the shift space such that the complexity of the closure of the associated points is less than or equal to that of (X, T). The result of Lindenstrauss and Weiss guarantees that if (X, T) has zero entropy, then the product of X with any irrational rotation on the circle has the small boundary property. By using Proposition 2.10 and some simple argument, we finish the proof that statement (3) implies statement (2), and hence the proof of Theorem 1.1.

While Theorem 1.1 does not provide a proof of the logarithmic Sarnak conjecture directly, it does indicate that a t.d.s. with polynomial mean complexity is important for the proof of the conjecture. So, it will be useful to understand the structure of a subshift with polynomial mean complexity. We remark that we do not know if the polynomial mean complexity for a subshift can be replaced by the polynomial block-complexity in Theorem 1.1, which is extensively studied in the literature.

For a t.d.s. (X, T) with a metric d, $\epsilon > 0$, and a $\rho \in \mathcal{M}(X, T)$, we let

$$S_n(d, T, \rho, \epsilon)$$

$$= \min\left\{ m \in \mathbb{N} : \text{there exists } x_1, x_2, \ldots, x_m \text{ s.t. } \rho\left(\bigcup_{i=1}^{m} B_{\overline{d}_n}(x_i, \epsilon)\right) > 1 - \epsilon \right\}.$$

It is clear that $S_n(d, T, \rho, \epsilon) \le S_n(d, T, X, \epsilon)$ for any $\rho \in \mathcal{M}(X, T)$ and $\epsilon > 0$. We say a $\rho \in \mathcal{M}(X, T)$ has *sub-linear mean measure complexity* if for any $\epsilon > 0$,

$$\liminf_{n \to +\infty} \frac{S_n(d, T, \rho, \epsilon)}{n} = 0. \tag{1.4}$$

We say (X, T) has *sub-linear mean measure complexity* if equation (1.4) holds for any $\rho \in \mathcal{M}(X, T)$. We emphasize that the sub-linear mean measure complexity is an invariant in the measure-theoretic category. One can refer to [11, Proposition 2.2] for details.

By using the fact that the two-term logarithmic Chowla conjecture holds [21], that is,

$$\lim_{N \to \infty} \frac{1}{\ln N} \sum_{n=1}^{N} \frac{\mu(n + h_1)\mu(n + h_2)}{n} = 0 \tag{1.5}$$

for any $0 \leq h_1 < h_2 \in \mathbb{N}$, and by using the method of the proof of Theorem 1.1 in [**11**], we have the following theorem.

THEOREM 1.2. *The logarithmic Sarnak conjecture holds for any t.d.s. with sub-linear mean measure complexity. Consequently, the conjecture holds for any t.d.s. with sub-linear mean complexity.*

We remark that, at this moment, we are not able to show that the logarithmic Sarnak conjecture holds for any t.d.s. with linear mean (measure) complexity. We also remark that if for any $k \in \mathbb{N}$ the $2k$-term logarithmic Chowla conjecture holds, that is,

$$\lim_{N \to \infty} \frac{1}{\ln N} \sum_{n=1}^{N} \frac{\mu(n + h_1)\mu(n + h_2) \ldots \mu(n + h_{2k})}{n} = 0 \qquad (1.6)$$

for any non-negative integer $0 \leq h_1 \leq h_2 \leq \cdots \leq h_{2k}$ with an odd number $j \in \{1, 2, \ldots, 2k\}$ such that $h_j < h_{j+1}$, then the logarithmic Sarnak conjecture holds for any t.d.s. with sub-polynomial (leading term cn^k) mean measure complexity by using the method of Theorem 1.2. Thus, by Theorem 1.1, we know that the logarithmic Sarnak conjecture holds if the logarithmic Chowla conjecture holds. In fact, the two conjectures are equivalent [**21**].

As an application of Theorem 1.2, one has the following result.

THEOREM 1.3. *Let C be a non-empty compact subset of* $[0, 1]$ *with packing dimension* < 1. *Then*

$$\lim_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} \sup_{\alpha \in C} |\mathbb{E}_{h \leq H} \mu(n + h)e(h\alpha)| = 0, \qquad (1.7)$$

where $e(t) := e^{2\pi i t}$ *for any* $t \in \mathbb{R}$.

We remark that in [**18**, Theorem 1.13], McNamara proved that equation (1.7) holds for a non-empty compact subset C of $[0, 1]$ with upper box dimension < 1. So Theorem 1.3 strengthens the result in [**18**].

We say a t.d.s. (X, T) has *sub-polynomial mean measure complexity* if for any $\tau > 0$ and $\rho \in \mathcal{M}(X, T)$,

$$\liminf_{n \to +\infty} \frac{S_n(d, T, \rho, \epsilon)}{n^\tau} = 0$$

for any $\epsilon > 0$. In [**11**], Huang, Wang, and Ye showed that the Sarnak conjecture holds for any t.d.s. with sub-polynomial mean measure complexity. As an application of the above result in [**11**], one has the following result.

THEOREM 1.4. *Let C be a non-empty compact subset of* $[0, 1]$ *with packing dimension* $= 0$. *Then,*

$$\lim_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \leq N} \sup_{\alpha \in C} |\mathbb{E}_{h \leq H} \mu(n + h)e(h\alpha)| = 0. \qquad (1.8)$$

The paper is organized as follows. In §2, we prove Theorem 1.1. In §3, we prove Theorem 1.3. In Appendixes A and B, we prove Theorems 1.2 and 1.4.

2. *Proof of Theorem 1.1*

In this section, we prove Theorem 1.1. As we said in the introduction, it remains to prove (2) $\implies$ (1) which is done in §2.1, and (3) $\implies$ (2) which is carried out in §2.2.

2.1. *Proof of statement (2) implies statement (1) in Theorem 1.1.* We have explained in the introduction that the starting point of the proof is Tao's result which gives an equivalent statement of the logarithmic Sarnak conjecture. We will first introduce the result, then derive some result concerning the complexity of polynomial sequences, and finally give the proof. Let us begin with basic notions related to nilmanifolds.

Let G be a group. For $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h, and we write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $s \geq 1$ be an integer. We say that G is *s-step nilpotent* if G_{s+1} is the trivial subgroup.

Recall that *an s-step nilmanifold* is a manifold of the form G/Γ, where G is a connected, simply connected s-step nilpotent Lie group, and Γ is a cocompact discrete subgroup of G. Tao shows that the logarithmic Sarnak conjecture is equivalent to the following conjecture [22].

Conjecture 2.1. For any $s \in \mathbb{N}$, an s-step nilmanifold G/Γ, a Lip-continuous function $F : G/\Gamma \to \mathbb{C}$, and $x_0 \in G/\Gamma$, one has

$$\lim_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} \sup_{g \in G} |\mathbb{E}_{h \leq H} \mu(n + h) F(g^h x_0)| = 0.$$

Let G/Γ be an m-dimensional nilmanifold (that is, G is a connected, simply connected s-step nilpotent Lie group with unit element e and Γ is a cocompact discrete subgroup of G) and let $G = G_1 \supset \cdots \supset G_s \supset G_{s+1} = \{e\}$ be the lower central series filtration. We will make use of the Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ of G together with the exponential map $\exp : \mathfrak{g} \to G$. Since G is a connected, simply connected s-step nilpotent Lie group, the exponential map is a diffeomorphism [1, 18]. A basis $\mathcal{X} = \{X_1, \ldots, X_m\}$ for the Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called a *Mal'cev basis* for G/Γ if the following four conditions are satisfied.

(1) For each $j = 0, \ldots, m - 1$, the subspace $\eta_j := \text{Span}(X_{j+1}, \ldots, X_m)$ is a Lie algebra ideal in $\mathfrak{g}$, and hence $H_j := \exp \eta_j$ is a normal Lie subgroup of G.
(2) For every $0 < i \leq s$, there is l_{i-1} such that $G_i = H_{l_{i-1}}$. Thus, $0 = l_0 < l_1 < \cdots < l_{s-1} \leq m - 1$.
(3) Each $g \in G$ can be written uniquely as $\exp(t_1 X_1) \exp(t_2 X_2) \ldots \exp(t_m X_m)$ for some $t_i \in \mathbb{R}$.
(4) Γ consists precisely of those elements which, when written in the above form, have all $t_i \in \mathbb{Z}$.

Note that such a basis exists [1, 8, 16]. Now we fix a Mal'cev basis $\mathcal{X} = \{X_1, \ldots, X_m\}$ of G/Γ. Define $\psi : G \to \mathbb{R}^m$ such that if $g = \exp(t_1 X_1) \ldots \exp(t_m X_m) \in G$, then

$$\psi(g) = (t_1, \ldots, t_m) \in \mathbb{R}^m.$$

Moreover, let $|\psi(g)| = \max_{1 \le i \le m} |t_i|$. The following metrics on G and G/Γ are introduced in [8].

Definition 2.2. We define $d : G \times G \to \mathbb{R}$ to be the largest metric such that $d(x, y) \le |\psi(xy^{-1})|$ for all $x, y \in G$. More explicitly, we have

$$d(x, y) = \inf \left\{ \sum_{i=1}^{n} \min\{|\psi(x_{i-1}x_i^{-1})|, |\psi(x_i x_{i-1}^{-1})|\} : x_0, \ldots, x_n \in G; x_0 = x, x_n = y \right\}.$$

This descends to a metric on G/Γ by setting

$$d(x\Gamma, y\Gamma) := \inf\{d(x', y') : x', y' \in G; x' = x \pmod{\Gamma}; y' = y \pmod{\Gamma}\}.$$

It turns out that this is indeed a metric on G/Γ (see [8]). Since d is right-invariant (that is, $d(x, y) = d(xg, yg)$ for all $x, y, g \in G$), we also have

$$d(x\Gamma, y\Gamma) = \inf_{\gamma \in \Gamma} d(x, y\gamma).$$

The following lemma appears in [3, Lemmas 7.5 and 7.6].

LEMMA 2.3. *Let G be a connected, simply connected s-step nilpotent Lie group. Then there exist real polynomials $P_1 : \mathbb{R}^3 \to \mathbb{R}$, $P_2 : \mathbb{R} \to \mathbb{R}$, and $P_3 : \mathbb{R}^2 \to \mathbb{R}$ with positive coefficients such that for $x, y, g, h \in G$:*

(1) $d(gx, gy) \le P_1(|\psi(g)|, |\psi(x)|, |\psi(y)|)d(x, y)$;
(2) $|\psi(g^n)| \le P_2(n)|\psi(g)|^{n_G}$, *where n_G is a positive constant determined by G;*
(3) $|\psi(gh)| \le P_3(|\psi(g)|, |\psi(h)|)$.

Let G be a connected, simply connected s-step nilpotent Lie group with unit element e and $G = G_0 = G_1$, $G_{i+1} = [G, G_i]$ be the lower central series filtration of G. It is clear that $\{e\} = G_{s+1} = G_{s+2} = \cdots$. By a *polynomial sequence adapted to the lower central series filtration*, we mean a map $\mathbf{g} : \mathbb{Z} \to G$ such that $\partial_{h_i}, \ldots, \partial_{h_1}\mathbf{g} \in G_i$ for all $i > 0$ and $h_1, \ldots, h_i \in \mathbb{Z}$, where

$$\partial_h \mathbf{f}(n) := \mathbf{f}(n + h)\mathbf{f}(n)^{-1}$$

for any map $\mathbf{f} : \mathbb{Z} \to G$ and $n, h \in \mathbb{Z}$. Let $\mathrm{Poly}(G)$ be the collection of all polynomial sequences of G adapted to the lower central series filtration. It is well known that a polynomial sequence $\mathbf{g} : \mathbb{Z} \to G$ adapted to the lower central series filtration has unique Taylor coefficients $g_j \in G_j$ for each $0 \le j \le s$ such that

$$\mathbf{g}(n) = g_0^{\binom{n}{0}} g_1^{\binom{n}{1}} \ldots g_s^{\binom{n}{s}},$$

where $\binom{n}{0} \equiv 1$ (see for example [9, Lemma B.9] and [10, p. 240, Theorem 8]). In this case, we say that $g_i \in G_i$ for $i = 0, 1, \ldots, s$ is the coefficients of $\mathbf{g}$.

Using Lemma 2.3(2) and (3), it is not hard to verify by induction that there exists a real polynomial $Q : \mathbb{R}^{s+2} \to \mathbb{R}$ with positive coefficients such that

$$|\psi(\mathbf{g}(n))| \le Q(n, |\psi(g_0)|, \ldots, |\psi(g_s)|) \tag{2.1}$$

for $n \in \mathbb{Z}_+$.

