CANONICAL FORMS FOR CERTAIN MATRICES UNDER UNITARY CONGRUENCE

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1. Introduction. If A is a matrix with complex elements and if $A = A^{\mathrm{T}}$ (where A^{T} denotes the transpose of A), there exists a non-singular matrix P such that $PAP^{\mathrm{T}} = D$ is a diagonal matrix (see **(3)**, for example). It is also true (see the principal result of **(5)**) that for such an A there exists a unitary matrix U such that $UAU^{\mathrm{T}} = D$ is a real diagonal matrix with non-negative elements which is a canonical form for A relative to the given U, U^{T} transformation. If $A = -A^{\mathrm{T}}$, it is known (see **(3)** or **(4)**) that there exists a non-singular matrix P such that PAP^{T} is a direct sum of a zero matrix (if present) and of 2×2 blocks of the form:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The present work is concerned with the following. First, a canonical form is obtained for a complex skew-symmetric matrix under a U, U^{T} transformation where U is a unitary complex matrix; this form is analogous to that of the symmetric matrix mentioned above. Thereafter, matrices with real quaternion elements are considered. For such an A the *-transpose (denoted by A^*) is defined and is seen to be a generalization of the transpose (of a complex matrix) for the non-commutative case which at the same time retains the properties of the ordinary transpose in the commutative case. Quaternion matrices of the form $A = A^*$ and $A = -A^*$ are considered, in turn, and results analogous to those mentioned above for complex matrices are obtained which justify this generalization.

2. A normal form for a complex symmetric matrix under unitary congruence. To obtain this form the following is employed:

LEMMA 1. *If A is a complex, unitary, skew-symmetric matrix there exists a complex unitary matrix U such that $UAU^{\mathrm{T}} = E$ is a direct sum of 2×2 matrices of the form*

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It is evident that A must be of even order since it is skew-symmetric and non-singular. Let $A = A_1 + i\,A_2$, where A_1 and A_2 are real matrices, so that $A_1 = -A_1^{\mathrm{T}}$ and $A_2 = -A_2^{\mathrm{T}}$. Since $AA^{\mathrm{CT}} = (A_1+iA_2)(A_1^{\mathrm{T}}-iA_2^{\mathrm{T}}) = I$,

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it follows that $A_1A_1^{\mathrm{T}}+A_2A_2^{\mathrm{T}} = I$ and $A_2A_1^{\mathrm{T}} = A_1A_2^{\mathrm{T}}$. The latter becomes $A_2A_1 = A_1A_2$. By a known theorem (see **(2)**, for example), there exists a real orthogonal matrix T such that $TA_1T^{\mathrm{T}} = E_1$ and $TA_2T^{\mathrm{T}} = E_2$ are direct sums of zeros and 2×2 matrices of the form

$$\text{(i)} \qquad \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

where $a > 0$ is real. Furthermore, it can be shown that, as in the present case, when A_1 and A_2 are both skew-symmetric, E_1 and E_2 can be regarded as conformable direct sums of 2×2 matrices of the above form, of 2×2 zero matrices, and of 1×1 zero matrices in such a way that whenever a single zero element appears in the direct sum of one, it appears in the same diagonal position in the other. (A 2×2 matrix of form (i) in one can correspond to a 2×2 zero matrix in the other, of course.) This may be seen as follows:

The statement is true or there is a first block (in E_1 or E_2) in the direct sum where it is not true; this would mean that there would be corresponding 3×3 diagonal blocks in E_1 and E_2, respectively, of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & -b & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $a \neq 0$ and $b \neq 0$. But since $A_2A_1 = A_1A_2$, the above matrices must commute and they do not. Hence E_1 and E_2 can be considered to be direct sums which are conformable as described above.

