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RELATIONS BETWEEN MAHLER'S MEASURE AND VALUES OF L-SERIES

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Introduction. Mahler's measure is a natural generalization of Jensen's formula to polynomials in several variables. Its definition is as follows:

$$M(p) = \exp\left[\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|p(e^{iv_1}, \ldots, e^{iv_n})|dv_1 \ldots dv_n\right].$$

The importance of Mahler's measure for polynomials in several variables lies in its connection to Lehmer's classical question which can be phrased in terms of Mahler's measure for polynomials in one variable:

Given $\epsilon > 0$, are there any polynomials p with integer coefficients in one variable for which $1 < M(p) < 1 + \epsilon$?

Surprisingly, Boyd **[1]** has shown that to answer this question, it is necessary to investigate the larger question involving polynomials in several variables.

An unexpected connection has also been discovered between Mahler's measure and values of L-series. Smyth has shown that

$$\log M(1 + z_1 + z_2) = \frac{3\sqrt{3}}{4\pi}L(2, \chi - 3) = L'(-1, \chi_{-3})$$

where χ_{-3} is the odd quadratic character of conductor 3. What makes this so tantalizing is that next to nothing is known about $L(s, \chi)$ when s and χ have opposite parity. In fact, Apery created a sensation by proving recently that $\zeta(3) = L(3, \chi_{\text{triv}})$ is irrational. On the other hand, when s and χ have the same parity, the values can be written in terms of generalized Bernoulli numbers and have great significance in the study of the associated algebraic number field.

Thus the above comments provide the source of motivation for this paper exploring the connections between Mahler's measures and values of L-series. We are able to find 6 polynomials with integer coefficients such that the log of their Mahler measure is a known constant times $L(2, \chi)$ for the 6 characters associated to the imaginary quadratic fields $\mathbf{Q}(\sqrt{-N})$ where $N = 3$, 4, 7, 8, 20 and 24. Our result when $N = 7$ is much deeper than the other ones and depends on a new identity involving $L(2, \chi_{-7})$.

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This identity is a special case of a more general relation involving sums of dilogarithms.

In Section 1 we begin by recalling some facts about L-series and polylogarithms. In Section 2, we introduce for each quadratic character χ a polynomial $g_\chi(x, y)$ with integer coefficients such that the log of Mahler's measure $M(g_\chi)$ is expressible in terms of the dilogarithm function. For the odd quadratic characters of conductors 3, 4, 7, 8, 20 and 24, we then show that

$$\log M(g_\chi) = (\text{a rational number}) \times L'(-1, \chi).$$

The most interesting example is $\chi = \chi_{-7}$, the character for the field $\mathbf{Q}(\sqrt{-7})$. No polynomial with integer coefficients was previously known which satisfies the above equality for this character. Our result follows as a consequence of some new identities involving dilogarithms and L-series in Sections 3 and 4. In the case of $\chi = \chi_{-7}$, we show

$$\text{(Ia)} \quad \sum_{n=1}^{\infty} \frac{\cos(n\theta)\,\chi(n)}{n^2} = -\frac{3}{4} L(2, \chi),$$

where $\cos(\theta) = -3/4$. We use this identity to obtain the following result:

$$\text{(Ib)} \quad \log M((y-1)^2(x^6 + x^5 + x + 1) + (y^2 + 5y + 1)(x^4 + x^2) + (y^2 + 12y + 1)x^3) = \frac{8}{7} L'(-1, \chi_{-7}).$$

In the final section, we see how these identities involving Mahler's measure and L-series values for quadratic characters can to some extent be generalized to arbitrary Dirichlet characters. Finally, I would like to thank Dr. Neal Koblitz for his many valuable suggestions and encouragement.

1. Definitions and classical results. To fix notation, recall that the Gauss sum for a non-trivial character χ is defined as

$$\tau(\chi) = \sum_{k=1}^{N-1} \chi(k)\zeta^k$$

where

$$\zeta = e^{2\pi i/N}.$$

The definitions of the Bernoulli numbers, generalized Bernoulli numbers, and Bernoulli polynomials are respectively:

$$B_n = n! \times \text{coefficient of } t^n \text{ in } \frac{t}{e^t - 1},$$

$$B_{n,\chi} = n! \times \text{coefficient of } t^n \text{ in } \sum_{\alpha=0}^{N} \frac{\chi(\alpha)te^{\alpha i}}{e^{Nt} - 1},$$

and

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}.$$

Then for $s = 1, 2, 3, \ldots$, if $\chi(-1) = (-1)^s$, we have

$$L(s, \chi) = (-1)^{s-1}\frac{\tau(\chi)}{2}\left(\frac{2\pi i}{N}\right)^s \frac{B_{s,\bar{\chi}}}{s!}.$$

In other words, $L(s, \chi)$ is known explicitly when s and χ have the same parity.

On the other hand, the only case when $L(s, \chi)$ is known explicitly if s and χ have opposite parity is when $s = 1$ and χ is even. Suppose, for example, that $K = \mathbf{Q}(\sqrt{N})$ is a real quadratic field with discriminant N and fundamental unit ϵ, and χ_N is the Dirichlet character of conductor N associated to this field. The Dirichlet class number formula states that

$$\text{(1a)} \quad L(1, \chi_N) = \frac{2h \log \epsilon}{\sqrt{N}}$$

where h is the class number of K. Alternatively, (1a) can be rewritten using the functional equation for $L(s, \chi)$. We now recall how this is done.

When χ is any primitive character, recall that the functional equation for $L(s, \chi)$ is

$$\text{(1b)} \quad \psi(s, \chi) = \omega(\chi)\psi(1 - s, \bar{\chi})$$

where

$$\omega(\chi) = (-i)^{\alpha} N^{-1/2}\tau(\chi)$$

and

$$\psi(s, \chi) = (N/\pi)^{s/2}\Gamma(\,(s + \alpha)/2)L(s, \chi).$$

Here $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$.

Using (1b) with χ even and non-trivial, we see that $L(0, \chi) = 0$ and

$$\text{(1c)} \quad L'(0, \chi) = \frac{\tau(\chi)}{2}L(1, \bar{\chi}).$$

If χ is odd, $L(-1, \chi) = 0$ and

(1d) $L'(-1, \chi) = \dfrac{-i\tau(\chi)N}{4\pi} L(2, \bar{\chi})$.

In the case of the even quadratic character χ_N,

(1e) $L'(0, \chi_N) = h \log \epsilon$.

We now proceed to write equation (1e) in terms of Mahler's measure. Recall that Mahler's measure for a polynomial p in n variables is defined as:

$$M(p) = \exp\left[\frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \log|p(e^{i\theta_1}, \dots, e^{i\theta_n})| d\theta_1 \dots d\theta_n\right].$$

In the one variable case, if

$$p(z) = \alpha_0 \prod (z - \alpha_i),$$

Jensen's formula shows that

$$M(p) = |\alpha_0| \prod_{|\alpha_i|>1} |\alpha_i|.$$

Let k be a real quadratic field and let p_ϵ be the irreducible polynomial for ϵ. Then the conjugate of ϵ is $\pm 1/\epsilon$ so that

(1f) $p_\epsilon(x) = (x - \epsilon)\left(x \mp \dfrac{1}{\epsilon}\right)$.

Since $\epsilon > 1$, it follows from Jensen's formula and (1f) that

(1g) $\log M(p_\epsilon) = \dfrac{1}{h} L'(0, \chi_N)$.

In analogy with (1g). T. Chinburg [**2**] conjectured that for each odd quadratic character χ, there exists a polynomial p in 2 variables with integer coefficients such that

(1h) $\log M(p) = (\text{a rational number}) \times L'(-1, \chi)$.

He has shown that there exists a rational function p with this property. The motivation for the conjecture comes from an example by Smyth (see [**1**]).

$$L'(-1, \chi_{-3}) = \log M(1 + x + y)$$

where $\chi_{-3}(n) = 0$, 1 or -1 depending on whether $n \equiv 0$, 1 or 2 mod 3, respectively. Smyth [**9**] has calculated Mahler's measure for a large class of polynomials. However, no polynomial with integer coefficients was known which, for example, satisfies (1h) for the character χ_{-7}. We produce such a polynomial as a consequence of our main results on linear relations among twisted L-series.

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A twisted L-series for any periodic function $f:\mathbf{N} \to \mathbf{C}$ is defined by

$$L_\mu(s, f) = \sum_{n=1}^{\infty} \frac{\mu^n f(n)}{n^s}$$

where μ is any complex number of absolute value $\leqq 1$. We will show that $L_\mu(s, \chi)$ can be analytically continued as a function of μ to the complex plane minus certain rays.

Recall that the polylogarithm function Li_s is defined as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

for $|z| < 1$ and $s = 1, 2, \dots$. Notice that

$$Li_1(z) = -\log(1 - z).$$

The branch of the log throughout this paper will be chosen so that

$$-\pi < \operatorname{Im}(\log(z)) \leqq \pi.$$

The formula

$$\text{(1i)} \quad Li_{s+1}(z) = \int_0^z \frac{Li_s(t)}{t} dt$$

where the path of the integral is taken to be the line from 0 to z if $z \notin [1, \infty)$ defines inductively an analytic extension of Li_s to the whole complex plane minus $[1, \infty)$. In addition the improper integral

$$\int_0^x \frac{Li_s(t)}{t} dt$$

along the real interval $[0, x]$ does converge so in fact (1i) defines $Li_{s+1}(z)$ for all $z \in \mathbf{C}$, $s = 1, 2, \dots$ (but it has a discontinuity if z crosses the ray $(1, \infty)$). A good reference for polylogarithms is [**3**].

Suppose μ is a complex number with $|\mu| < 1$, $\zeta = \zeta_N = e^{2\pi i/N}$, and χ is any character of conductor N. Then

$$\text{(1j)} \quad L_\mu(s, \chi) = \sum_{n=1}^{\infty} \frac{\mu^n \chi(n)}{n^s} = \frac{1}{\tau(\overline{\chi})} \sum_{k=1}^{N} \overline{\chi(k)} Li_s(\mu\zeta^k).$$

Now it is clear that even though the series for $L_\mu(s, \chi)$ only converges if μ is in the unit disc, $L_\mu(s, \chi)$ can be defined on all of $\mathbf{C}$ by (1j), as a function of μ for fixed $s > 1$ and χ. $L_\mu(s, \chi)$ is analytic on the complement of the union of the N rays

$$\{z | z = \tau/\zeta^k, \tau \geqq 1\} \quad \text{for } k = 1, 2, \dots, N.$$

2. Computations of Mahler's measure. In this section, for each character χ, a polynomial $f_\chi(x, y)$ is defined. Then $\log M(f_\chi)$ is computed in terms of a linear combination of $\{L_{\mu_i}(2, \chi)\}$ for various μ_i's. We will always assume χ is a primitive non-trivial Dirichlet character unless otherwise indicated.