We note that for $g, h \in G$, $\mathbf{g} : \mathbb{Z} \to G$ defined by $\mathbf{g}(n) = g^n h$ for each $n \in \mathbb{N}$ is a polynomial sequence adapted to the lower central series filtration since

$$\mathbf{g}(n) = g^n h = h^{\binom{n}{0}} (h^{-1} g h)^{\binom{n}{1}}.$$

For a non-empty subset K of G, we say $\mathbf{g} \in \mathrm{Poly}(G)$ *a polynomial sequence with coefficients in K*, if $g_i \in G_i \cap K$ for $i = 0, 1, \ldots, s$, where $\{g_i\}_{i=0}^{s}$ are the coefficients of $\mathbf{g}$. Green, Tao, and Ziegler proved the following lemma (see [**9**, Lemma C.1] and [**10**, p. 243, Proposition 12]).

LEMMA 2.4. *Let G be a connected, simply connected s-step nilpotent Lie group and Γ be a cocompact discrete subgroup of G. Then there exists a compact subset K of G such that any polynomial sequence $\mathbf{g} \in \mathrm{Poly}(G)$ can be factorized as $\mathbf{g} = \mathbf{g}'\gamma$, where $\mathbf{g}' \in \mathrm{Poly}(G)$ is a polynomial sequence with coefficients in K and $\gamma \in \mathrm{Poly}(G)$ is a polynomial sequence with coefficients in Γ.*

Let X be a separable metric space with metric d and Y be a non-empty subset of $X^{\mathbb{Z}}$. For any $\epsilon > 0$, we let $s_n(Y, \epsilon)$ be the minimal number such that there exist $x_i \in Y$, $1 \leq i \leq s_n(Y, \epsilon)$ satisfying that for any $y \in Y$, there exists $1 \leq i \leq s_n(Y, \epsilon)$ with $d(x_i(k), y(k)) < \epsilon$ for all $0 \leq k \leq n - 1$. Roughly speaking, $s_n(Y, \epsilon)$ is the minimal number of points which are ϵ-dense in $Y[0, n - 1] = \{(y_0, \ldots, y_{n-1}) : y = (y_i)_{i \in \mathbb{Z}} \in Y\}$.

Let G be a connected, simply connected s-step nilpotent Lie group and G/Γ be an s-step nilmanifold. For $K \subset G$, let $\mathrm{Poly}(K)$ be the collection of all polynomial sequences adapted to the lower central series filtration with coefficients in K. The map $\pi : \mathrm{Poly}(G) \to \{G/\Gamma\}^{\mathbb{Z}}$ is defined by

$$\pi(\mathbf{g})(n) = \mathbf{g}(n)\Gamma \quad \text{for all } n \in \mathbb{Z}.$$

Put $\mathrm{Poly}(G/\Gamma) = \pi(\mathrm{Poly}(G))$. We have the following.

PROPOSITION 2.5. *Let G/Γ be an s-step nilmanifold. Then there exists $k \in \mathbb{N}$ depending on G/Γ such that for each $\epsilon > 0$, we find $C(\epsilon) > 0$ depending on G/Γ and k satisfying $s_n(\mathrm{Poly}(G/\Gamma), \epsilon) \leq C(\epsilon)n^k$ for all $n \in \mathbb{N}$.*

To prove Proposition 2.5, we need the following lemma.

LEMMA 2.6. *Let G be a connected, simply connected s-step nilpotent Lie group and K be a non-empty compact subset of G. Then there is a real polynomial $P : \mathbb{R} \to \mathbb{R}$ depending on G and K such that*

$$d(\mathbf{g}(n), \widetilde{\mathbf{g}}(n)) \leq P(n) \max\{d(g_i, \widetilde{g}_i) : 0 \leq i \leq s\} \quad \text{for all } n \in \mathbb{N},$$

for any polynomials $\mathbf{g}(n) = g_0^{\binom{n}{0}} g_1^{\binom{n}{1}} \ldots g_s^{\binom{n}{s}}$ and $\widetilde{\mathbf{g}}(n) = \widetilde{g}_0^{\binom{n}{0}} \widetilde{g}_1^{\binom{n}{1}} \ldots \widetilde{g}_s^{\binom{n}{s}}$ adapted to the lower central series filtration with coefficients $g_0, g_1, \ldots, g_s, \widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_s \in K$.

Proof. Let P_1, P_2, P_3 be the real polynomials appearing in Lemma 2.3 and Q be the real polynomial appearing in equation (2.1). Since K is compact, $w = \max\{|\psi(g)| : g \in K\}$ is a positive real number. Put $\widetilde{Q}(n) = Q(n, w, w, \ldots, w)$ and $\widetilde{P}_2(n) = w^{n_G} P_2(n)$, where n_G is the constant appearing in Lemma 2.3(2).

Let $\mathbf{g}(n) = g_0^{\binom{n}{0}} g_1^{\binom{n}{1}} \ldots g_s^{\binom{n}{s}}$ and $\widetilde{\mathbf{g}}(n) = \widetilde{g}_0^{\binom{n}{0}} \widetilde{g}_1^{\binom{n}{1}} \ldots \widetilde{g}_s^{\binom{n}{s}}$ be two polynomials adapted to the lower central series filtration with coefficients $g_0, \ldots, g_s, \widetilde{g}_0, \ldots, \widetilde{g}_s \in K$. A simple computation yields

$$
\begin{aligned}
d(\mathbf{g}(n), \widetilde{\mathbf{g}}(n)) &\leq \sum_{i=0}^{s-1} d\big(g_0^{\binom{n}{0}} \ldots g_{i-1}^{\binom{n}{i-1}} \widetilde{g}_i^{\binom{n}{i}} \ldots \widetilde{g}_s^{\binom{n}{s}}, g_0^{\binom{n}{0}} \ldots g_i^{\binom{n}{i}} \widetilde{g}_{i+1}^{\binom{n}{i+1}} \ldots \widetilde{g}_s^{\binom{n}{s}}\big) \\
&= \sum_{i=0}^{s-1} d\big(g_0^{\binom{n}{0}} \ldots g_{i-1}^{\binom{n}{i-1}} \widetilde{g}_i^{\binom{n}{i}}, g_0^{\binom{n}{0}} \ldots g_i^{\binom{n}{i}}\big) \\
&\leq \sum_{i=0}^{s-1} P_1(|\psi(g_0^{\binom{n}{0}} \ldots g_{i-1}^{\binom{n}{i-1}})|, |\psi(\widetilde{g}_i^{\binom{n}{i}})|, |\psi(g_i^{\binom{n}{i}})|) d\big(\widetilde{g}_i^{\binom{n}{i}}, g_i^{\binom{n}{i}}\big) \\
&\leq \sum_{i=0}^{s-1} P_1(\widetilde{Q}(n), \widetilde{P}_2(\tbinom{n}{i}), \widetilde{P}_2(\tbinom{n}{i})) d\big(\widetilde{g}_i^{\binom{n}{i}}, g_i^{\binom{n}{i}}\big) \\
&\leq \widetilde{P}(n) \sum_{i=0}^{s-1} d\big(\widetilde{g}_i^{\binom{n}{i}}, g_i^{\binom{n}{i}}\big)
\end{aligned}
\tag{2.2}
$$

for all $n \in \mathbb{N}$, where $\widetilde{P}(n) = \sum_{i=0}^{s-1} P_1(\widetilde{Q}(n), \widetilde{P}_2(\tbinom{n}{i}), \widetilde{P}_2(\tbinom{n}{i}))$ is a polynomial of n.

Now we are going to show that there is a real polynomial $P_4 : \mathbb{R} \to \mathbb{R}$ such that $d(\widetilde{g}^n, g^n) \leq P_4(n) d(\widetilde{g}, g)$ for all $g, \widetilde{g} \in K$. In fact, it follows from the fact

$$
\begin{aligned}
d(\widetilde{g}^n, g^n) &\leq \sum_{i=0}^{n-1} d(\widetilde{g}^i g^{n-i}, \widetilde{g}^{i+1} g^{n-i-1}) = \sum_{i=0}^{n-1} d(\widetilde{g}^i g, \widetilde{g}^{i+1}) \\
&\leq \sum_{i=0}^{n-1} P_1(|\psi(\widetilde{g}^i)|, |\psi(\widetilde{g})|, |\psi(g)|) d(\widetilde{g}, g) \\
&\leq \sum_{i=0}^{n-1} P_1(\widetilde{P}_2(i), w, w) d(\widetilde{g}, g) \\
&\leq P_4(n) d(\widetilde{g}, g)
\end{aligned}
\tag{2.3}
$$

for all $n \in \mathbb{N}$, where $P_4(n) = \sum_{i=0}^{n-1} P_1(\widetilde{P}_2(i), w, w)$ is a real polynomial of n. Summing up, we obtain

$$
\begin{aligned}
d(\mathbf{g}(n), \widetilde{\mathbf{g}}(n)) &\overset{(2.2)}{\leq} \widetilde{P}(n) \sum_{i=0}^{s-1} d\big(\widetilde{g}_i^{\binom{n}{i}}, g_i^{\binom{n}{i}}\big) \\
&\overset{(2.3)}{\leq} \widetilde{P}(n) \sum_{i=0}^{s-1} P_4(\tbinom{n}{i}) d(\widetilde{g}_i, g_i) \\
&\leq P(n) \max\{d(g_i, \widetilde{g}_i) : 0 \leq i \leq s\}
\end{aligned}
$$

for all $n \in \mathbb{N}$, where $P(n) = \widetilde{P}(n) \sum_{i=0}^{s-1} P_4(\tbinom{n}{i})$ is a real polynomial of n. Then $P(n)$ is the real polynomial as required. This ends the proof of Lemma 2.6. $\qquad\square$

Now we are ready to prove Proposition 2.5.

Proof of Proposition 2.5. By Lemma 2.4, there exists a compact subset K of G such that any polynomial sequence $\mathbf{g}$ adapted to the lower central series filtration can be factorized as $\mathbf{g} = \mathbf{g}'\gamma$, where $\mathbf{g}'$ is a polynomial sequence adapted to the lower central series filtration with coefficients in K and γ is a polynomial sequence with coefficients in Γ. Since K is compact, by Lemma 2.6, there is a real polynomial $P : \mathbb{R} \to \mathbb{R}$ such that

$$d(\mathbf{g}(j), \widetilde{\mathbf{g}}(j)) \leq P(j) \max\{d(g_i, \widetilde{g}_i) : 0 \leq i \leq s\} \quad \text{for all } j \in \mathbb{N}, \qquad (2.4)$$

and any polynomials $\mathbf{g}, \widetilde{\mathbf{g}} \in \text{Poly}(G)$ with coefficients $g_0, \ldots, g_s, \widetilde{g}_0, \ldots, \widetilde{g}_s \in K$. It is not hard to see that there exists $k_0 \in \mathbb{N}$ and $C > 1$ such that

$$P(n) < Cn^{k_0} \quad \text{for all } n \in \mathbb{N}. \qquad (2.5)$$

Since K is compact, for $\epsilon > 0$, we let $N_\epsilon(K)$ be the smallest number of open balls of ratio ϵ needed to cover K. The upper Minkowski dimension or box dimension (see [17]) is defined by

$$\limsup_{\epsilon \to 0} \frac{-\log N_\epsilon(K)}{\log \epsilon}.$$

This dimension of K is not larger than the usual dimension of G since K is a subset of G. Hence, there exists a positive constant L such that

$$N_\epsilon(K) \leq L\left(\frac{1}{\min\{\epsilon, 1\}}\right)^{\dim(G)+1}. \qquad (2.6)$$

Set

$$k = k_0(s+1)(\dim(G)+1) \quad \text{and} \quad C(\epsilon) = \left(L\left(\frac{2C}{\min\{\epsilon, 1\}}\right)^{\dim(G)+1}\right)^{s+1} \quad \text{for } \epsilon > 0.$$

We are going to show that

$$s_n(G/\Gamma, \epsilon) \leq C(\epsilon)n^k$$

for $n \in \mathbb{N}$ and $\epsilon > 0$. To do this, let π be the projection from $\text{Poly}(K)$ to $\text{Poly}(G/\Gamma)$ defined by $\pi(\mathbf{g})(n) = \mathbf{g}(n)\Gamma$ for all $n \in \mathbb{Z}$. By Lemma 2.4, π is surjective and

$$d(\mathbf{g}(j), \widetilde{\mathbf{g}}(j)) \geq d(\pi(\mathbf{g})(j), \pi(\widetilde{\mathbf{g}})(j)) \quad \text{for all } j \in \mathbb{Z}.$$

Hence,

$$s_n(\text{Poly}(G/\Gamma), \epsilon) \leq s_n(\text{Poly}(K), \epsilon) \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \epsilon > 0. \qquad (2.7)$$