Therefore $T(A_1+iA_2)T^{\mathrm{T}} = E_1 + iE_2$ which is unitary (and non-singular); consequently, no 1×1 zero element can appear alone along the diagonal of E_1 and E_2 in the form described for each in the preceding paragraph. Therefore, E_1 and E_2 are each direct sums of 2×2 matrices of form (i) where $a \geqq 0$, so that $E_1+i\,E_2$ is a direct sum of 2×2 blocks of the form

$$\text{(ii)} \qquad E_0 = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$$

where α is non-zero complex. Since $E_1+i\,E_2$ is unitary, $\alpha\bar{\alpha} = 1$. Let $\alpha = e^{i\theta}$ and form the 2×2 unitary matrix

$$V = \begin{bmatrix} 0 & e^{-i\theta/2} \\ -e^{-i\theta/2} & 0 \end{bmatrix}.$$

Then VE_0V^{T} is a matrix of the form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

If S is an appropriate direct sum of such V (determined from each 2×2 matrix in the direct sum $E_1+i\,E_2$), then $ST(A_1+i\,A_2)T^{\mathrm{T}}S^{\mathrm{T}} = E$, the direct

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sum as described in the statement of the lemma, where $U = ST$ is a complex unitary matrix.

THEOREM 1. *If A is a complex skew-symmetric matrix, there exists a complex unitary matrix V such that $VAV^T = E \dotplus 0$ where E is a direct sum of 2×2 matrices of the form*

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix},$$

where $a > 0$ is real; and conversely.

Let $A = HU = UK \neq 0$ be a polar representation of A where H and K are hermitian and U is unitary. (It may be noted that each $a > 0$ described in the statement of the theorem is actually a characteristic root of H or K). Since $A = HU = UK = -A^T = -U^T H^T = -K^T U^T$, and since the hermitian polar matrix H is unique, it follows from $A = HU = -K^T U^T$ that $H = K^T$ or $H = -K^T$ (since $-K^T U^T$ is also a polar form of A). But since K is positive definite, K^T is also, and $H = -K^T$ cannot hold (since H would not be positive definite). Therefore $H = K^T$.

If A, skew-symmetric, is non-singular, it must be of even order; in any event, the rank of A is even. If $A = HU$, the rank of A = the rank of $H = r$, an even number.

For $H = K^T$ let V_1 be a complex unitary matrix such that $V_1 H V_1^{CT} = D = D_0 \dotplus 0$ (where 0 is absent if B is non-singular) where $D_0 = D_1 \dotplus D_2 \dotplus \ldots \dotplus D_k$, where $D_i = d_i I_i$ is a real diagonal scalar matrix, $d_i \neq d_j$ for $i \neq j$, and $d_1 > d_2 > \ldots > d_k > 0$. If A is non-singular, it is known (see (9)) that the polar representation is unique, so that $A = HU = K^T(-U^T)$ implies that $U = -U^T$. If A is singular, this need not be true (8); as a matter of fact, it cannot be true if A is of odd order since U is non-singular.

Consider the case where $A = HU$ is singular. Let $V_1 U V_1^{CT} = W$ and $V_1(-U^T) V_1^{CT} = W_1$; also let $V_1 K V_1^{CT} = V_1 H^T V_1^{CT} = M$. Then from

$$V_1 A V_1^{CT} = V_1 HU V_1^{CT} = V_1 UK V_1^{CT} = V_1(-U^T H^T) V_1^{CT}$$
$$= V_1(-K^T U^T) V_1^{CT}$$

it follows that $V_1 A V_1^{CT} = DW = WM = W_1 M = DW_1$. From $WM = W_1 M$ it follows, in turn, that

$$W(V_1 H^T V_1^{CT}) = W_1(V_1 H^T V_1^{CT}),$$

or

$$W V_1 V_1^T D V_1^C V_1^{CT} = W_1 V_1 V_1^T D V_1^C V_1^{CT},$$

so that $W V_1 V_1^T D = W_1 V_1 V_1^T D$. Since $DW = DW_1$ (and since D has rank r), W and W_1 have like first r rows, and so $W V_1 V_1^T$ and $W_1 V_1 V_1^T$ also have like first r rows; and from the last result in the preceding, $W V_1 V_1^T$ and $W_1 V_1 V_1^T$ also have like first r columns. Let $W V_1 V_1^T$ be of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & X \end{bmatrix}$$