To motivate our defintion of f_χ, suppose that χ is an even quadratic character. Then by (1a)

$$\text{(2a)} \quad \sqrt{N}L(1, \chi) = 2h \log \epsilon = \log \epsilon^{2h} = -\log \frac{\prod_{\chi(k)=1} (1 - \zeta^k)}{\prod_{\chi(k)=-1} (1 - \zeta^k)}.$$

The last equality is a standard result (see p. 199 of [**5**]). The products and sums over k in this section will run from 1 to N.

We define for any character χ which has even order,

$$p_\chi(z) = \prod_{\chi(k)=1} (1 - z\zeta^k),$$

$$\tilde{p}_\chi(z) = \prod_{\chi(k)=-1} (1 - z\zeta^k),$$

and

$$f_\chi(x, y) = y\tilde{p}_\chi(x) - p_\chi(x).$$

Because χ has even order, χ takes on the values 1 and -1 the same number of times. Therefore the degrees of p_χ and $\tilde{p}_\chi$ are equal. When χ is an odd quadratic character, we set

$$g_\chi = f_\chi \bar{f}_\chi$$

where the bar denotes complex conjugation of the coefficients of f_χ. Then $\tilde{p}_\chi = \bar{p}_\chi$ and it follows that $g_\chi \in \mathbf{Z}[x, y]$ and g_χ factors as

$$p_\chi(x)\bar{p}_\chi(x)\left(y - \frac{p_\chi(x)}{\bar{p}_\chi(x)}\right)\left(y - \frac{\bar{p}_\chi(x)}{p_\chi(x)}\right).$$

Comparing this to the factorization of p_ϵ in (1f) and to (2a) shows the analogy between the two variable polynomial g_χ and the one variable polynomial p_ϵ. Note that

$$\log M(g_\chi) = 2 \log M(f_\chi),$$

as follows directly from the definition.

It will be shown that

$$\log M(f_{\chi_{-N}}) = (\text{a rational}) \times L'(-1, \chi_{-N})$$

for $N = 3, 4, 8, 20, 24$, and 7. The case $N = 7$ will use a special identity to be proven in Section 4.

Let χ again be any character of even order. Since $f_\chi(0, 0) = -1$, we have

$$\log M(f_\chi) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\log|f_\chi(e^{i\theta_1}, e^{i\theta_2})|d\theta_2\right]d\theta_1.$$

Since

$$\log M(ax - b) = \log|\alpha| + \max\{0, \log|b/a|\},$$

computing the inner integral first gives

$$\log|\tilde{p}_\chi(e^{i\theta_1})| + \max\left\{0, \log\left|\frac{p_\chi(e^{i\theta_1})}{\tilde{p}_\chi(e^{i\theta_1})}\right|\right\}.$$

Then

$$\log M(f_\chi) = \log M(\tilde{p}_\chi) + \frac{1}{2\pi}\left\{\sum_{\chi(k)=\pm 1}\chi(k)\int_\Omega \log|1 - e^{i\theta_1}\zeta^k|d\theta_1\right\}$$

where $\Omega \subseteq [0, 2\pi]$ is defined as the set of all θ_1 for which

$$|p_\chi(e^{i\theta_1})| \geqq |\tilde{p}_\chi(e^{i\theta_1})|.$$

Since all the roots of $\tilde{p}_\chi$ lie on the unit circle,

$$\log M(\tilde{p}_\chi) = 0.$$

Suppose that Ω is written as a union of intervals $\cup_j[\tau_{2j-1}, \tau_{2j}]$. Set

$$\mu_j = e^{i\tau_j}.$$

Then

$$\text{(2b)}\quad \log M(f_\chi) = \frac{1}{2\pi}\left[\sum_{\chi(k)=\pm 1}\chi(k)\sum_j\int_{\tau_{2j-1}}^{\tau_{2j}}\log|1 - e^{i\theta_1}\zeta^k|d\theta_1\right].$$

If we let L_a (L_b) be the line segment from 0 to e^{ia}(e^{ib}) with $0 < a < b < 2\pi$, and A the arc on the unit circle between e^{ia} and e^{ib}, then

$$\begin{aligned} &Li_2(e^{ib}) - Li_2(e^{ia}) \\ &= \int_{L_b} - \log(1 - x)\frac{dx}{x} - \int_{L_a} - \log(1 - x)\frac{dx}{x} \\ &= \int_A - \log(1 - x)\frac{dx}{x} = -i\int_a^b \log(1 - e^{i\theta})\,d\theta, \end{aligned}$$

since $\log(1 - x)$ is analytic inside the unit circle. Thus (2b) becomes

(2c) $\log M(f_\chi)$

$$= \frac{1}{2\pi} \operatorname{Re}\left[\sum_{\chi(k)=\pm 1} i\chi(k) \sum_j \left[Li_2(\mu_{2j}\zeta^k) - Li_2(\mu_{2j-1}\zeta^k) \right] \right].$$

Using (1j) with χ a quadratic character, (2c) can be written in terms of twisted L-series. In fact

PROPOSITION 1. *If* $\chi = \chi_{-N}$ *is an odd quadratic character of conductor* N,

(2d) $$\log M(f_{\chi_{-N}}) = \frac{\sqrt{N}}{2\pi} \sum_j (-1)^{j+1} \operatorname{Re} L_{\mu_j}(2, \chi_{-N}).$$

Proof. Using (1j), factor out the Gauss sum $\tau(\chi) = i\sqrt{N}$ from (2c) and take the real part.

PROPOSITION 2. *When the conductor* $N = 3, 4, 8, 20,$ *or* 24,

$$\log M(g_{\chi_{-N}}) = 2 \log M(f_{\chi_{-N}}) = \frac{8 - 2\chi_{-N}(2)}{N} L'(-1, \chi_{-N}).$$

Proof. When $N = 3, 4, 8, 20,$ or 24, we will see that $\Omega = [0, \pi]$ with $\mu_1 = 1$ and $\mu_2 = -1$. Given that this is true,

(2e) $$\log M(f_{\chi_{-N}}) = \frac{1}{2\pi} \sqrt{N}(L_1(2, \chi_{-N}) - L_{-1}(2, \chi_{-N}))$$

by Proposition 1. It is an easy exercise to show that for any character χ,

(2f) $$L_{-1}(s, \chi) = (2^{1-s}\chi(2) - 1)L(s, \chi).$$

Using (2f), the right hand side of (2e) becomes

(2g) $$\frac{\sqrt{N}}{2\pi}\left[2 - \frac{\chi_{-N}(2)}{2}\right] L(2, \chi_{-N}).$$

Finally, using (1d) we simplify (2g) to

$$\log M(f_{\chi_{-N}}) = \frac{4 - \chi_{-N}(2)}{N} L'(-1, \chi_{-N}).$$

Therefore,

$$\log M(g_{\chi_{-N}}) = 2 \log M(f_{\chi_{-N}}) = \frac{8 - 2\chi_{-N}(2)}{N} L'(-1, \chi_{-N}).$$

To complete the calculation recall that we need to show how to obtain $\{\mu_i\}$, the end points of the region Ω. It is easy to show from the class number formula that if $\chi = \chi_{-N}$, $N \neq 3$ or 4, then $p_\chi(z)/\bar{p}_\chi(z)$ is real-valued for all z on the unit circle.

From this the $\{\mu_i\}$ must be roots of

$$\text{(2h)} \quad \frac{p_\chi(z)}{\bar{p}_\chi(z)} = \pm 1 \quad \text{or} \quad p_\chi(z) \pm \bar{p}_\chi(z) = 0$$

and must lie on the unit circle.

A computation reveals that the only such roots when $\chi = \chi_{-N}$ and $N = 8, 20$, or 24 are ± 1. A special computation for $N = 3$ and 4 reveals the same roots.

In order to determine whether Ω corresponds to $[0, \pi]$ or $[\pi, 2\pi]$, note that as z on the unit circle approaches ζ^r for some residue r, $\bar{p}_\chi$ approaches 0. Therefore

$$\frac{|p_\chi(z)|}{|\bar{p}_\chi(z)|} \geqq 1$$

on the upper half of the unit circle and $\Omega = [0, \pi]$. Thus $\mu_1 = 1$ and $\mu_2 = -1$.

The only other polynomial known which satisfies (1h) is $g_{\chi_{-7}}$, the polynomial given in equation (Ib) at the beginning of this section. The proof of (Ib) depends on (Ia) which in turn depends on the deeper results in the next two sections.

As a consequence of our main result, we shall prove

THEOREM 3. *Let χ be an odd quadratic character of conductor $N \neq 3$ or 4 and let $\Delta_\chi(t)$ be the discriminant of $f_\chi(x, t)$ considered as a polynomial in x. Suppose that for some real number $y < 1$, $\Delta_\chi(t) \neq 0$ for all t between 0 and y. Then for each t between 0 and y, $f_\chi(x, t) = 0$ has $M = \varphi(N)/2$ distinct roots $x = x_1(t), x_2(t), \ldots, x_M(t)$, all of which have absolute value one, and*

$$\sum_{i=1}^{M} \operatorname{Re} L_{x_i(t)}(2, \chi) = \frac{\hat{\mu}(N)}{2} L(2, \chi),$$

where $\hat{\mu}$ denotes the Moebius function from elementary number theory.

We now use Theorem 3 to derive the identity (Ia).

COROLLARY 4. *If*

$$\mu = \frac{-3 + \sqrt{-7}}{4},$$

then

$$\text{(2i)} \quad \operatorname{Re} L_\mu(2, \chi_{-7}) = -\frac{3}{4} L(2, \chi_{-7}).$$

Choosing $\chi = \chi_{-7}$, a calculation shows

$$p_\chi = 1 - \alpha x + \bar{\alpha} x^2 - x^3 = \bar{\tilde{p}}_\chi$$

where $\alpha = (-1 + \sqrt{-7})/2$ and

$$\Delta_\chi(t) = -7t^4 + 98t^3 - 133t^2 + 98t - 7.$$

$\Delta_\chi(t)$ has only 4 roots; they are approximately .08, 12.6, .67 $\pm$ .74i, so $\Delta_\chi(t) \neq 0$ for $t < 0$. Thus $|x_i(t)| \equiv 1$ for all $t \in (-\infty, 0)$ and all i by Theorem 3. Therefore if we choose $t = -1$, then

$$f_\chi(x, -1) = 2x^3 + x^2 - x - 2$$

has roots at $x = 1$, μ, and $\bar{\mu}$ where

$$\mu = \frac{-3 + \sqrt{-7}}{4}.$$

By Theorem 3,

$$L(2, \chi_{-7}) + 2 \operatorname{Re} L_\mu(2, \chi_{-7}) = -\frac{1}{2} L(2, \chi_{-7})$$

because $\operatorname{Re} L_\mu = \operatorname{Re} L_{\bar{\mu}}$ when χ is real-valued. Thus

$$\operatorname{Re} L_\mu(2, \chi_{-7}) = -\frac{3}{4} L(2, \chi_{-7}).$$

If we set $\cos(\theta) = -3/4$, then $\operatorname{Re}(\mu^n) = \cos(n\theta)$ and (1.2i) is seen to be equivalent to (Ia):

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)\,\chi(n)}{n^2} = -\frac{3}{4} L(2, \chi).$$

Surprisingly, no other relations (other than those which follow from the Kubert identities, see [**6**],) with this simple form are known for other characters besides χ_{-7}. We will examine reasons why χ_{-7} is so special in Section 5.

