Canad. Math. Bull. Vol. **20** (4), 1977

THE METRIZABILITY OF SPACES WHOSE DIAGONALS HAVE A COUNTABLE BASE

BY

JOHN GINSBURG

ABSTRACT. It is shown that the diagonal of X has a countable neighborhood base in $X \times X$ if and only if X is a metrizable space whose set of non-isolated points is compact.

The diagonal of X is the set $\Delta_X = \{(x, x) : x \in X\}$. A family $\mathcal{U}$ of open subsets of $X \times X$ is a *base for* Δ_X in $X \times X$ if every member of $\mathcal{U}$ contains Δ_X and every neighborhood of Δ_X in $X \times X$ contains a member of $\mathcal{U}$.

Based on a recent result due to Chaber [3] on countably compact spaces with G_δ-diagonals, and on Bing's well-known metrization theorem [1], we will give a short proof of the theorem stated in the abstract.

Our topological notation and terminology are standard. We will assume no separation axioms beyond T_2 of our spaces. Interesting results on neighborhoods of the diagonal can be found in Simon's paper [4] which has helped to motivate our proof.

Recall that, if $\mathcal{G}$ is an open cover of X and $p \in X$, then the star of p with respect to $\mathcal{G}$, denoted by $st(p, \mathcal{G})$, is the union of all the members of $\mathcal{G}$ which contain p.

1. LEMMA. *Let X be a Hausdorff space and let $\mathcal{U}$ be a base for the diagonal in $X \times X$. For $U \in \mathcal{U}$, we set $\mathcal{G}_U = \{G : G$ is open in X and $G \times G \subseteq U\}$. For $U \in \mathcal{U}$ and $S \subseteq X$, we set $U(S) = \{x \in X : (s, x) \in U$ for some $s \in S\}$.*

(a) *If A is closed in X then $\{U(A) : U \in \mathcal{U}\}$ is a base for the neighborhoods of A in X.*

(b) *For all $U \in \mathcal{U}$, $\mathcal{G}_U$ is an open cover of X and $st(x, \mathcal{G}_U) \subseteq U(x)$ for all $x \in X$. Thus $\{\mathcal{G}_U : U \in \mathcal{U}\}$ is a development for X.*

(c) *If A is closed in X, then $\{U \cap (A \times A) : U \in \mathcal{U}\} = \mathcal{U}_A$ is a base for the neighborhoods of Δ_A in $A \times A$.*

Proof. (a) If G is open and $A \subseteq G$, then $(G \times G) \cup (X - A) \times (X - A)$ is a neighborhood of the diagonal, so there is a member U of $\mathcal{U}$ such that $U \subseteq (G \times G) \cup (X - A) \times (X - A)$. For such U, clearly $U(A) \subseteq G$.

Received by the editors January 4, 1977 and, in revised form, March 17, 1977.
AMS Subj. Class. 54E35.
Key words and phrases: diagonal, neighborhood base, metrizability, compact.

(b) This result is essentially established in [2], and can be verified in a straight-forward manner. One first shows that $st(x, \mathcal{G}_U) \subseteq U(x)$. The sets $U(x)$ form a base at x by (a), and therefore so do the sets $st(x, \mathcal{G}_U)$. Thus $\{\mathcal{G}_U : U \in \mathcal{U}\}$ is a development for X.

(c) Let W be open in $A \times A$ such that $\Delta_A \subseteq W$. Find an open set W_1 in $X \times X$ such that $W_1 \cap (A \times A) = W$. Then $W_1 \cup (X-A) \times (X-A)$ is a neighborhood of Δ_X in $X \times X$, so there exists $U \in \mathcal{U}$ with $U \subseteq W_1 \cup (X-A) \times (X-A)$. Intersecting both sides of this inclusion with $A \times A$ gives $U \cap (A \times A) \subseteq W$, as desired.

2. THEOREM. *Let X be a Hausdorff space. The diagonal of X has a countable base in $X \times X$ if, and only if X is a metrizable space whose set of non-isolated points is compact.*

Proof. Let I be the set of isolated points of X, and let $\mathcal{U}$ be a base for Δ_X in $X \times X$ such that $|\mathcal{U}| \leq \aleph_0$. By 1.(a), X is first countable. The standard diagonalization argument which shows that the set of integers does not have a countable base in the real line can be easily extended to show that in a first countable space Y, no closed discrete countable set consisting of non-isolated points of Y has a countable base for its neighborhoods in Y. Since, by 1.(a), every closed set in X has a countable base in X, we see that $X-I$ can contain no closed discrete infinite set. That is, $X-I$ is countably compact. By 1.(c), every closed subset of X has a G_δ-diagonal. That is, if A is closed in X, then Δ_A is an intersection of countably many open subsets of $A \times A$. By Chaber's theorem [3], countably compact spaces with G_δ-diagonals are compact. Applying these remarks to the closed set $X-I$, we see that $X-I$ is compact. Now, it is easy to see that a space which is the union of a set of isolated points and a compact set is paracompact, and so X is paracompact. But, by 1.(b), X has a countable development. So X is a paracompact Moore space, and hence a collectionwise normal Moore space. By Bing's theorem [1], X is metrizable. We omit the elementary verification of the converse.

REFERENCES

1. R. H. Bing, *Metrization of topological spaces*, Canadian J. Math. **3** (1951), 175–186.

2. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–126.

3. J. Chaber, *Conditions which imply compactness in countably compact spaces*, to appear.

4. P. Simon, *A note on cardinal invariants of square*, Comment Math. Univ. Carolinae **14** (1973), 205–213.

UNIVERSITY OF MANITOBA,
DEPARTMENT OF MATHEMATICS,
WINNIPEG, MANITOBA.
R3T 2N2