For $\tau > 0$, we let E_τ be a finite subset of K such that

$$\sharp E_\tau \leq N_\tau(K) \quad \text{and} \quad K \subset \bigcup_{g \in E_\tau} B(g, \tau).$$

For $0 \leq i \leq s$, we let $E_\tau^{(i)}$ be a subset of $K \cap G_i$ such that

$$\sharp E_\tau^{(i)} \leq N_\tau(K) \quad \text{and} \quad K \cap G_i \subset \bigcup_{g \in E_\tau^i} B(g, 2\tau). \qquad (2.8)$$

Put P_τ to be the collection of all polynomial sequences **g** adapted to the lower central series filtration with coefficients $g_i \in E_\tau^{(i)}$, $i = 0, 1, \ldots, s$. Then for $n \in \mathbb{N}$ and $\epsilon > 0$,

$$\sharp P_{\epsilon/2Cn^{k_0}} = \prod_{i=0}^{s} \sharp E_{\epsilon/2Cn^{k_0}}^{(i)} \overset{(2.8),(2.6)}{\leq} \left(L \left(\frac{2Cn^{k_0}}{\min\{\epsilon, 1\}} \right)^{\dim(G)+1} \right)^{s+1} = C(\epsilon)n^k. \quad (2.9)$$

Now we fix $n \in \mathbb{N}$ and $\epsilon > 0$. By equation (2.8), for any polynomial sequence **g** $\in \text{Poly}(K)$ with coefficients $g_0, \ldots, g_s \in K$, we have that $g_i \in K \cap G_i$. Thus, there exists $\bar{\mathbf{g}} \in P_{\epsilon/2Cn^{k_0}}$ with coefficients $\bar{g}_0 \in E_{\epsilon/2Cn^{k_0}}^{(0)}, \ldots, \bar{g}_s \in E_{\epsilon/2Cn^{k_0}}^{(s)}$ such that

$$d(g_i, \bar{g}_i) \overset{(2.8)}{<} \frac{\epsilon}{Cn^{k_0}} \quad \text{for all } 0 \leq i \leq s.$$

Therefore,

$$d(\mathbf{g}(0), \bar{\mathbf{g}}(0)) = d(g_0, \bar{g}_0) < \frac{\epsilon}{Cn^{k_0}} < \epsilon$$

and for $1 \leq j \leq n-1$, one has

$$d(\mathbf{g}(j), \bar{\mathbf{g}}(j)) \overset{(2.4)}{\leq} P(j) \max\{d(g_i, \bar{g}_i) : 0 \leq i \leq s\} \overset{(2.5)}{<} Cj^{k_0} \times \frac{\epsilon}{Cn^{k_0}} \leq \epsilon.$$

Hence,

$$s_n(\text{Poly}(G/\Gamma), \epsilon) \overset{(2.7)}{\leq} s_n(\text{Poly}(K), \epsilon) \leq \sharp P_{\epsilon/2Cn^{k_0}} \overset{(2.9)}{\leq} C(\epsilon)n^k.$$

Since the above inequality holds for all $n \in \mathbb{N}$ and $\epsilon > 0$, we end the proof of Proposition 2.5. $\qquad \square$

With the above preparations, now we are in the position to prove Theorem 1.1.

Proof of (2) $\implies$ (1) in Theorem 1.1. Assume that Theorem 1.1(2) holds, that is, the logarithmic Sarnak conjecture holds for any t.d.s. with polynomial mean complexity. In what follows, we aim to show that the logarithmic Sarnak conjecture holds.

Assume the contrary that this is not the case, then by Tao's result [22], the Conjecture 2.1 does not hold. This means that there exist an $s \in \mathbb{N}$, an s-step nilmanifold G/Γ, a Lip-continuous function $F : G/\Gamma \to \mathbb{C}$, and an $x_0 \in G/\Gamma$ such that

$$\limsup_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} \sup_{g \in G} |\mathbb{E}_{h \leq H} \mu(n+h) F(g^h x_0)| > 0. \quad (2.10)$$

It is clear that $\|F\|_\infty := \max_{x \in G/\Gamma} |F(x)| > 0$. Without loss of generality, we assume that

$$\|F\|_\infty = 1. \quad (2.11)$$

Now we add an extra point p to the compact metric space G/Γ. We then extend the metric d on G/Γ to the space $G/\Gamma \cup \{p\}$ by letting $d(p, x) = 1$ for all $x \in G/\Gamma$. So, $(G/\Gamma \cup \{p\}, d)$ is also a compact metric space. Let $\widetilde{F} : (G/\Gamma \cup \{p\})^{\mathbb{Z}} \to \mathbb{C}$ be defined by $\widetilde{F}(z) = F(z(0))$ if $z(0) \in G/\Gamma$ and 0 if $z(0) = p$. It is clear that $\widetilde{F}$ is a continuous function and

$$\|\widetilde{F}\|_\infty = 1 \quad (2.12)$$

by equation (2.11).

In what follows, we will find a point $y \in (G/\Gamma \cup \{p\})^{\mathbb{Z}}$ such that

$$\limsup_{N \to \infty} |\mathbb{E}_{n \leq N}^{\log} \mu(n) \widetilde{F}(\sigma^n y)| > 0, \tag{2.13}$$

and the t.d.s. (X_y, σ) has polynomial mean complexity, where $\sigma : (G/\Gamma \cup \{p\})^{\mathbb{Z}} \to (G/\Gamma \cup \{p\})^{\mathbb{Z}}$ is the left shift and $X_y = \overline{\{\sigma^n y : n \in \mathbb{Z}\}}$ is a σ-invariant compact subset of $(G/\Gamma \cup \{p\})^{\mathbb{Z}}$. Clearly, this is a contradiction to our assumption and thus proves that statement (2) implies statement (1) in Theorem 1.1.

We divide the remaining proof into two steps.

Step 1. The construction of the point y. First, we note that

$$|z| \leq \sum_{j=0}^{3} \max \left\{ \text{Re}\left(e\left(\frac{j}{4}\right) z \right), 0 \right\}$$

for $z \in \mathbb{C}$. Thus, by equation (2.10), there is $\beta \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$ such that

$$\limsup_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} \max \left\{ \sup_{g \in G} \text{Re}(e(\beta)\mathbb{E}_{h \leq H} \mu(n+h) F(g^h x_0)), 0 \right\} > 0.$$

Thus, we can find $\tau \in (0, 1)$ with

$$E := \left\{ H \in \mathbb{N} : \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} \max \left\{ \sup_{g \in G} \text{Re}(e(\beta)\mathbb{E}_{h \leq H} \mu(n+h) F(g^h x_0)), 0 \right\} > \tau \right\}$$

is an infinite set. Moreover, putting $\sigma = \tau^2/200$ and by induction, we can find strictly increasing sequences $\{H_i\}_{i=1}^{\infty}$ of E and $\{N_i\}_{i=1}^{\infty}$ of natural numbers such that for each $i \in \mathbb{N}$, one has

$$H_i < \sigma N_i^{\sigma} < \frac{\sigma}{10} H_{i+1}^{\sigma}, \tag{2.14}$$

and there exist $g_{n,i} \in G$ for $1 \leq n \leq N_i$ satisfying

$$\mathbb{E}_{n \leq N_i}^{\log} \max\{\text{Re}(e(\beta)\mathbb{E}_{h \leq H_i} \mu(n+h) F(g_{n,i}^h x_0)), 0\} > \tau. \tag{2.15}$$

For $i \in \mathbb{N}$, let $M_i = \sum_{n=1}^{N_i}(1/n)$ and

$$S_i = \left\{ n \in [1, N_i] \cap \mathbb{Z} : \text{Re}(e(\beta)\mathbb{E}_{h \leq H_i} \mu(n+h) F(g_{n,i}^h x_0)) > \frac{\tau}{2} \right\}. \tag{2.16}$$

Then by equations (2.12) and (2.15), we have

$$\sum_{n \in S_i} \frac{1}{n} > \frac{\tau}{2} M_i. \tag{2.17}$$

Notice that $\lim_{N \to +\infty}(\sum_{n \leq N^{\sigma}}(1/n)/\sum_{n \leq N}(1/n)) = \sigma$. So, when $i \in \mathbb{N}$ is large enough, we have

$$\sum_{n \in S_i \setminus [1, N_i^{\sigma}]} \frac{1}{n} \overset{(2.17)}{>} \frac{\tau}{2} M_i - \sum_{n \leq N_i^{\sigma}} \frac{1}{n} > \frac{\tau}{2} M_i - 2\sigma M_i > \frac{\tau}{4} M_i.$$

Hence, we can select $S_i' \subset S_i \setminus [1, N_i^\sigma]$ with each gap not less than $2H_i$ and

$$\sum_{n \in S_i'} \frac{1}{n} > \frac{\tau M_i}{8H_i} \tag{2.18}$$

for $i \in \mathbb{N}$ large enough.

Define $y : \mathbb{Z} \to G/\Gamma \cup \{p\}$ such that

$$y(n + h) := g_{n,i}^h x_0 \quad \text{for } n \in S_i', h = 1, 2, \ldots, H_i, i \in \mathbb{N}$$

and $y(m) = p$ for $m \in \mathbb{Z} \setminus \bigcup_{i=1}^\infty \bigcup_{n \in S_i'} \{n + 1, n + 2, \ldots, n + H_i\}$.

Clearly, y is well defined since $N_{i+1} > N_i + H_i$ by equation (2.14). Then one has by equations (2.16) and (2.18) that

$$\mathrm{Re}\left(e(\beta) \sum_{n \in S_i'} \frac{1}{n} \sum_{h \le H_i} \mu(n + h) \widetilde{F}(\sigma^{n+h} y) \right) > \frac{\tau^2}{16} M_i$$

for $i \in \mathbb{N}$ large enough. This implies

$$\left| \sum_{n \in S_i'} \frac{1}{n} \sum_{h \le H_i} \mu(n + h) \widetilde{F}(\sigma^{n+h} y) \right| > \frac{\tau^2}{16} M_i \tag{2.19}$$

for $i \in \mathbb{N}$ large enough. Moreover, for $i \in \mathbb{N}$ large enough,

$$\left| \sum_{n \in S_i'} \frac{1}{n} \sum_{h \le H_i} \mu(n + h) \widetilde{F}(\sigma^{n+h} y) - \sum_{n \in S_i'} \sum_{h \le H_i} \frac{\mu(n + h) \widetilde{F}(\sigma^{n+h} y)}{n + h} \right|$$

$$\overset{(2.12)}{\le} \sum_{n \in S_i'} \sum_{h \le H_i} \left(\frac{1}{n} - \frac{1}{n + h} \right) \le \sum_{n \in S_i'} \sum_{h \le H_i} \frac{H_i}{n(n + H_i)}$$

$$\le \sum_{n \in S_i'} \frac{H_i}{n(N_i^\sigma + H_i)} \le \sum_{n \in S_i'} \frac{H_i}{n N_i^\sigma}$$

$$\overset{(2.14)}{\le} \sigma \sum_{n \in S_i'} \frac{1}{n} \le \frac{\tau^2}{32} M_i.$$

Combining this inequality with equation (2.19), one has

$$\left| \sum_{N_i^\sigma < n \le N_i + H_i} \frac{\mu(n) \widetilde{F}(\sigma^n y)}{n} \right| = \left| \sum_{n \in S_i'} \sum_{h \le H_i} \frac{\mu(n + h) \widetilde{F}(\sigma^{n+h} y)}{n + h} \right| \ge \frac{\tau^2 M_i}{32} \tag{2.20}$$

for $i \in \mathbb{N}$ large enough. Thus,

$$\left| \frac{1}{M_i} \sum_{n \le N_i} \frac{\mu(n) \widetilde{F}(\sigma^n y)}{n} \right|$$

$$\ge \frac{1}{M_i} \left| \sum_{N_i^\sigma < n \le N_i + H_i} \frac{\mu(n) \widetilde{F}(\sigma^n y)}{n} \right| - \frac{1}{M_i} \sum_{\substack{n \le N_i^\sigma \text{ or} \\ N_i < n \le N_i + H_i}} \left| \frac{\mu(n) \widetilde{F}(\sigma^n y)}{n} \right|$$

$$\overset{(2.20)}{\geq} \frac{\tau^2}{32} - \frac{\|\widetilde{F}\|_\infty}{M_i} \sum_{N_i < n \leq N_i + H_i} \frac{1}{n} - \frac{\|\widetilde{F}\|_\infty}{M_i} \sum_{n \leq N_i^\sigma} \frac{1}{n}$$

$$\overset{(2.12)}{\geq} \frac{\tau^2}{32} - \frac{H_i}{N_i} - 2\sigma \overset{(2.14)}{\geq} \frac{\tau^2}{32} - 3\sigma \overset{(2.14)}{\geq} \frac{\tau^2}{100}$$

for $i \in \mathbb{N}$ large enough. This deduces that

$$\limsup_{N \to \infty} |\mathbb{E}^{\log}_{n \leq N} \mu(n) \widetilde{F}(\sigma^n y)| \geq \frac{\tau^2}{100} > 0.$$

Therefore, y is the point as required.