where A_{11} is an $r \times r$ matrix. Since $DW = W_1M = W_1V_1V_1^{\mathrm{T}}DV_1^{\mathrm{C}}V_1^{\mathrm{CT}}$, therefore $DWV_1V_1^{\mathrm{T}} = W_1V_1V_1^{\mathrm{T}}D$. From this relation it follows, after equating corresponding elements and noting that $W_1V_1V_1^{\mathrm{T}}$ is of the same form as $WV_1V_1^{\mathrm{T}}$ except for X, that A_{12} and A_{21} are zero matrices. Then:

$$WV_1V_1^{\mathrm{T}} = A_{11} \dot{+} X, \qquad W_1V_1V_1^{\mathrm{T}} = A_{11} \dot{+} Y,$$

$$W = (A_{11}\dot{+}X)V_1^{\mathrm{C}}V_1^{\mathrm{T}} = V_1UV_1^{\mathrm{CT}},\ W_1 = (A_{11}\dot{+}Y)V_1^{\mathrm{C}}V_1^{\mathrm{CT}} = V_1(-U^{\mathrm{T}})V_1^{\mathrm{CT}},$$

$$U = V_1^{\mathrm{CT}}(A_{11} \dot{+} X)V_1^{\mathrm{C}},\ -U^{\mathrm{T}} = V_1^{\mathrm{CT}}(A_{11} \dot{+} Y)V_1^{\mathrm{C}}.$$

Therefore, $U^{\mathrm{T}} = V_1^{\mathrm{CT}}(A_{11}^{\mathrm{T}} \dot{+} X^{\mathrm{T}})V_1^{\mathrm{C}} = V_1^{\mathrm{CT}}(-A_{11} \dot{+} [-Y])V_1^{\mathrm{C}}$ and so $A_{11} = -A_{11}^{\mathrm{T}}$ and A_{11} must also be unitary (since U^{T} is) and $Y = -X^{\mathrm{T}}$ where X is unitary but otherwise arbitrary. So $V_1UV_1^{\mathrm{T}} = A_{11} \dot{+} X$ and $V_1(-U^{\mathrm{T}})V_1^{\mathrm{T}} = A_{11} \dot{+} Y$.

Then $V_1AV_1^{\mathrm{T}} = V_1HV_1^{\mathrm{CT}}V_1UV_1^{\mathrm{T}} = V_1(-U^{\mathrm{T}})V_1^{\mathrm{T}}V_1^{\mathrm{C}}H^{\mathrm{T}}V_1^{\mathrm{T}} = (D_0 \dot{+} 0)\cdot(A_{11} \dot{+} X) = (A_{11} \dot{+} Y)(D_0 + 0)$. This means that $V_1AV_1^{\mathrm{T}} = D_0A_{11} \dot{+} 0$ where $D_0A_{11} = A_{11}D_0$ is of (even) order r, and A_{11} is unitary and skew-symmetric. It follows that $A_{11} = A_1 \dot{+} A_2 \dot{+} \ldots \dot{+} A_k$, where A_i is of the order of D_i in $D_0 = D_1 \dot{+} D_2 \dot{+} \ldots \dot{+} D_k$, and that each A_i is unitary and skew-symmetric and hence of even order. From the lemma for each A_i there exists a complex unitary U_i such that $U_iA_iU_i^{\mathrm{T}}$ is a direct sum of the 2×2 matrices described in the lemma. If $U = U_1 \dot{+} \ldots \dot{+} U_k$, then $UV_1AV_1^{\mathrm{T}}U^{\mathrm{T}} = D_0E_0 \dot{+} 0$ where E_0 is a direct sum of 2×2 matrices of the form described in the lemma. Then D_0E_0 is the matrix E described in the theorem, and since UV_1 is unitary, the theorem has been obtained. If A is non-singular, the same proof holds and $D = D_0$, $U = V_1^{\mathrm{CT}}A_{11}V_1^{\mathrm{C}}$, etc., and 0 does not appear in the final form $E \dot{+} 0$.