Next we complete the proof of (Ib).

COROLLARY 5. *With the notation as above,*

$$\log M(g_{\chi_{-7}}) = \frac{8}{7} L'(-1, \chi_{-7}).$$

Proof. When $\chi = \chi_{-7}$, there are four roots of equation (2h), $p_\chi(z) \pm \tilde{p}_\chi(z) = 0$, with z on the unit circle. They are ± 1, $(-3 \pm \sqrt{-7})/4$. In fact,

$$\mu_1 = 1,\ \mu_2 = \mu = \frac{-3 + \sqrt{-7}}{4},\ \mu_3, = -1, \text{ and } \mu_4 = \bar{\mu}.$$

Therefore, by Proposition 1,

$$\log M(f_\chi)$$
$$= \frac{\sqrt{7}}{2\pi}\,\mathrm{Re}[L_1(2,\chi) + L_{-1}(2,\chi) - L_\mu(2,\chi) - L_{\bar{\mu}}(2,\chi)]$$
$$= \frac{\sqrt{7}}{\pi}L(2,\chi)$$

by using the identities (2i) and (2f). From (1d) it follows that

$$\log M(f_\chi) = \frac{4}{7}L'(-1,\chi)$$

and therefore

$$\log M(g_{\chi_{-7}}) = \frac{8}{7}L'(-1,\chi_{-7}).$$

Equation (Ib) follows by expanding g_χ in terms of powers of x and y. The next two sections will be devoted to proving Theorem 3.

3. Linear relations among dilogarithms. In this section we produce certain linear relations among dilogarithms. Theorem 11 gives the main result, which will be used in Section 4 to obtain linear relations among $\{L_{\mu_i}(2,\chi)\}$ for fixed Dirichlet characters χ. Theorem 14 gives the main result in the next section, and Theorem 3 then follows when χ is an odd quadratic character.

We begin by recalling a well-known linear relation involving polylogarithms, the multiplicative form of the Kubert identities. They have been studied by Kubert and in a recent paper by Milnor, see [**6**]. Here is a sketch of a derivation of these identities.

Set $\omega = \zeta_M$. Then

$$1 - x^M = \prod_{l=0}^{M-1}(1 - \omega^l x).$$

Take logarithms of both sides and integrate $s - 1$ times with respect to dx/x from 0 to x. This gives

$$\text{(3a)}\qquad \frac{1}{M^{s-1}}Li_s(x^M) = \sum_{l=0}^{M-1} Li_s(\omega^l x).$$

We have ignored the fact that $\log(ab) = \log(a) + \log(b)$ does not hold for all complex a and b because of the different branches of the log. Nevertheless, (3a) is easily seen to be valid as long as $|x| \leqq 1$ and when $s = 1$, $x \neq \omega^l$, $l = 0, \ldots, M - 1$.

These Kubert identities can be restated as a linear relation among twisted L-series valid for all fixed s and χ. To do this, replace x by $\zeta^k x$ in (3a), multiply by $\bar{\chi}(k)$, and sum from $k = 1$ to N to obtain

$$\text{(3b)} \quad \frac{\chi(M)}{M^{s-1}} L_{x^M}(s, \chi) = \sum_{l=0}^{M-1} L_{\omega^l x}(s, \chi).$$

If we set

$$x = e^{2\pi iy} \quad \text{and}$$

$$l_s(x) = Li_s(e^{2\pi iy}) = \sum_{n=1}^{\infty} \frac{e^{2\pi iny}}{n^s} = L_x(s, \chi_{\text{triv}}),$$

then (3a) results in the usual form of the Kubert identities:

$$\text{(3c)} \quad \frac{1}{M^{s-1}} l_s(My) = \sum_{i=0}^{M-1} l_s(y + i/M).$$

Milnor conjectures that all the **Q**-linear relations among l_s for rational y are generated by (3c). See **[6]** for a precise statement of this conjecture.

Suppose that one instead seeks, for fixed s and χ, **Q**-linear relations among $\{L_{\mu_i}(s, \chi)\}$ with algebraic μ_i. From (2i), we have one such relation:

$$L_\mu(2, \chi_{-7}) + L_{\bar{\mu}}(2, \chi_{-7}) = -\frac{3}{2} L(2, \chi_{-7})$$

where

$$\mu = \frac{-3 + \sqrt{-7}}{4}$$

is not a root of unity. It is fairly easy to see that this relation does not follow from (3b), no matter how x is chosen.

In this section we modify the previous argument. Our method is based on the work of Rogers [7] for dilogarithms. We begin by picking a collection of distinct non-zero complex numbers $\{\alpha_k, \beta_k\}$ where there are M α_k's and M β_k's chosen. Define

$$p(x) = \prod_{k=1}^{M} (1 - \alpha_k x) \quad \text{and} \quad \tilde{p}(x) = \prod_{k=1}^{M} (1 - \beta_k x).$$

Set

$$q(x, t) = t\tilde{p}_\chi(x) - p_\chi(x)$$

and let

(3d) $q(x, t) = d_M(t)x^M + \ldots + d_0(t) = 0$

define a multiple-valued algebraic function. Since $q(x, t)$ is a linear polynomial in t and p_χ and $\tilde{p}_\chi$ have no common factors, q is irreducible in $\mathbf{C}[x, t]$. For each complex number $t = y_0$ (except for a finite number of critical points), $q(x, y_0)$ is an Mth degree polynomial in x with M distinct roots $\{x_i(y_0)\}$. The critical points $c_1, c_2, \ldots, c_r$ are the values of y_0 such that either

$$d_M(y_0) = (-1)^M(y_0 \prod \beta_k - \prod \alpha_k) = 0$$

or $q(x, y_0)$ has a multiple root.

At this point we state the two main theorems in this paper:

THEOREM 11. *Suppose that* $\{\alpha_k, \beta_k\}$ *are distinct non-zero complex numbers such that*

$$\prod_{k=1}^{M} \alpha_k = \prod_{k=1}^{M} \beta_k.$$

Then if $y < 1$, *we have*

$$\sum_{i=1}^{M} \sum_{k=1}^{M} [Li_2(\beta_k x_i(y)) - Li_2(\alpha_k x_i(y))] - \sum_{i=1}^{M} \sum_{k=1}^{M} [Li_2(\beta_k/\alpha_i) - Li_2(\alpha_k/\alpha_i)]$$

$$= 2\pi i\left[\sum_j N_j(y) \log(d_j) + \sum_i \hat{N}_i(y) \log x_i(y)\right] - 4\pi^2 W(y)$$

for certain complex numbers $\{d_j\}$, not depending on y, and certain integers $\{W(y), N_j(y), \hat{N}_i(y)\}$.

THEOREM 14. *If* $y < 1$ *and* χ *is a "regular" (see below) primitive Dirichlet character of even order* m *and conductor* N, *then*

$$-\frac{2}{m} \sum_{i=1}^{M} \sum_{b\,\mathrm{odd}, b=1}^{m-1} L_{x_i}(y)(2, \hat{\chi}_b) + \frac{2}{m^2} \sum_{b\,\mathrm{odd}, b=1}^{m-1} \sum_{c=0}^{m-1} L(2, \hat{\chi}_b \bar{\hat{\chi}}_c)$$

$$= 2\pi i\left[\sum_j N_j(y) \log(d_j) + \sum_{i=1}^{M} \hat{N}_i(y) \log x_i(y)\right] - 4\pi^2 W(y).$$

The next two paragraphs outline the plan of the proofs of these two theorems and explain some of the terms and notation used in their statement.

The first step is to rigorously define the M roots $\{x_i(t)\}$ for all real t. Then an operator T_i is defined for each $i = 1, \dots, M$ so that

$$T_i(f)(y) = \int_0^y f(t)\frac{x_i'(t)}{x_i(t)}dt.$$

We take logarithms of both sides of $t\tilde{p}_\chi(t) = p_\chi(t)$ and apply T_i to each term. Except for some difficulties with the branch cut of the logarithm, this produces M equations:

$$T_i(\log)(y) + \sum_{k=1}^{M} Li_2(\beta_k x_i(y)) - Li_2(\beta_k/\alpha_i)$$

$$= \sum_{k=1}^{M} Li_2(\alpha_k x_i(y)) - Li_2(\alpha_k/\alpha_i)$$

for each $i = 1, \dots, M$. Lemma 6 and 7 handle the complications with the branch cuts. The actual expression is somewhat more complicated. Next, Lemmas 8, 9, and 10 show that $T_i(\log)(y)$ converges and if

$$\prod_{k=1}^{M} \alpha_k = \prod_{k=1}^{M} \beta_k,$$

then

$$\sum_{k=1}^{M} T_i(\log)(y) = 0.$$

Thus if we sum the M equations from $i = 1$ to M, we obtain a linear relation among dilogarithms. Theorem 11 gives the precise result.

In the next section, we use Theorem 11 to derive relations among twisted L series at $s = 2$ for various characters. Choosing a primitive Dirichlet character χ with conductor N and even order m, we set

$$\alpha_k = \zeta^{r_k} \text{ and } \beta_k = \zeta^{n_k}$$

where $\zeta = e^{2\pi i/N}$. The collection $\{r_k\}$ ($\{n_k\}$) is the set of $l \in \mathbf{Z}/N\mathbf{Z}$ such that $\chi(l) = 1$ ($\chi(l) = -1$). The condition that $\prod \alpha_k = \prod \beta_k$ can be expressed as a condition on χ; we then say that χ is regular. Lemma 12 shows that many characters are in fact regular. The next step is to rewrite sums of dilogarithms in terms of twisted L-series. Lemma 13 gives the necessary computations but in order to simplify the statement of Theorem 14, we introduce the notation

$$\hat{\chi}_b(n) = \sum_{k=1}^{N} \chi^b(k)\zeta^{kn}.$$

Even though χ is primitive, χ^b may not be primitive and therefore this Gauss sum is not necessarily equal to $\tau(\chi^b)\overline{\chi}^b(n)$. Theorem 14 then follows from Lemma 13 and Corollary 15 gives the special case when χ is an odd quadratic character. Lemma 16 and 17 refine the hypotheses of Corollary 15 so that the non-vanishing of $\Delta_\chi(t)$, the discriminant of $f_\chi(x, t)$ as a polynomial in x implies that the log terms in Theorem 14 vanish; this gives Theorem 3.