Step 2. (X_y, σ) has polynomial mean complexity. Recall that $X_y = \overline{\{\sigma^n y : n \in \mathbb{Z}\}}$ is a compact σ-invariant subset of $(G/\Gamma \cup \{p\})^{\mathbb{Z}}$. The metric on $(G/\Gamma \cup \{p\})^{\mathbb{Z}}$ is defined by

$$D(x, x') = \sum_{n \in \mathbb{Z}} \frac{d(x(n), x'(n))}{2^{|n|+2}} \tag{2.21}$$

for $x = (x(n))_{n \in \mathbb{Z}}, x' = (x'(n))_{n \in \mathbb{Z}} \in (G/\Gamma \cup \{p\})^{\mathbb{Z}}$. By Proposition 2.5, we can find $k > 1$ such that

$$\lim_{n \to +\infty} \frac{s_n(\text{Poly}(G/\Gamma), \epsilon)}{n^k} = 0 \quad \text{for all } \epsilon > 0. \tag{2.22}$$

Now we are going to show that

$$\liminf_{n \to +\infty} \frac{S_n(D, \sigma, X_y, \epsilon)}{n^{k+1}} = 0 \quad \text{for all } \epsilon > 0.$$

For $n \in \mathbb{Z}_+$ and $-n \leq q \leq n$, let $X_{n,q}$ be the collection of all points $z \in (G/\Gamma \cup \{p\})^{\mathbb{Z}}$ with

$$z(j) = \begin{cases} p & \text{if } -n \leq j < q, \\ \mathbf{g}(j)\Gamma & \text{if } q \leq j \leq n, \end{cases}$$

where $\mathbf{g}$ is some polynomial sequence of G adapted to the lower central series filtration; and let $X^*_{n,q}$ be the collection of all points $z \in (G/\Gamma \cup \{p\})^{\mathbb{Z}}$ with

$$z(j) = \begin{cases} \mathbf{g}(j)\Gamma & \text{if } -n \leq j < q, \\ p & \text{if } q \leq j \leq n, \end{cases}$$

where $\mathbf{g}$ is some polynomial sequence of G adapted to the lower central series filtration.

For $i \in \mathbb{N}$, put $t_i = [H_i/2]$, where $[u]$ is the integer part of the real number u. Then,

$$X_y \subset \bigcup_{-t_i \leq q \leq t_i} X_{t_i,q} \cup \bigcup_{-t_i \leq q \leq t_i} X^*_{t_i,q} \cup \{\sigma^j y : -H_i \leq j \leq H_i\}. \tag{2.23}$$

In fact, since $H_{j+1} > N_j + H_j$ for all $j \in \mathbb{N}$, one has

$$\sigma^n y \in \bigcup_{-t_i \leq q \leq t_i} X_{t_i,q} \cup \bigcup_{-t_i \leq q \leq t_i} X^*_{t_i,q} \cup \{\sigma^j y : -H_i \leq j \leq H_i\} \quad \text{for all } n \in \mathbb{Z} \tag{2.24}$$

by the construction of y. It is not hard to see that $X_{t_i,q}, X^*_{t_i,q}$ are all compact subsets of $(G/\Gamma \cup \{p\})^{\mathbb{Z}}$ for each $-t_i \leq q \leq t_i$ and $i \in \mathbb{N}$ by Lemma 2.4. Hence, the set in right part

of equation (2.23) is also a compact subset of $(G/\Gamma \cup \{p\})^{\mathbb{Z}}$. Now equation (2.23) follows from equation (2.24).

Now we fix $\epsilon > 0$. We have the following claim.

CLAIM. *For $i \in \mathbb{N}$ large enough, one has:*

(1) $S_{[t_i/2]}(D, \sigma, X_{t_i,q}, \epsilon) \le s_{t_i}(\mathrm{Poly}(G/\Gamma), \epsilon/2)$ *for all $q \in [-t_i, t_i] \cap \mathbb{Z}$;*

(2) $S_{[t_i/2]}(D, \sigma, X^*_{t_i,p}, \epsilon) \le s_{t_i}(\mathrm{Poly}(G/\Gamma), \epsilon/2)$ *for all $q \in [-t_i, t_i] \cap \mathbb{Z}$.*

Proof of the Claim. We prove part (1) first. For $i \in \mathbb{N}$ and $-t_i \le q \le t_i$, we let $\pi_{i,q} : \mathrm{Poly}(G/\Gamma) \to X_{t_i,q}$ be defined by

$$\pi_{i,q}(z)(j) = \begin{cases} p & \text{if } -t_i \le j < q, \\ z(j) & \text{otherwise,} \end{cases}$$

for $z \in \mathrm{Poly}(G/\Gamma)$. For $i \in \mathbb{N}$ large enough, if $z, \widetilde{z} \in \mathrm{Poly}(G/\Gamma)$ with $d(z(j), \widetilde{z}(j)) < \epsilon/2$ for all $-t_i \le j \le t_i$, then for $q \in [-t_i, t_i] \cap \mathbb{Z}$,

$$\bar{D}_{[t_i/2]}(\pi_{i,q}(z), \pi_{i,q}(\widetilde{z})) = \frac{1}{[t_i/2]} \sum_{l=0}^{[t_i/2]-1} D(\sigma^l z, \sigma^l \widetilde{z})$$

$$\overset{(2.21)}{=} \frac{1}{[t_i/2]} \sum_{l=0}^{[t_i/2]-1} \sum_{n \in \mathbb{Z}} \frac{d(z(n+l), \widetilde{z}(n+l))}{2^{|n|+2}}$$

$$\le \frac{1}{[t_i/2]} \sum_{l=0}^{[t_i/2]-1} \left(\sum_{|n| \le [t_i/2]} \frac{d(z(n+l), \widetilde{z}(n+l))}{2^{|n|+2}} + \sum_{|n| > [t_i/2]} \frac{d(z(n+l), \widetilde{z}(n+l))}{2^{|n|+2}} \right)$$

$$\le \frac{1}{[t_i/2]} \sum_{l=0}^{[t_i/2]-1} \left(\frac{\epsilon}{2} + \frac{\mathrm{diam}(G/\Gamma)}{2^{t_i/2}} \right) < \epsilon, \qquad (2.25)$$

where we use the fact $t_i \to +\infty$ as $i \to +\infty$ in the last inequality. Notice that the map $\pi_{i,q}$ is surjective for all $i \in \mathbb{N}$ and $-t_i \le q \le t_i$. By equation (2.25), for $i \in \mathbb{N}$ large enough, one has

$$S_{[t_i/2]}(D, \sigma, X_{t_i,q}, \epsilon) \le s_{t_i}\left(\mathrm{Poly}(G/\Gamma), \frac{\epsilon}{2}\right) \quad \text{for all } q \in [-t_i, t_i] \cap \mathbb{Z}.$$

By the similar arguments, one has part (2). This ends the proof of the Claim. $\square$

Hence, by the above Claim and equation (2.23), one has

$$S_{[t_i/2]}(D, \sigma, X_y, \epsilon) \le (2H_i + 1) + \sum_{q=-t_i}^{t_i} (S_{[t_i/2]}(D, \sigma, X_{t_i,q}, \epsilon) + S_{[t_i/2]}(D, \sigma, X^*_{t_i,q}, \epsilon))$$

$$\le (2H_i + 1) + (2t_i + 1)s_{t_i}\left(\mathrm{Poly}(G/\Gamma), \frac{\epsilon}{2}\right)$$

for $i \in \mathbb{N}$ large enough. Combining this with equation (2.22),

$$
\begin{aligned}
\liminf_{n \to +\infty} \frac{S_n(D, \sigma, X_y, \epsilon)}{n^{k+1}} &\le \liminf_{i \to +\infty} \frac{S_{[t_i/2]}(D, \sigma, X_y, \epsilon)}{[t_i/2]^{k+1}} \\
&\le \liminf_{i \to +\infty} \frac{(2H_i + 1) + (2t_i + 1)s_{t_i}(\text{Poly}(G/\Gamma), \epsilon/2)}{[t_i/2]^{k+1}} \\
&= 0,
\end{aligned}
$$

where we used the assumption $t_i = [H_i/2]$. This implies that (X_y, σ) has polynomial mean complexity, since the above inequality is true for all $\epsilon > 0$. This ends the proof of Step 2. □

Remark 2.7. In the proof above, we use dynamics on infinite products. Precisely, we show that if the logarithmic Sarnak's conjecture does not hold, one can find a point in an infinite product space $(G/\Gamma \cup \{p\})^{\mathbb{Z}}$ for which the logarithmic averages are almost equal to the short uniform averages of the corresponding space G/Γ. We remark that the idea behind the construction is similar to that in [4] where the authors also use dynamics on infinite products. They show that what they call the strong MOMO property is equivalent to Sarnak's conjecture (see Corollary 9).

2.2. *Proof of statement (3) implies statement (2) in Theorem 1.1.*
To get the proof, we first discuss a t.d.s. with the so-called small boundary property, then we obtain a key proposition for the proof, and finally we give the proof. We start with the notion of small boundary property.

For a t.d.s. (X, T), a subset E of X is called *T-small* (or simply small when there is no diffusion) if

$$
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_E(T^n x) = 0
$$

uniformly for $x \in X$. It is not hard to show that a closed subset E of X is small if and only if $\nu(E) = 0$ for all $\nu \in \mathcal{M}(X, T)$. For a subset U of X, we say U has a *small boundary* if ∂U is small. We say (X, T) has a *small boundary property* if for any $x \in X$ and any open neighborhood V of x, there exists an open neighborhood W of x such that $W \subset V$ and W has a small boundary. The following lemma indicates that when X has the small boundary property, then the logarithmic Sarnak conjecture can be verified through easier conditions.

LEMMA 2.8. *Let (X, T) be a t.d.s. with small the boundary property. Then the logarithmic Sarnak conjecture holds for (X, T) if and only if for any subset U of X with a small boundary, one has*

$$
\lim_{N \to +\infty} \mathbb{E}_{n \le N}^{\log} \mathbf{1}_U(T^n x)\mu(n) = 0 \tag{2.26}
$$

for all $x \in X$.

Proof. First, we assume that equation (2.26) holds for any subset U of X with small boundary and $x \in X$. For a given $f \in C(X)$ and fixed $\delta > 0$, let

$$
\epsilon = \epsilon(\delta) = \sup_{x, y \in X, d(x,y) < \delta} |f(x) - f(y)|.
$$

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a partition of X with diameter smaller than δ and each element of $\mathcal{P}$ has a small boundary. For $1 \le i \le k$, we fix points $x_i \in P_i$ and define $\bar{f}(x) = f(x_i)$ if $x \in P_i$. Then, $\bar{f}(x) = \sum_{i=1}^{k} f(x_i)\mathbf{1}_{P_i}(x)$ and by equation (2.26),

$$\lim_{N \to +\infty} \mathbb{E}_{n \le N}^{\log} \bar{f}(T^n x)\mu(n) = 0$$

for all $x \in X$. Since $\|\bar{f} - f\|_\infty \le \epsilon$, we have

$$\limsup_{N \to +\infty} |\mathbb{E}_{n \le N}^{\log} f(T^n x)\mu(n)|$$

$$\le \limsup_{N \to +\infty} |\mathbb{E}_{n \le N}^{\log} \bar{f}(T^n x)\mu(n)| + \limsup_{N \to +\infty} \mathbb{E}_{n \le N}^{\log} \|\bar{f} - f\|_\infty \cdot |\mu(n)|$$

$$\le \epsilon$$

for all $x \in X$. By taking $\delta \to 0$ and then $\epsilon \to 0$, one has

$$\lim_{N \to +\infty} \mathbb{E}_{n \le N}^{\log} f(T^n x)\mu(n) = 0$$

for all $x \in X$. This implies the logarithmic Sarnak conjecture holds for (X, T) since f is arbitrary.