The converse is immediate.

3. A normal form for a *-symmetric quaternion matrix under unitary congruence. If two matrices A and B have elements which lie in a non-commutative domain, among the properties of the transpose which do not hold (as they do in the commutative case) is that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. If a matrix A with real quaternion elements is written in the form $A = A_1 + jA_2$ (where A_1 and A_2 are complex matrices), then $A^{\mathrm{T}} = A_1^{\mathrm{T}} + jA_2^{\mathrm{T}}$. Also, by the conjugate transpose of A is meant the matrix $A^{\mathrm{CT}} = A_1^{\mathrm{CT}} + (jA_2)^{\mathrm{CT}} = A_1^{\mathrm{CT}} - jA_2^{\mathrm{T}}$ (where A_1^{CT} denotes the complex conjugate transpose of A).

If the *-transpose of the matrix A is defined to be the matrix $A^* = A_1^{\mathrm{T}} + A_2^{\mathrm{T}}j$, it is seen that this includes the ordinary transpose of a complex matrix as a special case. Among the properties of the *-transpose which can easily be verified are the following: $(A^*)^* = A$; $A^* = ij\, A^{\mathrm{CT}}ji$; $(A + B)^* = A^* + B^*$; $(AB)^* = B^*A^*$; if A is non-singular, $(A^*)^{-1} = (A^{-1})^*$; $(A^*)^{\mathrm{CT}} = (A^{\mathrm{CT}})^*$. Define A to be *-symmetric if $A = A^*$, and to be *-skew-symmetric if $A = -A^*$. In the following, canonical forms are found for such matrices under unitary congruence which are clearly generalizations of the theorems for the complex case stated in the two preceding sections.

The following lemma is first obtained:

LEMMA 2. *If U is a unitary quaternion matrix (that is, $UU^{CT} = I = U^{CT}U$) which is also *-symmetric ($U = U^*$), there exists a complex unitary matrix Z such that $ZUZ^T = D_0 + jD$ where D_0 and D are real diagonal matrices for which $D_0^2 + D^2 = I$.*

Let $U = U_1 + j\,U_2$, where U_1 and U_2 are complex matrices. Since $U = U_1 + j\,U_2 = U^* = U_1^T + U_2^T\,j$, it follows that $U_1 = U_1^T$ and $U_2 = U_2^{CT}$. Since, also, $UU^{CT} = (U_1 + j\,U_2)(U_1^{CT} - j\,U_2^T) = I$, $U_1U_1^{CT} + U_2^C\,U_2^T = I$ and $U_2U_1^{CT} = U_1^CU_2^T$ or, taking conjugates, $U_2^CU_1^T = U_1U_2^{CT}$ or $U_2^CU_1 = U_1U_2$. Let V be a complex unitary matrix such that $VU_2V^{CT} = D = D_1 \dotplus D_2 \dotplus \ldots \dotplus D_k$, where $D_i = d_iI_i$ for d_i real, $d_i \neq d_j$ for $i \neq j$, and where $d_1 > d_2 > \ldots > d_k$; also let $V^CU_1V^{CT} = N$. Since $U_2^CU_1 = U_1U_2$, $V^CU_2^CV^TV^CU_1V^{CT} = V^CU_1V^{CT}VU_2V^{CT}$ or $DN = ND$. Therefore $N = N_1 \dotplus N_2 \dotplus \ldots \dotplus N_k$ is a direct sum conformable to D. Since $N = N^T$, $N_i = N_i^T$ for all i; consequently, there is a complex unitary W_i for each N_i such that $W_iN_iW_i^T = D_{1i}$ is a real diagonal matrix. If $W = W_1 \dotplus W_2 \dotplus \ldots \dotplus W_k$, then $WNW^T = D_{11} \dotplus D_{12} \dotplus \ldots \dotplus D_{1k} = D_0$ is a direct sum of real diagonal matrices. Then $WV^C(U_1 + j\,U_2)V^{CT}W^T = W(N + j\,D)W^T = D_0 + j\,D$ where D_0 and D are real diagonal matrices and WV^C is a complex unitary matrix. Furthermore, since U, V, and W are each unitary, $D_0 + j\,D$ is also and $(D_0 + j\,D)(D_0 - j\,D) = D_0^2 + D^2 = I$; the lemma is then true (and the converse is also, incidentally).