We begin the proof of Theorem 11 by defining the various branches of the algebraic function defined by $q(x, t) = 0$. Suppose that y_0 is not a critical point. Then M distinct single-valued analytic functions $x_i(t)$ are determined by

$$q(x_i(t), t) \equiv 0$$

for all t in a neighborhood of y_0. Along any path γ in $\mathbf{C} - \{c_1, \dots, c_r\}$, each $x_i(t)$ can be analytically continued.

We choose $y_0 = 0$ and order these functions by setting $x_i(0) = 1/\alpha_i$ (each α_i is distinct). We then define the curve γ to lie along the real axis but missing any critical points. Specifically, let $c_1' < c_2' < \dots < c_s'$ denote the points in $\{c_1, \dots, c_r\}$ which are real. Since $d_M(0) \neq 0$ and there are M distinct roots when $t = 0$, 0 is not a critical point. Let $\delta_1, \dots, \delta_s$ be semi-circles such that

$$\delta_j = \delta_j(\epsilon) = \{z|\mathrm{Im}(z) \geqq 0, |z - c_j'| = \epsilon\}.$$

Assume that ϵ is chosen so that no other critical points lie between δ_j and the real axis. Define the path $\gamma = \gamma(\epsilon)$ to be

$$(-\infty, c_1' - \epsilon) \cup \delta_1 \cup \dots \cup \delta_s \cup (c_s' + \epsilon, \infty)$$

as in Figure 1.

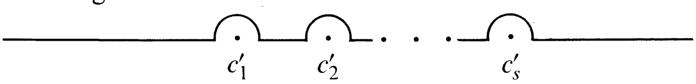

Figure 1. Analytic Continuation Path

We still denote by $x_i(t)$ its analytic continuation along γ. Letting $\epsilon \to 0$, we see that $x_i(t)$ is now defined unambiguously for all real $t \notin \{c_1', \dots, c_s'\}$. Furthermore, as long as $d_M(c_j') \neq 0$, we can set

$$x_i(c_j') = \lim_{\epsilon \to 0} x_i(c_j' + \epsilon)$$

because the M roots vary continuously as t varies. However if

$$y_0 = \prod \alpha_k / \prod \beta_k,$$

then $d_M(y_0) = 0$ and some of the M roots become "infinite" and disappear. Thus each function $x_i(t)$ is continuous on the real line minus

$$y_0 = \prod \alpha_k / \prod \beta_k$$

and is analytic near every real point $t \notin \{c'_1, \ldots, c'_s\}$.

From now on, fix a real number y and define I_y to be the interval between 0 and y, i.e., either $I_y = [0, y]$ or $[y, 0]$. We will assume y is chosen so that

$$y_0 = \prod \alpha_k / \prod \beta_k \notin I_y.$$

From $q(x, t) = 0$, it follows that for each i,

$$t = \frac{\prod_{k=1}^{M} (1 - \alpha_k x_i(t))}{\prod_{k=1}^{M} (1 - \beta_k x_i(t))}.$$

Taking absolute values and logarithms of both sides of the above equation produces

$$\operatorname{Re}\left\{ \sum_{k=1}^{M} [\log(1 - \alpha_k x_i(t)) - \log(1 - \beta_k x_i(t))] - \log(t) \right\} = 0.$$

However, the imaginary part is not as simple. For $t \neq 0$, define for each i and $t \in I_y$,

$$B_i(t) = \frac{1}{2\pi} \operatorname{Im}\left\{ \sum_{k=1}^{M} [\log(1 - \alpha_k x_i(t)) - \log(1 - \beta_k x_i(t))] - \log(t)] \right\}.$$

Thus B_i is an integer-valued step function which can only change value when x_i crosses the branch cut of one of the functions $\log(1 - \alpha_k x)$ or $\log(1 - \beta_k x)$. We have the following M equations:

$$\text{(3e)} \quad \sum_{k=1}^{M} [\log(1 - \alpha_k x_i(t)) - \log(1 - \beta_k x_i(t))] = \log t + 2\pi i B_i(t),$$

for $i = 1, 2, \ldots, M$.

We now define operators $\{T_i\}$ which will be applied to both sides of (3e) for each i.

Definition. If C is a piece-wise differentiable curve from $\mathbf{R}$ to $\mathbf{C}$ then for any integrable complex-valued function f for which the following integral is finite, we define the operator

$$T_C(f)(y) = \int_0^y f(t) \frac{C'(t)}{C(t)} dt.$$

For each $i = 1, 2, \ldots, M$, we will choose $C(t) = x_i(t)$ and apply the operator T_{x_i} (which we abbreviate as T_i) to both sides of equation (3e). Before doing so we need to set up some notation and several lemmas.

Definition. If C is a complex curve and R any complex ray, we say C *crosses* R at x if $x \in C \cap R$ and for all open intervals U along R with $x \in U$, $U \not\subseteq C \cap R$. In other works, x is a crossing point if C intersects R transversally, if C "touches" R at one point x, or if C initially contacts R at x and then continues along R.

Let us define

$$f_{\alpha,i}(t) = -\log(1 - \alpha x_i(t)) \quad \text{for all } i \quad \text{and}$$
$$\alpha \in \{\alpha_1, \dots, \alpha_M, \beta_1, \dots, \beta_M\}.$$

Let $\{t_i^j\}$, $j = 1, 2, \dots$, denote the set of values of t along the real axis such that $x_i(t)$ crosses any one of the rays

$$R_\alpha = \{z | z = r/\alpha, r > 1\}, \alpha \in \{\alpha_k, \beta_k\},$$

as in Figure 2.

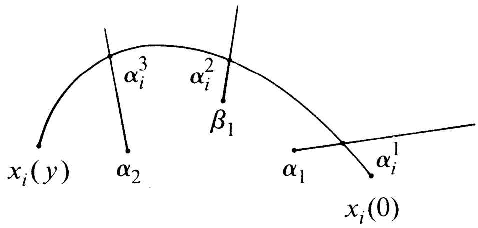

Figure 2. Crossing Points

Define $\alpha_i^j = x_i(t_i^j)$. Then we have

LEMMA 6. *Each function $x_i(t)$ crosses each ray R_α at most finitely many times. If y and $\{\alpha_k, \beta_k\}$ are algebraic, then the complex numbers $x_i(y)$ and $\{\alpha_i^j\}$ are algebraic as well for all i and j.*

Proof. We require the following definitions and two theorems from Lojasiewitz [**4**] which we state only for $\mathbf{R}^m$, $m = 1, 2, \dots$.

Definition. A set $E \subseteq \mathbf{R}^m$ will be called *semi-algebraic* if there exist a finite set of polynomials $\{p_r, q_s\}$ in m variables with complex coefficients such that

$$E = \{x \in \mathbf{R}^m | p_r(x) = 0 \text{ and } q_s(x) \leqq 0\}.$$

THEOREM (Seidenberg). *Suppose A and B are $\mathbf{R}^m$ and $\pi: A \times B \to A$ is projection onto the first coordinate. If E is contained in $A \times B$, and E is semi-algebraic, then so is $\pi(E)$.*

THEOREM (Lojasiewitz). *Any semi-algebraic set consists of a finite union of connected semi-algebraic sets.*

To prove Lemma 6, let us fix $\alpha \in \{\alpha_k, \beta_k\}$ and define $r_i^j = \alpha\alpha_i^j$. Then the points r_i^j which lie on the curve $\alpha x_i(t)$ and the ray $\{r > 1\}$ are among the solutions to

$$q(x/\alpha, t) = 0 \quad \text{for } x, t \in \mathbf{R}.$$

Therefore we apply the first theorem with $A = B = \mathbf{R}$ to the set

$$E = \{ (x, y) \,|\, q(x/\alpha, y) = 0, x \in A, y \in B\}.$$

$\pi(E)$ will be a semi-algebraic set by the first theorem and will consist of a finite union of intervals (which could be single points) by the second theorem.

Since the α_i^j's were crossing points of the curve x_i with the rays R_α, the collection $\{r_i^j\}$ lie among the end points of the intervals which make up $\pi(E)$. Therefore there are at most finitely many points where $x_i(t)$ crosses each R_α.

If we suppose that y and $\{\alpha_k, \beta_k\}$ are algebraic numbers, then the coefficients of $q(x, y)$ as a polynomial in x are algebraic as well. Thus each $x_i(y)$ is algebraic. Also, an inspection of the proofs of the two previous theorems from **[4]** shows that the polynomials which define $\pi(E)$ must have algebraic coefficients if $q(x, y)$ does. Because the $\{r_i^j\}$ are the end points of the intervals which make up $\pi(E)$, they are then defined by a set of polynomial equalities with algebraic coefficients and so must be algebraic numbers.

Since Lemma 6 implies the collection $\{\alpha_i^j\}$ is finite, let j run from 1 to ω_i for each i. We have

LEMMA 7. *Fix $\alpha \in \{\alpha_k, \beta_k\}$. For each $i = 1, 2, \dots, M$, the integral $T_i(f_{\alpha,i})(y)$ converges and there exists a collection of integers $\{M_{\alpha,i}^j(y)\}$, $j = 0, 1, \dots, \omega_i$, such that*

$$\text{(3f)} \quad T_i(f_{\alpha,i})(y) - [Li_2(\alpha x_i(y)) - Li_2(\alpha x_i(0))] = 2\pi i \sum_{j=1}^{\omega_i} M_{\alpha,i}^j(y) \log(\alpha a_i^j).$$

Proof. We first show that the integral $T_i(f_{\alpha,i})(y)$ converges. Looking at a more general situation, let $C(t)$ be any continuous function on I_y which is analytic in a neighborhood of all except a finite number of points in I_y and which crosses $[1, \infty)$ at most finitely many times. Further suppose that C is analytic at each point t_0 where $C(t_0) = 1$. We will first show that

$$\int_0^y - \log(1 - C(t)) \frac{C'(t)}{C(t)} dt$$

converges. Then setting $C(t) = \alpha x_i(t)$ will show that $T_i(f_{\alpha,i})(y)$ converges for each i.