Conversely, we assume that the logarithmic Sarnak conjecture holds for (X, T). Let U be a subset of X with small boundary. Fix $\delta > 0$. By a result of Shub and Weiss (see [20, p. 537]), we can find $\epsilon > 0$ such that for N large enough,

$$\frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{B(\partial U, \epsilon)}(T^n x) \le \frac{\delta}{2}$$

for all $x \in X$, where $B(\partial U, \epsilon) = \{y \in X : d(y, \partial U) < \epsilon\}$. Moreover, for N large enough,

$$\mathbb{E}_{n \le N}^{\log} \mathbf{1}_{B(\partial U, \epsilon)}(T^n x) = \frac{1}{M_N} \sum_{n=1}^{N} \frac{\mathbf{1}_{B(\partial U, \epsilon)}(T^n x)}{n}$$

$$= \frac{1}{M_N} \left(\frac{S_N(x)}{N} + \sum_{j=1}^{N-1} \frac{S_j(x)}{j} \frac{1}{j+1} \right) \qquad (2.27)$$

$$\le \delta$$

for all $x \in X$, where we simply write $M_N = \sum_{n=1}^{N}(1/n)$ and $S_j(x) = \sum_{n=1}^{j} \mathbf{1}_{B(\partial U, \epsilon)}(T^n x)$ for $j \in \mathbb{N}$.

Using Urysohn's lemma, there exists a continuous function $h : X \to \mathbb{R}$ with $0 \le h \le 1$ such that $h(x) = 1$ for $x \in U \setminus B(\partial U, \epsilon)$ and $h(x) = 0$ for $x \in X \setminus (U \cup B(\partial U, \epsilon))$. Since the logarithmic Sarnak conjecture holds for (X, T), one has

$$\lim_{N \to +\infty} \mathbb{E}_{n \le N}^{\log} h(T^n x)\mu(n) = 0$$

for all $x \in X$. Combining this equality with equation (2.27), we obtain

$$\limsup_{N \to +\infty} |\mathbb{E}_{n \leq N}^{\log} 1_U(T^n x)\mu(n)|$$

$$\leq \limsup_{N \to +\infty} |\mathbb{E}_{n \leq N}^{\log} h(T^n x)\mu(n)| + \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} |h(T^n x) - 1_U(T^n x)|$$

$$\leq \limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} 1_{B(\partial U, \epsilon)}(T^n x) \leq \delta$$

for all $x \in X$. By taking $\delta \to 0$, we have

$$\lim_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} 1_U(T^n x)\mu(n) = 0$$

for all $x \in X$. This ends the proof of Lemma 2.8. $\qquad\square$

The next lemma concerns the coding of a subset with small boundary.

LEMMA 2.9. *Let (X, T) be a t.d.s. and U be a subset of X with small boundary. For $x \in X$, we associate an $\hat{x} \in \{0, 1\}^{\mathbb{Z}}$ such that $\hat{x}(n) = 1$ if $T^n x \in U$ and $\hat{x}(n) = 0$ otherwise. Then for $\delta > 0$, there exist $\epsilon > 0$ and $N_\delta \in \mathbb{N}$ such that for all $N \geq N_\delta$ and any $x_1, x_2 \in X$ with $\bar{d}_N(x_1, x_2) < \epsilon$, one has*

$$\sharp\{0 \leq n \leq N - 1 : \hat{x_1}(n) \neq \hat{x_2}(n)\} \leq 2\delta N.$$

Proof. We fix an $\delta \in (0, +\infty)$ and a non-empty subset U of X with small boundary. By a result of Shub and Weiss (see [20, p. 537]), there exist $N_\delta \in \mathbb{N}$ and $\epsilon_0 \in (0, +\infty)$ such that

$$\sup_{x \in X, N \geq N_\delta} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{B(\partial U, \epsilon_0)}(T^n x) < \delta, \tag{2.28}$$

where $B(\partial U, \epsilon_0) = \{z \in X : d(z, z') < \epsilon_0 \text{ for all } z' \in \partial U\}$ if ∂U is not empty and $B(\partial U, \epsilon_0) = \emptyset$ if ∂U is empty.

We notice that $\overline{U \setminus B(\partial U, \epsilon_0)} \cap \overline{X \setminus U} = \emptyset$ and $\overline{(X \setminus U) \setminus B(\partial U, \epsilon_0)} \cap \overline{U} = \emptyset$. Thus, we can find $\epsilon \in (0, \delta^2)$ such that when $x, y \in X$ with $d(x, y) < \sqrt{\epsilon}$, if $x \in U \setminus B(\partial U, \epsilon_0)$ (respectively $x \in (X \setminus U) \setminus B(\partial U, \epsilon_0)$), then $y \in U$ (respectively $y \in X \setminus U$). We are to show that ϵ is the constant as required. We fix $N \geq N_\delta$ and $x_1, x_2 \in X$ with $\bar{d}_N(x_1, x_2) < \epsilon$. Set

$$C = \{0 \leq n \leq N - 1 : T^n x_1 \in B(\partial U, \epsilon_0)\}.$$

By equation (2.28), $\sharp C \leq \delta N$. Put

$$\mathcal{A} = \{0 \leq n \leq N - 1 : d(T^n x_1, T^n x_2) < \sqrt{\epsilon}\}.$$

One has $\sharp\mathcal{A} \geq (1 - \sqrt{\epsilon})N$ and $\hat{x_1}(n) = \hat{x_2}(n)$ for all $n \in \mathcal{A} \setminus C$. Therefore,

$$\sharp\{0 \leq n \leq N - 1 : \hat{x_1}(n) = \hat{x_2}(n)\} \geq \sharp\mathcal{A} - \sharp C \geq (1 - \sqrt{\epsilon} - \delta)N.$$

Since $\delta > \sqrt{\epsilon}$, one has

$$\sharp\{0 \leq n \leq N - 1 : \hat{x_1}(n) \neq \hat{x_2}(n)\} \leq 2\delta N.$$

This ends the proof of Lemma 2.9. $\qquad\square$

Recall that the metric on $\{0, 1\}^{\mathbb{Z}}$ is defined by

$$d(x, y) = \sum_{n \in \mathbb{Z}} \frac{|x(n) - y(n)|}{2^{|n|+2}} \tag{2.29}$$

for $x = (x(n))_{n \in \mathbb{Z}}, y = (y(n))_{n \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$. We have the following lemma which is key for the proof of statement (3) implies statement (2) in Theorem 1.1.

Now we show a key proposition for the proof of statement (3) implies statement (2) in Theorem 1.1.

PROPOSITION 2.10. *Let (X, T) be a t.d.s. and U be a subset of X with small boundary. For $x \in X$, we associate an $\hat{x} \in \{0, 1\}^{\mathbb{Z}}$ such that $\hat{x}(n) = 1$ if $T^n x \in U$ and 0 if $T^n x \in X \setminus U$. Then for each $\delta > 0$, we can find $\epsilon := \epsilon(\delta) > 0$ such that $S_N(d, \sigma, \hat{X}, \delta) \leq S_N(d, T, X, \epsilon)$ for $N \in \mathbb{N}$ large enough, where $\hat{X} = \overline{\{\hat{x} : x \in X\}}$ and $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is the left shift.*

Proof. We fix a $\delta > 0$ and a non-empty subset U of X with small boundary. We are to find $\epsilon \in (0, +\infty)$ such that $S_N(d, \sigma, \hat{X}, \delta) \leq S_N(d, T, X, \epsilon)$ for N large enough. To do this, we choose $L \in \mathbb{N}$ and $\delta' > 0$ such that

$$4\delta'L + \frac{2}{2^L} < \delta. \tag{2.30}$$

By Lemma 2.9, there exists $\epsilon := \epsilon(\delta') > 0$ such that for $N \in \mathbb{N}$ large enough and $x_1, x_2 \in X$ with $\bar{d}_N(x_1, x_2) < \epsilon$, one has

$$\sharp\{0 \leq n \leq N - 1 : \hat{x_1}(n) \neq \hat{x_2}(n)\} \leq 2\delta'N. \tag{2.31}$$

Fix $x_1, x_2 \in X$ with $\bar{d}_N(x_1, x_2) < \epsilon$ and put

$$\mathcal{C}_N = \{0 \leq n \leq N - 1 : \hat{x_1}(n + l) \neq \hat{x_2}(n + l) \text{ for some } -L + 1 \leq l \leq L - 1\}.$$

By equation (2.31), we have for $N \in \mathbb{N}$ large enough,

$$\sharp\mathcal{C}_N \leq 4\delta'LN.$$

Notice that $d(\sigma^n \hat{x_1}, \sigma^n \hat{x_2}) \leq 1$ for $n \in \mathcal{C}_N$. One has

$$
\begin{aligned}
\bar{d}_N(\hat{x_1}, \hat{x_2}) &= \frac{1}{N} \left(\sum_{n \in \mathcal{C}_N} d(\sigma^n \hat{x_1}, \sigma^n \hat{x_2}) + \sum_{n \in [0, N-1] \setminus \mathcal{C}_N} d(\sigma^n \hat{x_1}, \sigma^n \hat{x_2}) \right) \\
&\overset{(2.29)}{\leq} \frac{1}{N} \left(\sum_{n \in \mathcal{C}_N} 1 + \sum_{n \in [0, N-1] \setminus \mathcal{C}_N} \frac{2}{2^L} \right) \\
&= \frac{1}{N} \left(\sharp\mathcal{C}_N + \frac{2}{2^L}(N - \sharp\mathcal{C}_N) \right) \\
&\overset{(2.30)}{\leq} 4\delta'L + \frac{2}{2^L} < \delta.
\end{aligned}
$$

Therefore, $S_N(d, \sigma, \hat{X}, \delta) \leq S_N(d, T, X, \epsilon)$ for $N \in \mathbb{N}$ large enough and ϵ is the constant as required. This ends the proof of Proposition 2.10. $\square$

For a t.d.s. (X, T), Lindenstrauss and Weiss [15] introduced the notion of *mean dimension*, denoted by $\mathrm{mdim}(X, T)$. It is well known that for a t.d.s. (X, T), if $h_{\mathrm{top}}(T) < \infty$ or

the topological dimension of X is finite, then $\text{mdim}(X, T) = 0$ (see [**15**, Definition 2.6 and Theorem 4.2]).

Now we are ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1: (3) $\implies$ (2). Assume that Theorem 1.1(3) holds. Now we are going to show that Theorem 1.1(2) holds. Assume the contrary that Theorem 1.1(2) does not hold, then there exists a t.d.s. (X, T) with polynomial mean complexity such that the logarithmic Sarnak conjecture does not hold for (X, T).

Let (Y, S) be an irrational rotation on the circle. Then $(X \times Y, T \times S)$ has polynomial mean complexity as well as zero mean dimension and admits a non-periodic minimal factor (Y, S). Hence, $(X \times Y, T \times S)$ has small boundary property by [**14**, Theorem 6.2]. Since the logarithmic Sarnak conjecture does not hold for (X, T), neither does $(X \times Y, T \times S)$. By Lemma 2.8, there is a subset U of $X \times Y$ with small boundary and $w \in X \times Y$ such that

$$\limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} \mathbf{1}_U \big((T \times S)^n w \big) \mu(n) > 0.$$

Combining this with Proposition 2.10, the $\{0, 1\}$-symbolic system $(\overline{\{\hat{z} : z \in X \times Y\}}, \sigma)$ has polynomial mean complexity and

$$\limsup_{N \to +\infty} \mathbb{E}_{n \leq N}^{\log} F_0(\sigma^n \hat{w}) \mu(n) > 0,$$

where $F_0(\hat{z}) = \hat{z}(0)$ for $z \in X \times Y$, which contradicts the assumption that Theorem 1.1(3) holds. This ends the proof of (3)$\implies$(2) in Theorem 1.1, and hence the proof of Theorem 1.1. $\qquad\square$

3. *Proof of Theorem 1.3*

In this section, we will prove Theorem 1.3. First, we recall the definition of packing dimension. Let X be a metric space endowed with a metric d and E be a subset of X. We say that a collection of balls $\{U_n\}_{n \in \mathbb{N}} \subset X$ is a δ-*packing* of E if the diameter of the balls is not larger than δ, they are pairwise disjoint, and their centers belong to E. For $\alpha \in \mathbb{R}$, the α-*dimensional pre-packing measure of E* is given by

$$P(E, \alpha) = \lim_{\delta \to 0} \sup \left\{ \sum_{n \in \mathbb{N}} \text{diam}(U_n)^\alpha \right\},$$

where the supremum is taken over all δ-packings of E. The α-*dimensional packing measure of E* is defined by

$$p(E, \alpha) = \inf \left\{ \sum_{i \in \mathbb{N}} P(E_i, \alpha) \right\},$$

where the infimum is taken over all covers $\{E_i\}_{i \in \mathbb{N}}$ of E. Finally, we define the *packing dimension of E* by

$$\text{Dim}_P E = \sup\{\alpha : p(E, \alpha) = +\infty\} = \inf\{\alpha : p(E, \alpha) = 0\}.$$

For $x \in [0, 1]$ and $r > 0$, let $B(x, r) = \{y \in [0, 1], |x - y| < r\}$. To prove Theorem 1.3, we need several lemmas. We begin with the following lemma (see [**5**]).