THEOREM 2. *If A is a *-symmetric quaternion matrix, there exists a quaternion unitary matrix U such that $UAU^* = D$ is a real diagonal matrix with non-negative diagonal elements; and conversely.*

This is clearly an analogue of the theorem for the complex case mentioned in §1, above; and its proof proceeds as does the proof for the complex case given in (**7**, p. 36). If $A = HV = VK$ is the polar form of the quaternion matrix A (see (**6**)), the proof follows the same pattern except that *-transpose replaces T-transpose and the elements involved are quaternion (though the matrix D is still a real diagonal matrix). It is then found that for $A = HV = VH^*$, there exists (**7**, p. 37) a quaternion unitary matrix U such that $UAU^* = UHU^{CT}UVU^* = UVU^*(U^*)^{CT}H^*U^* = DW = WD$ where D is a real diagonal matrix as there described and $W = UVU^* = W^*$ is now a quaternion unitary matrix. Since D is real diagonal with like roots arranged together along the diagonal, $W = W_1 \dotplus W_2 \dotplus \ldots \dotplus W_t$ is a direct sum conformable to that of D (as a direct sum of scalar matrices) and each $W_i = W_i^*$ is unitary; it may be noted that if $D = D_1 \dotplus 0$ (as in (**7**)) and if 0 is present, W_t will be chosen to have these properties also. By the preceding lemma, a complex unitary Z_i can be chosen so that $Z_iW_iZ_i^* = Z_iW_iZ_i^T = D_{0i} + j\,D_{1i}$ where

D_{0i} and D_{1i} are real diagonal with the properties given. If $Z = Z_1 \dotplus Z_2 \dotplus \ldots \dotplus Z_t$, then $ZUAU^*Z^* = ZDWZ^* = DZWZ^* = D(D_a + j\,D_b) = D_c + j\,D_d$ where D_c and D_d are real diagonal and ZU is a quaternion unitary matrix.

To obtain the form given in the theorem, an additional step is required. If $\alpha = a + i\,b$, a and b real, is any complex number, since it is a 1×1 matrix and is equal to its transpose, there exists a complex unitary (number) $u = u_1 + i\,u_2$ so that $u\alpha u^{\mathrm{T}} = u\alpha u = r$, a real number. If j replaces i in this relation, the result still holds (since only j and real numbers are involved); therefore, if $\alpha = a + j\,b$ is any diagonal element of $D_c + j\,D_d$, there exists a quaternion unitary $u = u_1 + j\,u_2$ so that $u\alpha u^* = r$ is real. If this is applied to each diagonal element, the form described in the theorem can be obtained under the transformation required.

The converse follows immediately and the form is a canonical form, the diagonal elements being the characteristic roots of the hermitian polar matrix of A.

4. A normal form for a *-skew-symmetric matrix under unitary congruence. For this case there is the following lemma:

LEMMA 3. *If A is a *-skew-symmetric, unitary quaternion matrix, there exists a unitary complex matrix V such that VAV^{T} is a direct sum of 1×1 matrices of the form $+\,j\,i$ and $-j\,i$, and of 2×2 matrices of the form*

$$\begin{bmatrix} jri & a \\ -a & -jri \end{bmatrix}$$

where $a^2 + r^2 = 1$ and $a > 0$ and r are real numbers.