Since $\log(1 - t)/t$ is continuous except at $t = 1$, we can bound this integral if for each t_0 with $C(t_0) = 1$, we can bound

$$\int_{t_0}^{t_0+\delta} - \log(1 - C(t))\frac{C'(t)}{C(t)}dt$$

for all sufficiently small δ. This integral is bounded as long as $C'(t)/C(t)$ is bounded near each point t_0 and the two integrals

$$I_1 = \int_0^{\delta} - \arg(1 - C(t + t_0))dt \quad \text{and}$$

$$I_2 = \int_0^{\delta} - \log|1 - C(t + t_0)|dt$$

converge for δ sufficiently small.

To prove I_i converges, choose δ small enough so that for all t between t_0 and $t_0 + \delta$, $C(t)$ never crosses $[1, \infty)$ and $C'(t)/C(t)$ is defined and continuous. The former is possible since C crosses $[1, \infty)$ only finitely many times and the latter is possible since C is analytic near each t_0. It follows by our choice of δ that the integrand of I_1 is continuous and therefore I_1 converges.

For the second integral I_2, note that for t sufficiently near t_0,

$$1 > |1 - C(t + t_0)| > c|t|^b$$

for some b and $c > 0$. Since

$$\int_0^{\delta} \log c + b \log|t|dt$$

is bounded for all δ, we conclude that I_2 converges.

Now if we let $C(t) = \alpha x_i(t)$, it follows that $T_i(f_{\alpha,i})(y)$ converges as well.

To prove equation (3f), suppose that $C(t)$ is as before and $C(0) = z_0$, $C(y) = z_1$. Then

$$\int_0^{y} - \log(1 - C(t))\frac{C'(t)}{C(t)}dt - [Li_2(C(y)) - Li_2(C(0))]$$

$$= \int_{z_0}^{z_1} - \log(1 - t)\frac{dt}{t} + \int_0^{z_0} - \log(1 - t)\frac{dt}{t}$$

$$- \int_0^{z_1} - \log(1 - t)\frac{dt}{t}$$

where the first integral is along the curve C, the second along the line L_0 from 0 to z_0, and the third along the line L_1 from 0 to z_1 as in Figure 3.

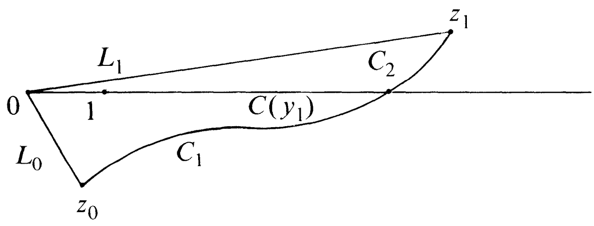

Figure 3. Integration Path

Combining the first two integrals, we obtain

$$\text{(3g)} \quad \int_{L_0 \cup C} -\log(1-t)\frac{dt}{t} - \int_{L_1} -\log(1-t)\frac{dt}{t}$$

with the endpoints as before.

If none of the curves C, L_0, and L_1 crosses the branch cut $[1, \infty)$, then the integrands are analytic inside $C \cup L_0 \cup L_1$ and (3g) is 0. Now suppose that only the curve C crosses $[1, \infty)$ and it crosses transversally exactly once at $C(y_1)$ as pictured in Figure 3. Divide C into arcs C_1 and C_2 and let $\hat{C}$ be the real interval $[1, C(y_1)]$. Let

$$g(t) = \lim_{x \to 0^+} \log(1 - (t + ix))$$

and

$$h(t) = \lim_{x \to 0^-} \log(1 - (t + ix)).$$

Then $g(t) - h(t) = 2\pi i$ for $t \geqq 1$ and by modifying the paths of the integrals in (3g), it follows that (3g) equals

$$\text{(3h)} \quad \pm\left\{\int_{\hat{C}} g(t)\frac{dt}{t} - \int_{\hat{C}} h(t)\frac{dt}{t}\right\}$$
$$= \pm 2\pi i \log(C(y_1))$$

for some choice of $\pm$ sign, depending on which of L_1 or $L_0 \cap C$ lies in the upper or lower half plane.

If C crosses $[1, \infty)$ several times, decompose C into a finite union of curves $\cup C_j$ where the end points of each C_j are consecutive crossing points, say $C(y_j)$ and $C(y_{j+1})$, along $[1, \infty)$. Then applying the argument above shows that (3g) equals

$$2\pi i \sum_j M^j \log(C(y_j))$$

for some integers M^j. Finally, it is easy to see that this conclusion is still valid if either C does not intersect $[1, \infty)$ transversally at some points y_j, or L_0 or L_1 intersects $[1, \infty)$.

Therefore, if for each $\alpha \in \{\alpha_k, \beta_k\}$ and $i = 1, \dots, M$ we set $C(t) = \alpha x_i(t)$, then $C(y_j) \in \{\alpha\alpha_i^j\}$ and we can choose integers $\{M_{\alpha,i}^j(y)\}$ so that we obtain (3f):

$$T_i(f_{\alpha,i})(y) - [Li_2(\alpha x_i(y)) - Li_2(\alpha x_i(0))]$$

$$= 2\pi i \sum_{j=0}^{\omega_i} M_{\alpha,i}^j(y) \log(\alpha\alpha_i^j).$$

LEMMA 8. *If $t \neq 1$, then $x_i(t) \neq 0$ for all $i = 1, 2, \dots, M$. Further, suppose*

$$\prod_{k=1}^{M} \alpha_k = \prod_{k=1}^{M} \beta_k.$$

Then

$$\prod_{i=1}^{M} x_i(t) = 1/\prod_{k=1}^{M} \alpha_k;$$

that is, the product of these functions of t is constant.

Proof. By gathering coefficients of powers of x in equation (3d), we see that

$$-q(x, t) = (-1)^M(\prod \alpha_k - \prod \beta_k t)x^M + \dots + (1 - t) = 0.$$

If $t \neq 1$, then the constant term of $q(x, t)$ is not zero. Therefore none of the roots $\{x_i\}$ can be 0. Given our hypothesis, we also have

$$\prod_{i=1}^{M} x_i(t) = (-1)^M \frac{(1 - t)}{(-1)^M(\prod \alpha_k - t\prod \beta_k)} = \frac{1}{\prod \alpha_k}.$$

The lemma follows.

Recall that B_i was defined by

$$B_i(t) = \frac{1}{2\pi} \operatorname{Im} \left\{ \sum_{k=1}^{M} [\log(1 - \alpha_k x_i(t)) - \log(1 - \beta_k x_i(t))] - \log(t)] \right\}.$$

We also have

LEMMA 9. *If $1 \notin I_y$, then the integral $T_i(\log)(y)$ converges and*

$$\text{(3i)} \quad T_i(B_i)(y) = \sum_{j=1}^{\omega_i} S_i^j(y) \log(\alpha_i^j) + \hat{N}_i(y) \log x_i(y) + \hat{M}_i(y) \log(\alpha_i) + 2\pi i W_i(y)$$

for some integers $\{S_i^j(y), \hat{N}_i(y), \hat{M}_i(y), W_i(y)\}$ *which depend on y. (Note: the number i in the factor* $2\pi i$ *should not be confused with the subscript i.)*

Proof. We first show that $T_i(\log)(y)$ must converge. Since $1 \notin I_y$, $y < 1$ and $x_i(t) \neq 0$ on I_y by Lemma 8. We suppose that $y < 0$. Choose $\delta > y$ so that for all t in I_y with $\delta < t < 0$, $x_i'(t)/x_i(t)$ is defined and continuous. Set

$$B = \max\left\{\sup_{[y,\delta]} |\log(t)|, \sup_{[\delta,0]} \left|\frac{x_i'(t)}{x_i(t)}\right|\right\}.$$

Recall that log was defined so that

$$-\pi < \operatorname{Im} \log(t) \leqq \pi.$$

We see that $|T_i(\log)(y)|$ is less than or equal to

$$\left|\int_0^\delta \log(t) \frac{x_i'(t)}{x_i(t)} dt\right| + \left|\int_\delta^{-1} \log(t) \frac{x_i'(t)}{x_i(t)} dt\right| + \left|\int_{-1}^{y} \log(t) \frac{x_i'(t)}{x_i(t)} dt\right|$$

if $y < -1$ (if $-1 < y < 0$, remove the third integral and let the limits of the second be from δ to y). The first integral is

$$\leqq B||\delta| - \delta|\log|\delta| + i\pi|\delta||.$$

Using the inequality $|a| + |b| \geqq |a + bi|$, the second integral is seen to be

$$\leqq \left|\int_\delta^{-1} \log|t| \operatorname{Re} \frac{x_i'(t)}{x_i(t)} dt - \pi \int_\delta^{-1} \operatorname{Im} \frac{x_i'(t)}{x_i(t)} dt\right| + \left|\int_\delta^{-1} \log|t| \operatorname{Im} \frac{x_i'(t)}{x_i(t)} dt + \pi \int_\delta^{-1} \operatorname{Re} \frac{x_i'(t)}{x_i(t)} dt\right|.$$

Notice that $\log|t|$ is negative on the entire range of integration. Therefore, we can bound these four integrals if we can bound

$$\int_c^d \frac{x_i'(t)}{x_i(t)} dt$$

for all real c and $d < 1$.

Let $C(t)$ be the curve $x_i(t)$ from $x_i(c)$ to $x_i(d)$. Then the techniques in Lemma 6 easily show that $C(t)$ crosses the ray $(-\infty, 1]$ at most finitely many times. Therefore

$$\int_c^d \frac{x_i'(t)}{x_i(t)} dt = \int_C \frac{dt}{t} = \log x_i(d) - \log x_i(c) + 2\pi i W_i$$

where W_i is an integer depending on the winding number of C around 0. This shows the second integral converges. Similarly, the third integral converges and an easier argument shows that $T_i(\log)(y)$ converges if $y > 0$.

To prove (3i), recall that $B_i(t)$ was an integer-valued step function which could only change value if $t \in \{t_i^j\}$, i.e., if t is a point where $x_i(t)$ crosses one of the rays R_α. Let $\Psi_{[a,b]}$ be the characteristic function on the interval $[a, b]$. Then

$$T_i(\Psi_{[a,b]})(y) = \log x_i(b) - \log x_i(a) + 2\pi i W_i$$

if $[a, b] \subseteq I_y$. Here W_i is an integer which depends on the winding number of $x_i(t)$ around 0 as t runs from a to b. W_i will be 0 if $x_i(t)$ never crosses $(-\infty, 0)$. By expressing B_i as a finite sum of characteristic functions, it follows that then there exists integers $W_i(y)$, $S_i^j(y)$, $j = 1, \ldots, \omega_i$, $\hat{N}_i$ and $\hat{M}_i$ such that

$$\text{(3j)} \quad T_i(B_i)(y) = \sum_{j=1}^{\omega_i} S_i^j(y) \log(\alpha_i^j) + 2\pi i W_i(y) + \hat{N}_i(y) \log x_i(y) - \hat{M}_i(y) \log x_i(0).$$

Since $x_i(0) = 1/\alpha_i$, the lemma follows.