LEMMA 3.1. *Let μ be a Borel probability measure on* $[0, 1]$. *Then,*

$$\mathrm{Dim}^*\mu = \inf\{\mathrm{Dim}_P E : E \subset [0, 1] \text{ with } \mu(E^c) = 0\},$$

where $\mathrm{Dim}^*\mu = \mathrm{ess\ sup\ lim\ sup}_{r \to 0}(\log \mu(B(x, r)))/\log r$.

We also need the following lemma [**17**, Theorem 2.1].

LEMMA 3.2. *Let $\mathcal{B} = \{B(x_i, r_i)\}_{i \in \mathcal{I}}$ be a family of open balls in* $[0, 1]$. *Then there exists a finite or countable subfamily $\mathcal{B}' = \{B(x_i, r_i)\}_{i \in \mathcal{I}'}$ of pairwise disjoint balls in $\mathcal{B}$ such that*

$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{i \in \mathcal{I}'} B(x_i, 5r_i).$$

Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. Recall that $e(t) = e^{2\pi t}$ for any $t \in \mathbb{R}$. We will prove the following lemma by using Lemmas 3.1 and 3.2. Define a metric d on $[0, 1] \times \mathbb{T}$ such that $d((x_1, z_1), (x_2, z_2)) = \max\{|x_1 - x_2|, |z_1 - z_2|\}$ for $(x_1, z_1), (x_2, z_2) \in [0, 1] \times \mathbb{T}$.

LEMMA 3.3. *Let C be a compact subset of* $[0, 1]$ *with $\mathrm{Dim}_P C < \tau$ for some given $\tau > 0$. Then the t.d.s. $T : C \times \mathbb{T} \to C \times \mathbb{T}$ defined by $T(x, e(y)) = (x, e(y + x))$ satisfies for any $\rho \in \mathcal{M}(C \times \mathbb{T}, T)$ and any $\epsilon > 0$,*

$$\liminf_{n \to +\infty} \frac{S_n(d, T, \rho, \epsilon)}{n^\tau} = 0.$$

Proof. Fix a constant τ_0 with $\mathrm{Dim}_P C < \tau_0 < \tau$. For a given $\rho \in \mathcal{M}(C \times \mathbb{T}, T)$, let m be the projection of ρ onto the first coordinate. Fix $\epsilon \in (0, 1)$. To prove Lemma 3.3, it suffices to demonstrate

$$\liminf_{n \to +\infty} \frac{S_n(d, T, \rho, \epsilon)}{n^\tau} = 0.$$

First we note that $m(C) = 1$. Using Lemma 3.1, one has $\mathrm{Dim}^*m < \tau_0$ and there exist a subset $\widetilde{C}$ of C and a constant $r_\epsilon \in (0, 1)$ such that:
(1) $\widetilde{C}$ is compact and $m(\widetilde{C}) > 1 - \epsilon$;
(2) $m(B(x, r)) > r^{\tau_0}$ for $0 < r \leq r_\epsilon$ and $x \in \widetilde{C}$.

For any given integer $n > \epsilon/10r_\epsilon$, set $\mathcal{B}_n = \{B(x, \epsilon/10n)\}_{x \in \widetilde{C}}$. By Lemma 3.2, there exist pairwise disjoint balls $\mathcal{B}'_n = \{B(x_i, \epsilon/10n)\}_{i \in \mathcal{I}_n}$ in $\mathcal{B}$ such that

$$\widetilde{C} \subset \bigcup_{i \in \mathcal{I}'_n} B\left(x_i, \frac{\epsilon}{2n}\right).$$

Since $\epsilon/10n < r_\epsilon$, one deduces that

$$m\left(B\left(x, \frac{\epsilon}{10n}\right)\right) > \left(\frac{\epsilon}{10n}\right)^{\tau_0} \quad \text{for all } x \in \widetilde{C}.$$

Therefore, $\mathcal{I}_n$ is finite since elements in $\mathcal{B}'_n$ are pairwise disjoint. Precisely,

$$\sharp \mathcal{I}_n \leq \left(\frac{10n}{\epsilon}\right)^{\tau_0}.$$

Now we put

$$E_\epsilon = \left\{\left(x_i, e\left(\frac{\epsilon j}{4\pi}\right)\right) : i \in \mathcal{I}_n \text{ and } j \in \left\{0, 1, \ldots, \left[\frac{4\pi}{\epsilon}\right]\right\}\right\},$$

where $[4\pi/\epsilon]$ is the integer part of $4\pi/\epsilon$. Then, for $n > \epsilon/10 r_\epsilon$, it is not hard to verify that

$$B_{\tilde{d}_n}\left(\left(x_i, e\left(\frac{\epsilon j}{4\pi}\right)\right), \epsilon\right) \supset B\left(x_i, \frac{\epsilon}{2n}\right) \times \left\{e(t) : \left|t - \frac{\epsilon j}{4\pi}\right| < \frac{\epsilon}{4\pi}\right\}$$

for $i \in \mathcal{I}_n$ and $j \in \{0, 1, \ldots, [4\pi/\epsilon]\}$. This implies that for $n > \epsilon/10 r_\epsilon$, one has

$$\rho\left(\bigcup_{y \in E_\epsilon} B_{\tilde{d}_n}(y, \epsilon)\right) \geq \rho\left(\bigcup_{i \in \mathcal{I}_n} B\left(x_i, \frac{\epsilon}{2n}\right) \times \mathbb{T}\right) = m\left(\bigcup_{i \in \mathcal{I}_n} B\left(x_i, \frac{\epsilon}{2n}\right)\right)$$

$$\geq m(\widetilde{C}) \geq 1 - \epsilon,$$

and

$$S_n(d, T, \rho, \epsilon) \leq \sharp E_\epsilon \leq \sharp \mathcal{I}_n \times \frac{4\pi}{\epsilon} \leq \left(\frac{10n}{\epsilon}\right)^{\tau_0} \times \frac{4\pi}{\epsilon}.$$

By the fact $\tau_0 < \tau$, one has

$$\liminf_{n \to +\infty} \frac{S_n(d, T, \rho, \epsilon)}{n^\tau} = 0.$$

This ends the proof of Lemma 3.3. $\qquad\square$

Now let $p = (0, 0)$ be the origin of $\mathbb{C}$. For a sequence $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$, let

$$\text{Gen}(y) = \left\{\mu \in \mathcal{M}((\mathbb{T} \cup \{p\})^{\mathbb{Z}}, \sigma) : \frac{1}{N_i - M_i} \sum_{M_i < n \leq N_i} \delta_{\sigma^n y} \to \mu \text{ for } N_i - M_i \to +\infty\right\},$$

where $\sigma : (\mathbb{T} \cup \{p\})^{\mathbb{Z}} \to (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$ is the left shift. Put $X_y = \overline{\{\sigma^n y : n \in \mathbb{Z}\}}$. Then (X_y, σ) is a subsystem of $((\mathbb{T} \cup \{p\})^{\mathbb{Z}}, \sigma)$. It is not hard to see that for $\mu \in \text{Gen}(y)$, $\mu(X_y) = 1$, and thus we can identify $\text{Gen}(y)$ with $\mathcal{M}(X_y, \sigma)$. We have

LEMMA 3.4. *Let C be a non-empty compact subset of $[0, 1]$ and $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$. Assume that the pair (y, C) meets the following property.*

Property $(\ast)$—there exist $\{m_1 < n_1 < m_2 < n_2 \ldots\} \subset \mathbb{Z}$, $\{\theta_k\}_{k \geq 1} \subset C$, and $\{\phi_k\}_{k \geq 1} \subset [0, 1]$ such that:

(1) $\lim_{i \to \infty} n_i - m_i = +\infty$;
(2) $y(j) = p$ for $j \in \mathbb{Z} \setminus \bigcup_{i \in \mathbb{N}}[m_i, n_i)$;
(3) $y(m_i + j) = e(\phi_i + j\theta_i)$ for all $i \geq 1$ and $0 \leq j < n_i - m_i$.

Then, any element in $\text{Gen}(y)$ supports on the compact subset

$$\widetilde{C} = \{(ze(i\theta))_{i \in \mathbb{Z}} \in \mathbb{T}^{\mathbb{Z}} : \theta \in C, z \in \mathbb{T}\} \cup \{p\}^{\mathbb{Z}}.$$

Proof. Assume that (y, C) meets Property $(*)$ and set

$$Z = \{z \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}} : z(-1) = p, z(0) \in \mathbb{T}\}.$$

It is clear that $X_y \setminus \bigcup_{n \in \mathbb{Z}} \sigma^n Z \subset \widetilde{C}$. To prove the lemma, it is enough to show that $\mu(\widetilde{C}) = 1$ for all $\mu \in \mathrm{Gen}(y)$. Since $\mathrm{Gen}(y) = \mathcal{M}(X_y, \sigma)$, it is enough to show that $\mu(Z) = 0$ for all $\mu \in \mathrm{Gen}(y)$.

Now we fix a $\mu \in \mathrm{Gen}(y)$. Then there exist $M_1 < N_1, M_2 < N_2, \ldots$ such that $\lim_{i \to +\infty} N_i - M_i = +\infty$ and

$$\lim_{i \to +\infty} \frac{1}{N_i - M_i} \sum_{M_i < n \le N_i} \delta_{\sigma^n y} = \mu.$$

Since Z is an open subset of $(\mathbb{T} \cup \{p\})^{\mathbb{Z}}$, we have

$$\begin{aligned}
\mu(Z) &\le \liminf_{i \to +\infty} \frac{1}{N_i - M_i} \sum_{M_i < n \le N_i} \delta_{\sigma^n y}(Z) \\
&= \liminf_{i \to +\infty} \frac{\sharp\{M_i < n \le N_i : \sigma^n y \in Z\}}{N_i - M_i} \\
&= \liminf_{i \to +\infty} \frac{\sharp\{M_i < n \le N_i : y(n-1) = p, y(n) \in \mathbb{T}\}}{N_i - M_i} \\
&= \liminf_{i \to +\infty} \frac{\sharp\{j \in \mathbb{N} : M_i < m_j \le N_i\}}{N_i - M_i} = 0,
\end{aligned}$$

where the last equality follows from Property $(*)(1)$. This ends the proof of Lemma 3.4. □

The next lemma follows easily from the previous ones.

LEMMA 3.5. *Assume that C is a non-empty compact subset of $[0, 1]$ with $\mathrm{Dim}_P C < \tau$ and $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$. If (y, C) meets Property $(*)$ as in Lemma 3.4, then the t.d.s. (X_y, σ) satisfies*

$$\liminf_{n \to +\infty} \frac{S_n(d, T, \rho, \epsilon)}{n^\tau} = 0$$

for all $\epsilon > 0$ and $\rho \in \mathcal{M}(X_y, \sigma)$.

Proof. Fix a pair (y, C) which meets Property $(*)$ as in Lemma 3.4. Then all measures in $\mathrm{Gen}(y)$ support on a compact set,

$$\widetilde{C} = \{(ze(i\theta))_{i \in \mathbb{Z}} \in \mathbb{T}^{\mathbb{Z}} : \theta \in C, z \in \mathbb{T}\} \cup \{p\}^{\mathbb{Z}}.$$

It is clear that $\widetilde{C}$ is a σ-invariant compact subset of $(\mathbb{T} \cup \{p\})^{\mathbb{Z}}$, that is, $(\widetilde{C}, \sigma)$ is a t.d.s. Notice that $(\widetilde{C}, \sigma)$ is a factor of $(C \times \mathbb{T} \cup \{p\}, T)$, where $T : C \times \mathbb{T} \cup \{p\} \to C \times \mathbb{T} \cup \{p\}$ with $T(p) = p$ and $T(x, e(y)) = (x, e(y + x))$ for $(x, e(y)) \in C \times \mathbb{T}$. The lemma is immediately from Lemma 3.3. □

The final lemma we need is the following one.

LEMMA 3.6. *If there exist a non-empty compact subset C of $[0, 1]$ and $\beta \in \mathbb{R}$ such that*

$$\limsup_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}^{\log}_{n \leq N} \max \left\{ \mathrm{Re} \left(\sup_{\alpha \in C} e(\beta) \mathbb{E}_{h \leq H} \mu(n + h) e(h\alpha) \right), 0 \right\} > 0, \quad (3.1)$$

then there is $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$ such that (y, C) meets Property $(*)$ as in Lemma 3.4 and

$$\limsup_{N \to \infty} |\mathbb{E}^{\log}_{n \leq N} \mu(n) \widetilde{F}(\sigma^n y)| > 0, \quad (3.2)$$

where $\widetilde{F} : (\mathbb{T} \cup \{p\})^{\mathbb{Z}} \to \mathbb{C}$ is the continuous function defined by $\widetilde{F}(z) = z(0)$ if $z(0) \in \mathbb{T}$ and 0 if $z(0) = p$.