Since $A = A_1 + j\,A_2 = -A^* = -(A_1^{\mathrm{T}} + A_2^{\mathrm{T}}\,j)$, it follows that $A_1 = -A_1^{\mathrm{T}}$ and $A_2 = -A_2^{\mathrm{CT}}$. Since $AA^{\mathrm{CT}} = I = A^{\mathrm{CT}}A$, it follows, among other relations, that $A_2A_1^{\mathrm{CT}} = A_1^{\mathrm{C}}A_2^{\mathrm{T}}$ and $A_1^{\mathrm{T}}A_2 = A_2^{\mathrm{T}}A_1$. Since A_2 is skew-hermitian, let U be a complex unitary matrix such that $UA_2U^{\mathrm{CT}} = D = D_1 \dotplus D_2 \dotplus D_3 \dotplus \ldots \dotplus D_k$ is a direct sum of $D_s = ir_sI_s$ (where r_s is real), that is, of pure imaginary scalar matrices, arranged as follows: $r_s \neq r_t$ if $s \neq t$; if ir_s and $-ir_s$ are roots of A_2, their corresponding blocks appear successively on the diagonal; all such successive pairs of blocks, if present, appear first in D; and $D_k = 0$ if 0 is a root of A_2. Let $U^{\mathrm{C}}A_1U^{\mathrm{CT}} = M$.

From $A_2A_1^{\mathrm{CT}} = A_1^{\mathrm{C}}A_2^{\mathrm{T}}$ it follows that

$$UA_2U^{\mathrm{CT}}UA_1^{\mathrm{CT}}U^{\mathrm{T}} = UA_1^{\mathrm{C}}U^{\mathrm{T}}U^{\mathrm{C}}A_2^{\mathrm{T}}U^{\mathrm{T}},$$

or $DM^{\mathrm{CT}} = M^{\mathrm{C}}D^{\mathrm{T}}$; taking conjugates, $D^{\mathrm{C}}M^{\mathrm{T}} = MD^{\mathrm{CT}}$ or $-D(-M) = M(-D)$ or $DM = -MD$ (since $M^{\mathrm{T}} = -M$). Therefore, $D^2M = DDM = -D(MD) = MD^2$. Let $D = (D_1 \dotplus D_2) \dotplus \ldots \dotplus (D_{t-1} \dotplus D_t) \dotplus D_{t+1} \dotplus \ldots \dotplus D_k$ where the parentheses contain the successive pairs described earlier. Then $M = M_{12} \dotplus \ldots \dotplus M_{t-1,t} \dotplus M_{t+1} \dotplus \ldots \dotplus M_k$ where M_{rs} is of the

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dimension of $D_r \dotplus D_s$, M_i is of the dimension of D_i, and all M_{rs} and M_i are complex skew-symmetric (since M is). Furthermore, since $-DM = MD$, it follows that $-(D_r \dotplus D_s)M_{rs} = M_{rs}(D_r \dotplus D_s)$ and $-D_iM_i = M_iD_i$ for all M_{rs} and M_i involved. Finally, it may be noted that $U^{C}A\,U^{CT} = U^{C}(A_1 + j\,A_2)U^{CT} = U^{C}A_1U^{CT} + j\;UA_2U^{CT} = M + j\,D$ must be *-skew symmetric and unitary. (Note that U is complex and $U^{C}A\,(U^{C})^* = U^{C}A\,U^{CT}$ is *-skew symmetric since A is also.)

(a) Consider, first, any relation $-(D_r \dotplus D_s)M_{rs} = M_{rs}(D_r \dotplus D_s)$ and, for convenience, the case where $r = 1$ and $s = 2$. Let $D_1 \dotplus D_2 = riI_1 \dotplus (-ri)I_2$ where I_1 and I_2 are, respectively, $p \times p$ and $q \times q$ identity matrices, $r \neq 0$, and assume, for specificity, that $p \leqq q$. Let M_{12} be of the form

$$\begin{bmatrix} M_1 & M_3 \\ -M_3^{T} & M_2 \end{bmatrix}$$

where M_1 and M_2 are, respectively, $p \times p$ and $q \times q$ matrices. From the relation $-(D_1 \dotplus D_2)M_{12} = M_{12}(D_1 \dotplus D_2)$, it follows that M_2 and M_1 are zero matrices (since $r \neq 0$). Now M_3 may be a zero matrix or it may not; before proceeding further, consider the latter case.