LEMMA 10. *If*

$$\prod_{k=1}^{M} \alpha_k = \prod_{k=1}^{M} \beta_k,$$

then provided $y < 1$,

$$\sum_{i=1}^{M} T_i(\log)(y) = 0.$$

Proof. By Lemma 9, $T_i(\log)(y)$ converges for each i if $y < 1$. Clearly,

$$\sum_{i=1}^{M} T_i(\log)(y) = \int_0^y \log(t) \sum_{i=1}^{M} \left[\frac{x_i'(t)}{x_i(t)}\right] dt.$$

Using the product rule, it is easy to see that

$$\sum_{i=1}^{M} \frac{x_i'(t)}{x_i(t)} = \frac{\left(\prod_{i=1}^{M} x_i(t)\right)'}{\prod_{i=1}^{M} x_i(t)}.$$

By Lemma 8, $\prod x_i(t)$ is constant, so the ratio above is 0 for all values of t. Therefore

$$\sum_{i=1}^{M} T_i(\log)(y) = 0 \quad \text{if } y < 1.$$

If we apply the operator T_i to equation (3e), Lemmas 7, 8 and 9 show that for each $i = 1, 2, \dots, M$,

$$\text{(3k)} \quad T_i\left[\sum_k [\log(1 - \alpha_k x_i(t)) - \log(1 - \beta_k x_i(t))]\right]$$
$$- T_i(\log)(y) - 2\pi i T_i(B_i)(y)$$
$$= \sum_k [Li_2(\beta_k x_i(t)) - Li_2(\alpha_k x_i(t))]$$
$$- \sum_k [Li_2(\beta_k x_i(0)) - Li_2(\alpha_k x_i(0))]$$
$$- T_i(\log)(y) - 2\pi i \hat{N}_i(y) \log x_i(y) + 4\pi^2 W_i(y)$$
$$- 2\pi i\left[\hat{M}_i(y) \log(\alpha_i) + \sum_j S_i^j(y) \log(\alpha_i^j)\right]$$
$$- 2\pi i\left[\sum_{k,j} M^j_{\alpha_k,i}(y) \log(\alpha_k \alpha_i^j) - \sum_{k,j} M^j_{\beta_k,i}(y) \log(\beta_k \alpha_i^j)\right] = 0.$$

Summing (3k) over all values of i, the term

$$\sum_i T_i(\log)(y)$$

will disappear by Lemma 10. Let us also reorganize the indexing of the sums appearing in the last two lines of (3k). Let

$$\{d_j\} = \bigcup_{i,j,k} \{\alpha_k, \beta_k, \alpha_i^j\}$$

and choose integers $\{N_j(y)\}$ such that the last two lines of (3k) become

$$2\pi i \sum_j N_j(y) \log(d_j).$$

Finally, set

$$\sum_i W_i(y) = W(y).$$

The main theorem of this section has now been proven:

THEOREM 11. *Suppose that* $\{\alpha_k, \beta_k\}$ *are distinct non-zero complex numbers such that*

$$\prod_{k=1}^{M} \alpha_k = \prod_{k=1}^{M} \beta_k.$$

Then if $y < 1$, *we have*

$$(31)\quad \sum_{i=1}^{M} \sum_{k=1}^{M} [Li_2(\beta_k x_i(y)) - Li_2(\alpha_k x_i(y))]$$

$$- \sum_{i=1}^{M} \sum_{k=1}^{M} [Li_2(\beta_k/\alpha_i) - Li_2(\alpha_k/\alpha_i)]$$

$$= 2\pi i\left[\sum_j N_j(y) \log(d_j) + \sum_i \hat{N}_i(y) \log x_i(y)\right] - 4\pi^2 W(y)$$

for certain integers $\{N_j(y), \hat{N}_i(y), W(y)\}$ *depending on* y. *Furthermore, if* $\{\alpha_k, \beta_k\}$ *and* y *are all algebraic numbers, then each* d_j *and* $x_i(y)$ *is algebraic as well.*

Theorem 11 shows that certain linear combinations of dilogarithms can be expressed in terms of elementary functions. It does not seem possible to derive this result directly using the functional equations in Roger's paper. In any case, he does not consider how the branch cuts involved affect his identities.

In the next section we will choose the α_k's and β_k's so as to obtain a linear relation satisfied by $\{L_{\mu_i}(2, \chi)\}$ for fixed χ, in many cases without any logarithm terms.

4. Twisted *L*-series identities. Let χ be any Dirichlet character of conductor N and even order m with m dividing $\varphi(N)$. Let

$$M = \varphi(N)/m = \#\{k \in \mathbf{Z}/N\mathbf{Z} | \chi(k) = 1\}$$
$$= \#\{k \in \mathbf{Z}/N\mathbf{Z} | \chi(k) = -1\}.$$

We set

$$\zeta = e^{2\pi i/N}, \ \{\alpha_k\} = \{\zeta^k\}_{\chi(k)=1}, \text{ and } \{\beta_k\} = \{\zeta^k\}_{\chi(k)=-1}.$$

That is,

$$p(z) = p_\chi(z) = \prod_{\chi(k)=1} (1 - z\zeta^k)$$

and

$$\tilde{p}(z) = \tilde{p}_\chi(z) = \prod_{\chi(k)=1} (1 - z\zeta^k).$$

Note that both p_χ and $\tilde{p}_\chi$ have degree $M = \varphi(N)/m$. Then set

$$f_\chi(x, y) = q(x, y) = y\tilde{p}_\chi(x) - p_\chi(x).$$

Notice that this agrees with the definition of f_χ given in Section 2.

We will now investigate when the condition

$$\prod_{k=1}^{M} \alpha_k = \prod_{k=1}^{M} \beta_k$$

in Lemma 8 is satisfied.

To simplify notation, define r_k (resp. n_k) to be the k^{th} value of $l = 1, 2, \dots N$ such that $\chi(l) = 1$, (resp. -1). In this case of a quadratic character, r_k (resp. n_k) denotes the k^{th} residue (resp. non-residue).

Definition. Set

$$L = \sum_{k=1}^{M} r_k \quad \text{and} \quad L' = \sum_{k=1}^{M} n_k.$$

Then we will call a character χ *regular* if N divides $L - L'$.

It follows immediately that if χ is regular, then

$$\prod \alpha_k = \prod \zeta^{r_k} = \prod \zeta^{n_k} = \prod \beta_k,$$

so the conditions of Lemma 8 are satisfied. For example, if χ is even, $L - L' = 0$, and this shows that all even characters are regular.

LEMMA 12. (a) *If N is prime and χ is odd, χ is regular if and only if the order of χ is not $N - 1$.*

(b) *The odd quadratic character χ_{-N} is regular if $N \neq 3$ or 4.*

Proof. For the first statement, let g be a primitive root mod N. χ is odd, so

$$L - L' \equiv 2L \bmod N,$$

and χ is regular if and only if $N|L$. Let m be the order of χ.

$$L \equiv \sum_{k=0}^{(N-1)/m-1} (g^m)^k = \frac{1 - g^{N-1}}{1 - g^m} \equiv 0 \bmod N$$

as long as $m \neq N - 1$. If $m = N - 1$, then $L = 1$, and so $N \nmid L$.

The second statement follows from the Dirichlet class number formula:

$$h = \frac{1}{N}\left|\sum_{k=1}^{N} k\chi(k)\right| = \frac{1}{N}|L - L'|$$

if χ is the character for any imaginary quadratic field

$$\mathbf{Q}(\sqrt{-N}) \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3}).$$

Given that χ is regular, the conditions of Theorem 11 are satisfied and we obtain

$$\text{(4a)}\quad \sum_i \sum_k [Li_2(\zeta^{n_k}x_i(y)) - Li_2(\zeta^{r_k}x_i(y))]$$
$$- \sum_i \sum_k [Li_2(\zeta^{n_k - r_i}) - Li_2(\zeta^{r_k - r_i})]$$
$$= 2\pi i\left[\sum_j N_j(y)\log d_j + \sum_i \hat{N}_i(y)\log x_i(y)\right] - 4\pi^2 W(y)$$

for all $y < 1$.

We next investigate when the left hand side of (4a) can be expressed as a sum of twisted L-series.

Definition. If χ is any primitive character of conductor N, define

$$\hat{\chi}_b(n) = \sum_{k=1}^{N} \chi^b(k)\zeta^{kn}.$$

If χ is primitive and $(b, \varphi(N)) = 1$, it is easy to show that χ^b is primitive also. A well-known property of Gauss sums for primitive characters shows that

$$\hat{\chi}_b(n) = \tau(\bar{\chi}^b)\bar{\chi}^b(n) \quad \text{for all } n.$$

Here τ denotes the Gauss sum. However, $\hat{\chi}$ is not necessarily a multiple of the character $\bar{\chi}^b$ if $(b, \varphi(N)) > 1$, but it is periodic with period dividing N. Even though $\hat{\chi}_b$ is not a character, $L_\mu(s, \hat{\chi}_b)$ can be analytically continued as $L_\mu(s, \chi)$ was, namely

$$L_\mu(s, \hat{\chi}_b) = \sum_{k=1}^{N} \chi^b(k)Li_2(\mu\zeta^k).$$

It is easy to see that if $|\mu| < 1$, this definition agrees with the original definition of a twisted L-series for the periodic function $f = \hat{\chi}_b$. Then, for example when χ is primitive and $\hat{\chi} = \hat{\chi}_1$,

$$L(s, \hat{\chi}) = \tau(\chi)L(s, \bar{\chi}).$$

LEMMA 13. *Let χ be a primitive Dirichlet character of conductor N and even order m. Let*

$$M = \varphi(N)/m = \#\{k|\chi(k) = 1\}.$$

Recall that

$$\{r_k\} = \{l|\chi(l) = 1\} \text{ and } \{n_k\} = \{l|\chi(l) = -1\}.$$

For any $\mu \in \mathbf{C}$ and any $s = 2, 3, \ldots,$

$$\text{(4c)}\quad \sum_{k=1}^{M} [Li_s(\mu\zeta^{r_k}) - Li_s(\mu\zeta^{n_k})] = \frac{2}{m}\sum_{b\text{odd}, b=1}^{m-1} L_\mu(s, \hat{\chi}_b)$$

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and

(4d) $$\sum_{i=1}^{M} \sum_{k=1}^{M} [Li_s(\zeta^{r_k - r_i}) - Li_s(\zeta^{n_k - r_i})]$$
$$= \frac{2}{m^s} \sum_{c=0}^{m-1} \sum_{b\text{odd},b=1}^{m-1} L(s, \hat{\chi}_b \bar{\hat{\chi}}_c).$$

Proof. It is easy to see that

$$\sum_{b\text{odd},b=1}^{m-1} \chi^b(k) = \begin{cases} m/2\chi(k) & \text{if } \chi(k) = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore if b runs over the odd integers from 1 to $m - 1$ in the following sums,

$$\frac{2}{m} \sum_{b\text{odd}} L_\mu(s, \hat{\chi}_b)$$
$$= \sum_{k=1}^{N} \left(\frac{2}{m} \sum_{b\text{odd}} \chi^b(k)\right) Li_s(\mu\zeta^k)$$
$$= \sum_{k=1}^{M} [Li_s(\mu\zeta^{r_k}) - Li_s(\mu\zeta^{n_k})].$$

This proves (4c).