Proof. By the assumption in equation (3.1) and the similar arguments as in the proof of Theorem 1.1 (2) $\implies$ (1), we can find $\tau \in (0, 1)$, strictly increasing sequences $\{H_i\}_{i \in \mathbb{N}}, \{N_i\}_{i \in \mathbb{N}}$ of natural numbers, series $\{\alpha_{i,j}\}_{j=1}^{N_i} \subset \mathbb{R}$, $i = 1, 2, 3 \ldots$, and $\beta \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$ such that for each $i \in \mathbb{N}$, one has

$$H_i < \sigma N_i^{\sigma} < \frac{\sigma}{10} H_{i+1}^{\sigma} \quad \text{where } \sigma = \frac{\tau^2}{200} \quad (3.3)$$

and

$$\mathbb{E}^{\log}_{n \leq N_i} \max\{\mathrm{Re}(e(\beta) \mathbb{E}_{h \leq H_i} \mu(n + h) e(h\alpha_{n,i})), 0\} > \tau. \quad (3.4)$$

For $i \in \mathbb{N}$, let $M_i = \sum_{n=1}^{N_i} (1/n)$ and

$$S_i = \left\{ n \in [1, N_i] : \mathrm{Re}(e(\beta) \mathbb{E}_{h \leq H_i} \mu(n + h) e(h\alpha_{n,i})) > \frac{\tau}{2} \right\}. \quad (3.5)$$

Then by equation (3.4), one has

$$\sum_{n \in S_i} \frac{1}{n} > \frac{\tau}{2} M_i. \quad (3.6)$$

Notice that $\lim_{N \to +\infty} (\sum_{n \leq N^{\sigma}} (1/n) / \sum_{n \leq N} (1/n)) = \sigma$. We have

$$\sum_{n \in S_i \setminus [1, N_i^{\sigma}]} \frac{1}{n} \overset{(3.6)}{>} \frac{\tau}{2} M_i - \sum_{n \leq N_i^{\sigma}} \frac{1}{n} > \frac{\tau}{2} M_i - 2\sigma M_{i-1} \overset{(3.3)}{>} \frac{\tau}{4} M_i$$

for $i \in \mathbb{N}$ large enough. Then we can choose $S_i' \subset S_i \setminus [1, N_i^{\sigma}]$ such that each gap in S_i' is not less than $2H_i$ and

$$\sum_{n \in S_i'} \frac{1}{n} > \frac{\tau M_i}{8 H_i} \quad (3.7)$$

for $i \in \mathbb{N}$ large enough. Define $y : \mathbb{Z} \to \mathbb{T} \cup \{p\}$ such that

$$y(j) = e((j - n)\alpha_{n,i}) \quad \text{if } j \in [n + 1, n + H_i] \quad \text{for some } i \geq 1 \quad \text{and} \quad n \in S_i',$$

and $y(j) = p$ for other j, where p is the zero of $\mathbb{C}$. It is not hard to see that y is well defined and meets Property $(*)$.

Now we are going to show that equation (3.2) holds. Combining equations (3.5) with (3.7), one has

$$\mathrm{Re}\left(e(\beta)\sum_{n\in S_i'}\sum_{h\leq H_i}\frac{\mu(n+h)\widetilde{F}(\sigma^{n+h}y)}{n}\right) > \frac{\tau}{2}\times H_i\times\sum_{n\in S_i'}\frac{1}{n} > \frac{\tau^2}{16}M_i \qquad (3.8)$$

for $i\in\mathbb{N}$ large enough. Then,

$$\left|\sum_{n\in S_i'}\sum_{h\leq H_i}\frac{\mu(n+h)\widetilde{F}(\sigma^{n+h}y)}{n}-\sum_{n\in S_i'}\sum_{h\leq H_i}\frac{\mu(n+h)\widetilde{F}(\sigma^{n+h}y)}{n+h}\right|$$

$$\leq\sum_{n\in S_i'}\sum_{h\leq H_i}\left(\frac{1}{n}-\frac{1}{n+h}\right)\leq\sum_{n\in S_i'}\sum_{h\leq H_i}\frac{H_i}{n(n+H_i)}$$

$$\leq\sum_{n\in S_i'}\frac{H_i}{nN_i^{\sigma}}\overset{(3.3)}{\leq}\sigma\sum_{n\in S_i'}\frac{1}{n}\overset{(3.3)}{\leq}\frac{\tau^2}{32}M_i$$

for $i\in\mathbb{N}$ large enough. Combining this inequality with equation (3.8), one has

$$\left|\sum_{N_i^{\sigma}<n\leq N_i+H_i}\frac{\mu(n)\widetilde{F}(\sigma^n y)}{n}\right|=\left|\sum_{n\in S_i'}\sum_{h\leq H_i}\frac{\mu(n+h)\widetilde{F}(\sigma^{n+h}y)}{n+h}\right|$$

$$\geq\mathrm{Re}\left(e(\beta)\sum_{n\in S_i'}\sum_{h\leq H_i}\frac{\mu(n+h)y(n+h)}{n}\right)-\frac{\tau^2 M_i}{32}$$

$$\geq\frac{\tau^2 M_i}{32}$$

for $i\in\mathbb{N}$ large enough. Thus,

$$|\mathbb{E}_{n\leq N_i}^{\log}\mu(n)\widetilde{F}(\sigma^n y)|\geq\left|\frac{1}{M_i}\sum_{n\leq N_i+H_i}\frac{\mu(n)\widetilde{F}(\sigma^n y)}{n}\right|-\frac{1}{M_i}\sum_{\substack{N_i<n\leq N_i+H_i\text{ or}\\n\leq N_i^{\sigma}}}\left|\frac{\mu(n)\widetilde{F}(\sigma^n y)}{n}\right|$$

$$\geq\frac{\tau^2}{32}-2\sigma-\frac{H_i}{N_i}\overset{(3.3)}{\geq}\frac{\tau^2}{100}>0$$

for $i\in\mathbb{N}$ large enough. Therefore, y is the point as required. This ends the proof of Lemma 3.6. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Assume that Theorem 1.3 is not valid. Then there exists a non-empty compact subset C of $[0,1]$ with $\mathrm{Dim}_P C<1$ such that

$$\limsup_{H\to+\infty}\limsup_{N\to+\infty}\mathbb{E}_{n\leq N}^{\log}\sup_{\alpha\in C}|\mathbb{E}_{h\leq H}\mu(n+h)e(h\alpha)|>0.$$

Thus, we can find $\beta\in\{0,\frac{1}{4},\frac{2}{4},\frac{3}{4}\}$ such that

$$\limsup_{H\to+\infty}\limsup_{N\to+\infty}\mathbb{E}_{n\leq N}^{\log}\max\left\{\sup_{\alpha\in C}\mathrm{Re}(e(\beta)\mathbb{E}_{h\leq H}\mu(n+h)e(h\alpha)),0\right\}>0.$$

By Lemma 3.6, there is $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$ such that (y, C) meets Property $(*)$ as in Lemma 3.4 and

$$\limsup_{N \to \infty} |\mathbb{E}^{\log}_{n \leq N} \mu(n) \widetilde{F}(\sigma^n y)| > 0, \tag{3.9}$$

where $\widetilde{F} : X_y \to \mathbb{R}$ is a continuous function defined by $\widetilde{F}(z) = z(0)$ if $z(0) \in \mathbb{T}$ and 0 if $z(0) = p$. Then, by Lemma 3.5 and the assumption $\mathrm{Dim}_P C < 1$, the t.d.s. (X_y, σ) satisfies

$$\liminf_{n \to +\infty} \frac{S_n(d, \sigma, \rho, \epsilon)}{n} = 0 \quad \text{for any } \epsilon > 0 \quad \text{and} \quad \rho \in \mathcal{M}(X_y, \sigma).$$

By Theorem 1.2,

$$\lim_{N \to \infty} \mathbb{E}^{\log}_{n \leq N} \mu(n) \widetilde{F}(\sigma^n y) = 0.$$

This conflicts with equation (3.9) and the theorem follows. We end the proof of Theorem 1.3. $\qquad \square$

A. *Appendix. Proof of Theorem 1.2*

In this appendix, we prove Theorem 1.2 following the arguments of the proof of [11, Theorem 1.1'].

Let (X, T) be a t.d.s. with a metric d and sub-linear mean measure complexity. To prove that the logarithmic Sarnak conjecture holds for (X, T), it is sufficient to show

$$\limsup_{i \to +\infty} \left| \frac{1}{\sum_{n=1}^{N_i} (1/n)} \sum_{n=1}^{N_i} \frac{\mu(n) f(T^n x)}{n} \right| < 7\epsilon \tag{A.1}$$

for any $\epsilon \in (0, 1)$ and $f \in C(X)$ with $\max_{z \in X} |f(z)| \leq 1$, $x \in X$ and $\{N_1 < N_2 < N_3 < \cdots\} \subseteq \mathbb{N}$ such that the sequence $\mathbb{E}^{\log}_{n \leq N_i} \delta_{T^n x}$ weakly* converges to a Borel probability measure ρ.

To this aim, we will find $L \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_m\} \subset X$ and $j_n \in \{1, 2, \ldots, m\}$ for $n = 1, 2, 3, \ldots$ such that that for large i,

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{\mu(n) f(T^n x)}{n} - \frac{1}{M_i} \sum_{n=1}^{N_i} \left(\frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^\ell x_{j_n})}{n} \right) \right| < 5\epsilon \tag{A.2}$$

and

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \left(\frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^\ell x_{j_n})}{n} \right) \right| < 2\epsilon. \tag{A.3}$$

It is clear that equation (A.1) follows by equations (A.2) and (A.3). Equations (A.2) and (A.3) will be proved in Lemmas A.1 and A.2) respectively, where we write $M_i = \sum_{n=1}^{N_i} (1/n)$ for $i \in \mathbb{N}$.

To prove the two lemmas, we first choose $\epsilon_1 > 0$ such that $\epsilon_1 < \epsilon^2$ and

$$|f(y) - f(z)| < \epsilon \quad \text{when } d(y, z) < \sqrt{\epsilon_1}. \tag{A.4}$$

Since $\mathbb{E}^{\log}_{n \leq N_i} \delta_{T^n x}$ weakly* converges to ρ, it is not hard to verify $\rho \in \mathcal{M}(X, T)$. So, the measure complexity of (X, d, T, ρ) is sub-linear by the assumption of the theorem, and thus there exists $L > 0$ such that

$$m = S_L(d, T, \rho, \epsilon_1) < \epsilon L. \tag{A.5}$$

This means that there exist $x_1, x_2, \ldots, x_m \in X$ such that

$$\rho\Big(\bigcup_{i=1}^{m} B_{\overline{d}_L}(x_i, \epsilon_1)\Big) > 1 - \epsilon_1 > 1 - \epsilon^2.$$

Put $U = \bigcup_{i=1}^{m} B_{\overline{d}_L}(x_i, \epsilon_1)$ and $E = \{n \in \mathbb{N} : T^n x \in U\}$. Then U is open and so

$$\liminf_{i \to +\infty} \frac{1}{M_i} \sum_{n \in E \cap [1, N_i]} \frac{1}{n} = \liminf_{i \to +\infty} \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{\delta_{T^n x}(U)}{n} \geq \rho(U) > 1 - \epsilon_1. \tag{A.6}$$

For $n \in E$, we choose $j_n \in \{1, 2, \ldots, m\}$ such that $T^n x \in B_{\overline{d}_L}(x_{j_n}, \epsilon_1)$. Hence, for $n \in E$, we have $\overline{d}_L(T^n x, x_{j_n}) < \epsilon_1$, that is,

$$\frac{1}{L} \sum_{\ell=0}^{L-1} d\big(T^\ell(T^n x), T^\ell(x_{j_n})\big) < \epsilon_1,$$

and so we have

$$\#\{\ell \in [0, L-1] : d(T^\ell(T^n x), T^\ell x_{j_n}) \geq \sqrt{\epsilon_1}\} < L\sqrt{\epsilon_1} < L\epsilon. \tag{A.7}$$

Thus, for $n \in E$,

$$\begin{aligned}
\frac{1}{L} \sum_{\ell=0}^{L-1} &|f(T^\ell(T^n x)) - f(T^\ell x_{j_n})| \\
&\leq \frac{1}{L}\big(\epsilon\#\{\ell \in [0, L-1] : d(T^\ell(T^n x), T^\ell x_{j_n}) < \sqrt{\epsilon_1}\} \\
&\quad + 2\#\{\ell \in [0, L-1] : d(T^\ell(T^n x), T^\ell x_{j_n}) \geq \sqrt{\epsilon_1}\}\big) \\
&< 3\epsilon,
\end{aligned} \tag{A.8}$$

by using the inequality in equation A.4, equation (A.7), and the assumption $\max_{x \in X} |f(x)| \leq 1$.