If M_3, a $p \times q$ matrix, is not zero, by a theorem of Eckert and Young **(1)** it follows that there exist complex unitary matrices V and W, of orders $p \times p$ and $q \times q$, respectively, such that $VM_3W = D$ is a $p \times q$ diagonal matrix with non-negative real elements (at least one of which is not 0 here) along the diagonal. (A $p \times q$ matrix is diagonal if the only non-zero elements are of the form a_{ii}.) Form the matrix

$$X = \begin{bmatrix} 0 & W^{T} \\ V & 0 \end{bmatrix}$$

which is complex unitary. Then $X(M_{12} + j\,D_{12})X^{T} = XM_{12}X^{T} + j\,X^{C}D_{12}X^{T}$ is a matrix of the form

$$\begin{bmatrix} 0 & -D^{T} \\ D & 0 \end{bmatrix} + j\begin{bmatrix} D_2 & 0 \\ 0 & D_1 \end{bmatrix}$$

where D is the above-mentioned $p \times q$ diagonal matrix. Let $N_1 = XM_{12}X^{T}$ and $N_2 = X^{C}D_{12}X^{T}$, and note that the dimension of $D_2 = q \geqq p =$ dimension of D_1, that D_1 and D_2 have non-0 diagonal elements, and D has at least one non-zero diagonal element; also, let the non-0 diagonal elements of D appear first along the diagonal. Consider N_1 and N_2 and perform the following operations on them: interchange the $q + 1$st column of N successively with the qth, $q - 1$st, $q - 2$nd, . . . , 2nd so that the $q + 1$st column becomes the second column and all succeeding columns are in the same order as before; and also perform the same row operations. This can be accomplished by a real orthogonal simularity transformation and there result from N_1 and N_2, respectively, the matrices

$$\begin{bmatrix} 0 & a_1 & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -D_3^{\mathrm{T}} \\ 0 & 0 & D_3 & 0 \end{bmatrix} \qquad \begin{bmatrix} -ri & 0 & 0 & 0 \\ 0 & ri & 0 & 0 \\ 0 & 0 & -riI_3 & 0 \\ 0 & 0 & 0 & riI_4 \end{bmatrix}$$

where I_3 and I_4 are, respectively, identity matrices of order $q - 1$ and $p - 1$, respectively. If the same procedure is applied to the lower right blocks (ignoring the first two rows and columns of each), it can be seen that a series of such steps provides a real orthogonal matrix Y such that the matrix $YX(M_{12} + j\,D_{12})X^{\mathrm{T}}Y^{\mathrm{T}}$ is a direct sum of 2×2 blocks of the form

$$\begin{bmatrix} -jri & a_t \\ -a_t & jri \end{bmatrix}$$

(where a_t and r are non-zero and real), and of single elements $-jri$ and $+jri$. But since YX is complex unitary, so is this direct sum, and so each 2×2 block and jri must be unitary. This means that $r^2 + a_t^2 = 1$ and $r^2 = 1$; but since $a_t \neq 0$, this can only mean that jri and $-jri$ cannot appear singly in the direct sum. Therefore $YX(M_{12} + j\,D_{12})X^{\mathrm{T}}Y^{\mathrm{T}}$ is a direct sum of 2×2 blocks of the above form where $r^2 + a_t^2 = 1$, $r \neq 0$ and $a_t \neq 0$. (If in the above $p \geqq q$, the roles of $+jri$ and $-jri$ are interchanged, but a simple (and allowable) operation at the close can still place the element $-jri$ in the $1 - 1$ position.)

All of the above in (a) occurs if M_3 is not a zero matrix. If $M_3 = 0$, then $M_{12} + j\,D_{12} = j\,D_{12} = j\,(D_1 \dotplus D_2)$ which is a direct sum with diagonal elements $+\,j\,r\,i$, $r^2 = 1$; in this case no X and Y are required.

Therefore in $U^{\mathrm{C}}A\,U^{\mathrm{CT}} = M + j\,D$, each $M_{rs} + j\,(D_r \dotplus D_s)$ can be treated as above depending on whether or not M_{rs} is a zero matrix.