To prove (4d), we expand both sides using the series for $Li_s(x)$. The nth term of the left hand side is

(4e) $$\left\{\sum_i \zeta^{-r_i n}\right\}\left\{\sum_k \zeta^{r_k n} - \zeta^{n_k n}\right\}/n^s.$$

But an easy calculation shows

$$\sum_{i=1}^{M} \zeta^{-r_i n} = \frac{1}{m} \sum_{c=0}^{m-1} \bar{\hat{\chi}}_c(n)$$

and

$$\sum_{k=1}^{M} \zeta^{r_k n} - \zeta^{n_k n} = \frac{2}{m} \sum_{k=1}^{N} \sum_{b\text{odd}} \chi^b(k)\zeta^{kn}.$$

Using the above equalities, (4e) becomes

$$\frac{2}{m^2} \sum_{c=0}^{m-1} \sum_{b\text{odd},b=1}^{m-1} \hat{\chi}_b(n)\bar{\hat{\chi}}_c(n)/n^s$$

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which is the nth term of the right hand side of (4d).

Our main result in this section can now be proven:

THEOREM 14. *If $y < 1$ and χ is a regular primitive Dirichlet character of even order m and conductor N, then*

$$\text{(4f)}\quad -\frac{2}{m}\sum_i \sum_{b\,\mathrm{odd},b=1}^{m-1} L_{x_i(y)}(2, \hat{\chi}_b) + \frac{2}{m^2}\sum_{b\,\mathrm{odd},b=1}^{m-1}\sum_{c=0}^{m-1} L(2, \hat{\chi}_b\bar{\hat{\chi}}_c)$$
$$= 2\pi i\left[\sum_j N_j(y)\log(d_j) + \sum_{i=1}^{M}\hat{N}_i(y)\log x_i(y)\right] - 4\pi^2 W(y)$$

where

$$\{d_j\} = \bigcup_{i,j,k}\{\zeta^{\tau_k}, \zeta^{n_k}, \alpha_i^j\}$$

and $\{W(y), N_j(y), \hat{N}_i(y)\}$ are certain integers depending on y. Furthermore if y is algebraic, then all the d_j's are algebraic as well.

Proof. Simply apply Lemma 13 with $s = 2$ to equation (4a).

COROLLARY 15. *Let $y < 1$ and let $\chi = \chi_{-N}$ be an odd quadratic character of conductor $N > 4$. Suppose that $|x_i(t)| \equiv 1$ for all t between 0 and y and each i. Then*

$$\text{(4g)}\quad \sum_{i=1}^{M}\mathrm{Re}\, L_{x_i(y)}(2, \chi) - \frac{\hat{\mu}(N)}{2}L(2, \chi) = 0$$

where $\hat{\mu}$ is the Moebius function.

Proof. Since χ is quadratic, m is 2 and the indices b and c run from 1 to 1 and 0 to 1, respectively. An argument from elementary number theory shows that

$$\sum_{(k,N)=1}\zeta_N^k = \hat{\mu}(N)$$

where $\hat{\mu}(N)$ is the Moebius function of N defined as $(-1)^v$ if $N = p_1p_2\dots p_v$, where each p_i is a distinct prime, $\hat{\mu}(N) = 0$ if N is divisible by a square. Therefore, $\hat{\chi}_0(n) = \hat{\mu}(N)$ if $(n, N) = 1$, $\hat{\chi}_1\hat{\chi}_0 \equiv \hat{\mu}(N)\tau(\chi)\chi$, and (4f) becomes

$$\text{(4h)}\quad -\sum_i \tau(\chi)L_{x_i(y)}(2, \chi) + \frac{\hat{\mu}(N)}{2}\tau(\chi)L(2, \chi) - \frac{1}{2}\tau(\chi)^2 L(2, \chi^2)$$
$$= 2\pi i\left\{\sum_j N_j(y)\log d_j + \sum_i \hat{N}_i(y)\log x_i(y)\right\} - 4\pi^2 W(y).$$

Dividing (4h) by $-\tau(\chi) = -i\sqrt{N}$ and taking real parts gives

$$\sum_{i=1}^{M} \text{Re } L_{x_i(y)}(2, \chi) - \frac{\hat{\mu}(N)}{2} L(2, \chi)$$

$$= -\frac{2\pi}{\sqrt{N}} \left\{ \sum_j N_j(y) \log|d_j| + \sum_i \hat{N}_i(y) \log|x_i(y)| \right\}.$$

By hypothesis, $|x_i(t)| \equiv 1$ for all t between 0 and y. Therefore $|d_j| = 1$ for all j and $|x_i(y)| = 1$ for all i. Thus, the right hand side of equation (4h) is 0.

The following two lemmas will show that the hypothesis about the absolute values of the algebraic functions $x_i(t)$ can be replaced by a simple condition on the zeroes of $\Delta_\chi(t)$, the discriminant of $f_\chi(x, t)$ as a polynomial in x.

LEMMA 16. *If χ is any regular odd character and y is a real number, then $1/\bar{x}$ is a root of $q_\chi(x, y) = 0$ whenever x is.*

Proof. Because χ is odd, $\tilde{p}(x) = \bar{p}(x)$. Then

$$\frac{p(1/\bar{x})}{\bar{p}(1/\bar{x})} = \frac{\prod(1 - \zeta^r/\bar{x})}{\prod(1 - \zeta^n/\bar{x})}$$

(using a convenient abbreviation of notation)

$$= \frac{\prod(\bar{x} - \zeta^r)}{\prod(\bar{x} - \zeta^n)}$$

$$= \zeta^{L-L'} \frac{\prod(1 - \zeta^{-r}\bar{x})}{\prod(1 - \zeta^{-n}\bar{x})}$$

$$= \overline{\left(\frac{p(x)}{\bar{p}(x)}\right)} = \bar{y}.$$

The last step used the regularity of χ to show that $\zeta^{L-L'} = 1$. Since y is real, we have shown that $1/\bar{x}$ is a root if x is.

LEMMA 17. *Let χ be a regular odd quadratic character. If $\Delta_\chi(t) \neq 0$ for all $t \in I_y$, then $|x_i(t)| \equiv 1$ for all i and all $t \in I_y$.*

Proof. By assumption $\Delta_\chi(t) \neq 0$ on I_y, so there are no real critical points of the algebraic functions defined by $f_\chi(x, t) = 0$ on this interval and therefore each $x_i(t)$ is analytic near every point of I_y. Suppose that for some $x(t) \in \{x_i(t)\}$, there exists a point $y_0 \in I_y$ such that $|x(y_0)| \neq 1$. By Lemma 16, the function $1/\bar{x}(t)$ satisfies

$$f_\chi\left(\frac{1}{\bar{x}(t)}, t\right) = 0.$$

By Lemma 8, $x(t) \neq 0$ on I_y. Therefore, $1/\bar{x}(t)$ is analytic and must be equal to one of the M algebraic functions $\{x_i(t)\}$ on the interval I_y. Furthermore,

$$x(y_0) \neq 1/\bar{x}(y_0).$$

This implies $x(t)$ and $1/\bar{x}(t)$ are distinct; in fact, since $\Delta_\chi(t) \neq 0$ on I_y, $x(t) \neq 1/\bar{x}(t)$ for all $t \in I_y$. But $|x(0)| = 1$ so $x(0) = 1/\bar{x}(0)$, a contradiction. We conclude that $|x_i(t)| \equiv 1$ on the entire interval I_y.

Lemma 17 applied to Corollary 15 completes the proof of Theorem 3 which we restate below.

THEOREM 3. *Let χ be an odd quadratic character of conductor $N \neq 3$ or 4 and let $\Delta_\chi(t)$ be the discriminant of $f_\chi(x, t)$ considered as a polynomial in x. Suppose that for some real number $y < 1$, $\Delta_\chi(t) \neq 0$ for all t between 0 and y. Then for each t between 0 and y, $f_\chi(x, t) = 0$ has $M = \varphi(N)/2$ distinct roots $x = x_1(t), x_2(t), \dots, x_M(t)$, all of which have absolute value one, and*

$$\text{(4i)} \quad \sum_{i=1}^{M} \operatorname{Re} L_{x_i(t)}(2, \chi) = \frac{\hat{\mu}(N)}{2} L(2, \chi).$$

Since 0 is not a root of $\Delta_\chi(t)$ and $x_i(0) \neq 0$ for all i, there must exist an open interval around 0 where the conditions of Theorem 3 are satisfied. Since there exist infinitely many real algebraic numbers in every open interval, we also have

COROLLARY 18. *There exist infinitely many collections $\{\mu_i\}$ of $\varphi(N)$ algebraic numbers for each odd quadratic character χ of conductor N such that*

$$\text{(4j)} \quad \sum_{i=1}^{\varphi(N)} L_{\mu_i}(2, \chi) = \hat{\mu}(N) L(2, \chi).$$

Proof. Choose t to be a real algebraic number sufficiently close to 0 and let

$$\mu_{2i} = x_i(t) \text{ and } \mu_{2i-1} = \overline{x_i(t)}.$$

Then each μ_i is algebraic and since χ is real,

$$L_{\mu_{2i-1}} + L_{\mu_{2i}} = 2 \operatorname{Re} L_{x_i(t)}.$$

The corollary then follows from Theorem 3.