For each $n \notin E$, we simply set $j_n = 1$.

We first establish Lemma A.1.

LEMMA A.1. *For all sufficiently large i,*

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{\mu(n) f(T^n x)}{n} - \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n + \ell) f(T^\ell x_{j_n})}{n} \right| < 5\epsilon.$$

Proof. As $\max_{x \in X} |f(x)| \leq 1$, it is not hard to see that

$$\limsup_{i \to +\infty} \left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{\mu(n) f(T^n x)}{n} - \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^{n+\ell} x)}{n} \right| = 0. \quad (A.9)$$

By equation (A.6), once i is large enough,

$$\frac{1}{M_i} \sum_{n \in E \cap [1,N_i]} \frac{1}{n} > 1 - \epsilon^2 > 1 - \epsilon. \quad (A.10)$$

Now,

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^{n+\ell} x)}{n} - \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^\ell x_{j_n})}{n} \right|$$

$$\leq \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{|f(T^\ell(T^n x)) - f(T^\ell x_{j_n})|}{n}$$

$$\leq \frac{1}{M_i} \sum_{n \in [1,N_i] \setminus E} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{|f(T^\ell(T^n x)) - f(T^\ell x_{j_n})|}{n}$$

$$+ \frac{1}{M_i} \sum_{n \in E \cap [1,N_i]} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{|f(T^\ell(T^n x)) - f(T^\ell x_{j_n})|}{n}$$

$$< \frac{2}{M_i} \sum_{n \in [1,N_i] \setminus E} \frac{1}{n} + \frac{3\epsilon}{M_i} \sum_{n \in E \cap [1,N_i]} \frac{1}{n} \quad \text{(by equation (A.8))}$$

$$< \frac{2}{M_i} \sum_{n \in [1,N_i] \setminus E} \frac{1}{n} + 3\epsilon.$$

Combining this inequality with equation (A.10), when i is large enough,

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^{n+\ell} x)}{n} - \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^\ell x_{j_n})}{n} \right|$$

$$< 5\epsilon. \quad (A.11)$$

So the lemma follows by equations (A.9) and (A.11). This ends the proof of Lemma A.1. □

Now we proceed to show Lemma A.2.

LEMMA A.2. *For all sufficiently large i,*

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell) f(T^\ell x_{j_n})}{n} \right| < 2\epsilon.$$

Proof. By Cauchy's inequality,

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell)f(T^\ell x_{j_n})}{n} \right|^2$$

$$\leq \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{n} \left| \frac{1}{L} \sum_{\ell=0}^{L-1} \mu(n+\ell)f(T^\ell x_{j_n}) \right|^2$$

$$\leq \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{n} \sum_{j=1}^{m} \left| \frac{1}{L} \sum_{\ell=0}^{L-1} \mu(n+\ell)f(T^\ell x_j) \right|^2$$

$$\leq \frac{1}{L^2} \sum_{j=1}^{m} \sum_{\ell_1=0}^{L-1} \sum_{\ell_2=0}^{L-1} \frac{f(T^{\ell_1}x_j)\overline{f}(T^{\ell_2}x_j)}{M_i} \sum_{n=1}^{N_i} \frac{\mu(n+\ell_1)\mu(n+\ell_2)}{n}.$$

Note that $M_i \approx \log N_i$. Since the two-term logarithmic Chowla conjecture holds [21], we have

$$\lim_{i\to\infty} \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{\mu(n+\ell_1)\mu(n+\ell_2)}{n} = 0$$

for any $0 \leq \ell_1 \neq \ell_2 \leq L-1$. Combining this equality with the fact that $\max_{x\in X} |f(x)| \leq 1$, one has that for sufficiently large i,

$$\left| \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{L} \sum_{\ell=0}^{L-1} \frac{\mu(n+\ell)f(T^\ell x_{j_n})}{n} \right|^2$$

$$< \epsilon + \sum_{j=1}^{m} \frac{1}{L^2} \sum_{\ell=0}^{L-1} \sum_{j=1}^{m} \frac{|f(T^\ell x_j)\overline{f}(T^\ell x_j)|}{M_i} \sum_{n=1}^{N_i} \frac{|\mu(n+\ell)\mu(n+\ell)|}{n}$$

$$\leq \epsilon + \frac{m}{L^2} \sum_{l=0}^{L-1} \frac{1}{M_i} \sum_{n=1}^{N_i} \frac{1}{n}$$

$$= \epsilon + \frac{m}{L}$$

$$\overset{(A.5)}{<} 2\epsilon.$$

This ends the proof of Lemma A.2.   □

B. *Appendix. Proof of Theorem 1.4*

In this appendix, we prove Theorem 1.4. As in the proof of Theorem 1.3, we let p be the zero of $\mathbb{C}$. For a sequence $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$, we put $X_y = \overline{\{\sigma^n y : n \in \mathbb{Z}\}}$, where σ is the left shift. To this aim, we give a lemma first.

LEMMA B.1. *If there exist a non-empty compact subset C of $[0, 1]$ and $\beta \in \mathbb{R}$ such that*

$$\limsup_{H\to+\infty} \limsup_{N\to+\infty} \mathbb{E}_{n\leq N} \max \left\{ \sup_{\alpha\in C} \mathrm{Re}(e(\beta)\mathbb{E}_{h\leq H}\mu(n+h)e(h\alpha)), 0 \right\} > 0, \qquad (B.1)$$

then there is $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$ such that (y, C) meets Property $(*)$ in Lemma 3.4 and

$$\limsup_{N \to \infty} |\mathbb{E}_{n \le N} \mu(n) \widetilde{F}(\sigma^n y)| > 0, \tag{B.2}$$

where $\widetilde{F} : X_y \to \mathbb{C}$ is a continuous function defined by $\widetilde{F}(z) = z(0)$ if $z(0) \in \mathbb{T}$ and 0 if $z(0) = p$.

Proof. It follows by a similar arguments of the proof of Lemma 3.4. □

Now we are going to prove Theorem 1.4.

Proof of Theorem 1.4. Assume the contrary that Theorem 1.4 does not hold. Then there exists a non-empty compact subset C of $[0, 1]$ such that $\mathrm{Dim}_P C = 0$ and

$$\limsup_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \le N} \sup_{\alpha \in C} |\mathbb{E}_{h \le H} \mu(n + h) e(h\alpha)| > 0.$$

Thus, there is $\beta \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$ with

$$\limsup_{H \to +\infty} \limsup_{N \to +\infty} \mathbb{E}_{n \le N} \max \left\{ \sup_{\alpha \in C} \mathrm{Re}(e(\beta) \mathbb{E}_{h \le H} \mu(n + h) e(h\alpha)), 0 \right\} > 0.$$

By Lemma B.1, there is $y \in (\mathbb{T} \cup \{p\})^{\mathbb{Z}}$ such that (y, C) meets Property $(*)$ in Lemma 3.4 and

$$\limsup_{N \to \infty} |\mathbb{E}_{n \le N} \mu(n) \widetilde{F}(\sigma^n y)| > 0, \tag{B.3}$$

where $\widetilde{F} : X_y \to \mathbb{R}$ is a continuous function defined by $\widetilde{F}(z) = z(0)$ if $z(0) \in \mathbb{T}$ and 0 if $z(0) = p$. By Lemma 3.5, the t.d.s. (X_y, σ) satisfies

$$\liminf_{n \to +\infty} \frac{S_n(d, \sigma, \rho, \epsilon)}{n^\tau} = 0 \quad \text{for any } \epsilon > 0, \tau > 0 \quad \text{and} \quad \rho \in \mathcal{M}(X_y, \sigma),$$

since $\mathrm{Dim}_P C = 0$. Using the result of [11], one has

$$\limsup_{N \to \infty} |\mathbb{E}_{n \le N} \mu(n) \widetilde{F}(\sigma^n y)| = 0.$$

This conflicts with equation (B.3) and the theorem follows. This ends the proof of Theorem 1.4. □

References

[1] L. J. Corwin and F. P. Greenleaf. *Representations of Nilpotent Lie Groups and their Applications. Part I. Basic Theory and Examples (Cambridge Studies in Advanced Mathematics, 18)*. Cambridge University Press, Cambridge, 1990.

[2] H. Davenport. On some infinite series involving arithmetical functions II. *Q. J. Math.* **8**(1937), 313–320.

[3] P. Dong, S. Donoso, A. Maass, S. Shao and X. Ye. Infinite-step nilsystems, independence and complexity. *Ergod. Th. & Dynam. Sys.* **33**(1) (2013), 118–143.

[4] E. H. El Abdalaoui, J. Kułaga-Przymus, M. Lemańczyk and T. de la Rue. Möbius disjointness for models of an ergodic system and beyond. *Israel J. Math.* **228** (2018), 707–751.

[5] A. Fan, K. Lau and H. Rao. Relationships between different dimensions of a measure. *Monatsh. Math.* **135** (2002), 191–201.

[6] S. Ferenczi, J. Kułaga-Przymus and M. Lemańczyk. Sarnak's conjecture: what's new. *Ergodic Theory and Dynamical Systems in Their Interactions with Arithmetics and Combinatorics, CIRM Jean-Morlet Chair, Fall 2016 (Lecture Notes in Mathematics, 2213)*. With a forward by P. Sarnak. Eds. S. Ferenczi, J. Kułaga-Przymus and M. Lemańczyk. Springer, Cham, 2018, pp. 163–235.

[7] N. Frantzikinakis and B. Host. The logarithmic Sarnak conjecture for ergodic weights. *Ann. of Math. (2)* **187**(3) (2018), 869–931.

[8] B. Green and T. Tao. The quantitative behaviour of polynomial orbits on nilmanifolds. *Ann. of Math. (2)* **175**(2) (2012), 465–540.

[9] B. Green, T. Tao and T. Ziegler. An inverse theorem for the Gowers $U^{s+1}[N]$-norm. *Ann. of Math. (2)* **176**(2) (2012), 1231–1372.

[10] B. Host and B. Kra. *Nilpotent Structures in Ergodic Theory (Mathematical Surveys and Monographs, 236)*. American Mathematical Society, Providence, RI, 2018.

[11] W. Huang, Z. Wang and X. Ye, Measure complexity and Möbius disjointness. *Adv. Math.* **347**(2019), 827–858.

[12] A. Kanigowski, M. Lemańczyk and M. Radziwiłł. Rigidity in dynamics and Möbius disjointness. *Fund. Math.* **255**(3) (2021), 309–336.

[13] J. Kułaga-Przymus and M. Lemańczyk. Sarnak's conjecture from the ergodic theory point of view. *Encyclopedia of Complexity and Systems Science*, to appear. *Preprint*, arXiv:2009.04757, 2020.

[14] E. Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Publ. Math. Inst. Hautes Études Sci.* **89** (1999), 227–262.

[15] E. Lindenstrauss and B. Weiss. Mean topological dimension. *Israel J. Math.* **115** (2000), 1–24.

[16] A. I. Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Transl.* **1951**(39) (1951), 33 pp.

[17] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability (Cambridge Studies in Advanced Mathematics, 44)*. Cambridge University Press, Cambridge, 1995.

[18] R. McNamara. Sarnak's conjecture for sequences of almost quadratic word growth. *Ergod. Th. & Dynam. Sys.* **41**(10) (2021), 3060–3115.

[19] P. Sarnak. Möbius randomness and dynamics. *Not. S. Afr. Math. Soc.* **43**(2) (2012), 89–97.

[20] M. Shub and B. Weiss. Can one always lower topological entropy? *Ergod. Th. & Dynam. Sys.* **11**(3) (1991), 535–546.

[21] T. Tao. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum Math. Pi* **4** (2016), e8, 36 pp.

[22] T. Tao. Equivalence of the logarithmically averaged Chowla and Sarnak conjectures. *Number Theory-Diophantine Problems, Uniform Distribution and Applications*. Springer, Cham, 2017, pp. 391–421.

[23] T. Tao and J. Teräväinen. Odd order cases of the logarithmically averaged Chowla conjecture. *J. Théor. Nombres Bordeaux* **30**(3) (2018), 997–1015.

[24] T. Tao and J. Teräväinen. The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures. *Duke Math. J.* **168**(11) (2019), 1977–2027.