(b) Consider any relation $-D_iM_i = M_iD_i$ where D_i is a non-0 pure imaginary scalar matrix. Then $M_i = -M_i$ so M_i is a zero matrix and $M_i + j\,D_i = j\,D_i$ which has diagonal elements jri, $r^2 = 1$.

(c) If $D_k = 0$ is present in $UA_2U^{\mathrm{CT}} = D$, then $M_k + j\,D_k = M_k = -M_k$,$^{\mathrm{T}}$ a complex unitary matrix. By Lemma 1 there exists a complex unitary matrix U such that $UA\,U^{\mathrm{T}} = E$ is a direct sum as described in the lemma.

If the results of (a), (b), and (c) are combined, it is evident that a complex unitary matrix W can be constructed so that $WU^{\mathrm{C}}A\,U^{\mathrm{CT}}W^{\mathrm{T}} = W(M + jD)W^{\mathrm{T}}$ is a direct sum of 2×2 matrices of the form

$$\begin{bmatrix} jri & a \\ -a & -jri \end{bmatrix}$$

(where $a^2 + r^2 = 1$, $a > 0$ and r are real) and of 1×1 matrices of the form ji and $-ji$.

THEOREM 3. *If A is a *-skew-symmetric quaternion matrix, there exists a quaternion unitary matrix V such that $VA\,V^* = E \dotplus 0$ where E is a direct*

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sum of 1×1 *matrices of the form* kji *and* $-kji$, $k > 0$ *real, and of* 2×2 *matrices of the form*

$$\begin{bmatrix} sji & t \\ -t & -sji \end{bmatrix}$$

where $t > 0$ *and* s *are real.*

The proof follows the pattern of that of Theorem 1. If $A = 0$, the result is trivial. If $A \neq 0$, let $A = HU = UK$ be a polar representation of A. If *-transpose replaces T-transpose in the earlier proof, it is evident that $H = K^*$. Here, however, the rank of a *-skew-symmetric matrix is not necessarily even (as the preceding lemma shows). If the earlier proof is followed, it is seen eventually that, using the same letters, $U = V_1^{CT}(A_{11} \dotplus X) V_1^{*CT}$ and $U^* = -V_1^{CT}(A_{11} \dotplus Y) V_1^{*CT}$ so that $U^* = V_1^{CT}(A_{11}^* \dotplus X^*) V_1^{CT} = -V_1^{CT}(A_{11} \dotplus Y) V_1^{*CT}$ and, since $V_1^{CT*} = V_1^{*CT}$, $A_{11}^* = -A_{11}$ is quaternion unitary. Then $V_1 A V_1^* = V_1 H V_1^{CT} V_1 U V_1^* = (D_1 \dotplus 0)(A_{11} \dotplus X) = (D_1 A_{11} \dotplus 0) = V_1(-U^*) V_1^* V_1^{*CT} H^* V_1^* = (A_{11} \dotplus Y)(D_1 \dotplus 0) = (A_{11} D_1 \dotplus 0)$. Since $D_1 A_{11} = A_{11} D_1$, A_{11} is a direct sum, $A_1 \dotplus A_2 \dotplus \ldots \dotplus A_k$, (of *-skew-symmetric, unitary quaternion matrices) conformable to the direct sum of D_1. For each A_i there exists, by the preceding lemma, a complex unitary matrix W_i so that $W_i A_i W_i^T$ has the form described in the lemma. If $W = W_1 + W_2 \dotplus \ldots \dotplus W_k \dotplus I$ (where I is of the order of 0 in $D_1 \dotplus 0$), $W V_1 A V_1^* W^T$ is then a direct sum of 1×1 matrices of the form kji and $-kji$ ($k > 0$ is real), of 2×2 matrices of the form

$$\begin{bmatrix} jrci & ac \\ -ac & -jrci \end{bmatrix}$$

where $ac > 0$ is real, and of a zero matrix. (WV_1 is a unitary quaternion matrix.)

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