We note that equation (4i) seems to be valid whenever

$$|x_i(t)| \equiv 1 \quad \text{for all } i,$$

regardless of the hypotheses of Theorem 3. However, this has not been proven. In fact, there are many examples when the numbers $\{x_i(t)\}$ are all roots of unity. For example, if $t = 0$, $x_i(0) = \zeta^{-r_i}$ for all i. In this case, the resulting identity (4i) does follow from the Kubert identities. There are other examples when $t \neq 0$. Here is one with $\chi = \chi_{-7}$, $t \sim -4.79$ and $\zeta = e^{2\pi i/12}$:

$$\text{(4k)} \quad \frac{1}{2}L(2, \chi) + L_{\zeta^3}(2, \chi) + L_{\zeta^{-3}}(2, \chi) + L_{\zeta^2}(2, \chi) + L_{\zeta^{-2}}(2, \chi) + L_{\zeta^5}(2, \chi) + L_{\zeta^{-5}}(2, \chi) = 0.$$

Using (3b) with $x = 1$ and $M = 3$, 4, 6 and 12, we find that (4k) does indeed follow from the Kubert identities after some simplification.

This leads to the following question which we have not been able to resolve:

If t is chosen so that the roots of $f_\chi(x, t)$ are all roots of unity, then is equation (4i) always valid and does this identity follow from the Kubert identities?

5. Extensions of previous results. In this section, we first show that our method of producing polynomials g_χ such that

$$\log M(g_\chi) = (\text{a rational number}) \times L'(-1, \chi)$$

will apparently only work for those odd quadratic characters mentioned before, namely χ_{-3}, χ_{-4}, χ_{-7}, χ_{-8}, χ_{-20}, and χ_{-24}. We start by investigating the number of points $z = \mu_i$ on the unit circle such that

$$|p_\chi(z)/\tilde{p}_\chi(z)| = 1.$$

Let this number be n_χ. Recall that we computed Mahler's measure from equation (2d) by expressing each of the n_χ terms Re $L_{\mu_i}(2, \chi)$ as a rational times $L(2, \chi)$. Thus it seems reasonable to look at all odd characters of even order with a small value of n_χ.

LEMMA 19. *Let N be the conductor of an odd Dirichlet character χ with even order. If ν_χ equals the number of sign changes in the sequence* $\{\chi(k) | \chi(k) = \pm 1\}$ *for $k = 1, 2, \ldots, N + 1$, then $n_\chi \geqq \nu_\chi$.*

Proof. As z approaches ζ^r on the unit circle,

$$\left|\frac{p_\chi(z)}{\tilde{p}_\chi(z)}\right| \to 0.$$

As $z \to \zeta^n$, the ratio approaches ∞. Thus there must be a point on the unit circle between ζ^r and ζ^n where the ratio is 1. This corresponds to a sign change in the sequence of χ values.

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PROPOSITION 20. *Fix positive even integers m and B. Then there exists at most a finite number of odd Dirichlet characters χ of order m such that $\nu_{\chi} = B$.*

Proof. We will bound the conductors N of the characters which satisfy the hypotheses. Since χ has even order m, χ takes on the values ± 1 $2\varphi(N)/m$ times as k runs from 1 to N. Therefore, for a given B, the condition $\nu_{\chi} = B$ implies there exist integers a and b with $1 \leqq a < b \leqq N + 1$ such that at all values k between a and b, either $\chi(k)$ is never -1 and $\chi(k)$ equals 1 at least $2\phi(N)/(mB)$ times or $\chi(k)$ is never 1 and $\chi(k)$ equals -1 at least $2\phi(N)/(mB)$ times. In other words, there must be a string of consecutive 1's or -1's of length $2\phi(N)/(mB)$ in the sequence $\{\chi(k) \,|\chi(k) = \pm 1\}$ for $k = 1$ to $N + 1$.

Let $\chi^{(i)}$ denote the i^{th} conjugate of χ in $\mathbf{Q}(\chi) = \mathbf{Q}(\zeta_m)$. Then

$$\sum_i \chi^{(i)}(k) = 0$$

unless $\chi(k) = \pm 1$. The Polya-Vinogradov inequality, [**10**], pp. 143-147, implies that

$$\left| \sum_{k=a}^{b} \chi(k) \right| < \sqrt{N} \log(N).$$

Therefore

$$\varphi(m) \left| \sum_{\chi(k)=\pm 1, k=a}^{b} \chi(k) \right| = \left| \sum_i \sum_{k=a}^{b} \chi^{(i)}(k) \right| < \varphi(m) \sqrt{N} \log(N).$$

By our choice of a and b:

$$\text{(5a)} \quad \frac{2\varphi(N)}{mB} \leqq \left| \sum_{\chi(k)=\pm 1, k=a}^{b} \chi(k) \right| < \sqrt{N} \log(N).$$

Elementary estimates show

$$\varphi(N) > \frac{cN}{\log(N)}$$

for some $c > 0$. Thus

$$\frac{2c\sqrt{N}}{mB} < (\log(N))^2.$$

Since $\sqrt{N}$ grows faster than any power of $\log(N)$, for N sufficiently large this inequality is false. Thus N is bounded and the number of characters is finite.

The estimates used above are much too crude to be practical however. We could have used the character sum estimates of Burgess, but they involve undetermined constants. Instead, by obtaining sharper estimates for odd quadratic characters, we prove

PROPOSITION 21. *The only primitive odd quadratic characters* χ *with* $n_\chi = 2$ *are* χ_{-3}, χ_{-4}, χ_{-8}, χ_{-20}, *and* χ_{-24}. *The only such* χ *with* $n_\chi = 4$ *are* χ_{-7} *and* χ_{-15}.

Proof. When χ is an odd quadratic character, it is easy to show the inequalities:

$$\frac{\varphi(N)}{2\sqrt{N}} < \log(N)$$

if $\nu_\chi = 2$ and

$$\frac{\varphi(N) + 2}{4\sqrt{N}} < \log(N)$$

if $\nu_\chi = 4$. If N is the conductor of χ, N is the absolute value of the discriminant of the associated quadratic field. Thus $N = N'$, $4N'$, or $8N'$ where N' is odd and square-free. We will find the largest conductor N such that

$$\frac{\varphi(N) + 2}{4\sqrt{N}} < \log(N).$$

This will suffice for both cases $\nu_\chi = 2$ and 4.

First we will bound $w(N')$, the number of prime factors of N'. Let

$$N' = \prod p_i^{n_i}.$$

Then

$$\varphi(N') = \prod p_i^{n_i - 1} \prod (p_i - 1)$$

so

$$\frac{\prod p_i^{n_i - 1} \prod (p_i - 1) + 2}{4 \prod p_i^{n_i/2}} < \log(\prod p_i^{n_i}).$$

The left hand side increases faster than the right as n_i increases. Therefore, assume that $n_i = 1$ for all i. Clearly the left hand side is greater than or equal to

$$\frac{2 \times 4 \times 6 \times \ldots + 2}{\sqrt{3}\sqrt{5}\sqrt{7}\ldots}.$$

A calculation reveals that if $\omega(N') > 4$, the above inequality is false. Hence $\omega(N') \leqq 4$.

If $N = 4N'$, $\varphi(N) = 2\varphi(N')$, and

$$\frac{\varphi(N') + 1}{4\sqrt{N'}} < \log(4N').$$

A similar calculation shows that $\omega(N') \leqq 4$. This also follows if $N = 8N'$.

Secondly, a bound for N' will be obtained. Suppose that $N = N'$. Since $\omega(N') \leqq 4$,

$$\varphi(N') = N' \prod(1 - 1/p_i) \geqq N'\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \geqq \frac{32}{77}N'.$$

Therefore

$$\frac{\frac{16}{77}\sqrt{N'} + 1}{2\sqrt{N'}} < \log(N').$$

A calculation shows $N' < 7360$. If $N = 4N'$, $N' < 10510$. Finally, if $N = 8N'$, $N' < 5260$. All quadratic characters with conductor N in these ranges were examined by computer. The final result is the statement of Proposition 21.

The one character we have not examined previously is $\chi = \chi_{-15}$. There are four roots of $p_\chi(z) \pm \tilde{p}_\chi(z) = 0$ in this case: μ_1, μ_2 and their complex conjugates. However, none of these are roots of unity and it does not appear possible to isolate the two unknown terms in equation (2d), namely $\mathrm{Re}\, L_{\mu_1}(2, \chi)$ and $\mathrm{Re}\, L_{\mu_2}(2, \chi)$, as we did for the single term $\mathrm{Re}\, L_\mu(2, \chi)$ when $\chi = \chi_{-7}$.

Since we have found so few examples when Mahler's measure gives a rational times $L'(-1, \chi)$, let us instead try to find polynomials f_χ in two variables whose Mahler measure is a linear combination of derivatives of L functions with rational or even algebraic coefficients. We can extend the previous work directly to certain characters of even order m without much additional work.

PROPOSITION 22. *Let χ be an odd character of even order m and conductor N such that χ^b is primitive for all odd b less than m. Suppose that the region Ω defined in Section 2 as the set of $\theta \in [0, 2\pi]$ such that*

$$|p_\chi(e^{i\theta})| \geqq |\tilde{p}_\chi(e^{i\theta})|$$

consists of the interval $[0, \pi]$ with $\mu_1 = 1$ and $\mu_2 = -1$. Then

$$\text{(5b)} \quad \log M(f_\chi) = \frac{4}{mN} \mathrm{Re}\left\{ \sum_{b\,\mathrm{odd}, b=1}^{m-1} (2 - \bar{\chi}^b(2)/2) L'(-1, \chi^b) \right\}.$$

In particular, (5b) *holds if the conductor N of χ is a Fermat prime and $m = \varphi(N)$ or if $N = 2^a$ for $a > 2$ and the order of χ is $\varphi(N)/2 = 2^{a-2}$.*

Proof. Using equation (4c) and the fact that χ^b is primitive, we simplify (2c) and obtain

$$\text{(5c)} \quad \log M(f_\chi) = \frac{1}{m\pi} \operatorname{Re} \sum_{b \text{ odd}, b=1}^{m-1} i\tau(\chi^b)[-L_1(2, \bar{\chi}^b) + L_{-1}(2, \bar{\chi}^b)].$$

The term L_{-1} can be simplified using (1.2f), producing

$$\text{(5d)} \quad \log M(f_\chi) = \frac{1}{m\pi} \operatorname{Re} \sum_{b \text{ odd}, b=1}^{m-1} (\bar{\chi}^b(2)/2 - 2) i\tau(\chi^b) L(2, \bar{\chi}^b).$$

Finally, the functional equation (1d) allows (5d) to be expressed in terms of L', as in (5b).

If N is a Fermat prime, then $\varphi(N)$ is a power of 2 and it is clear that $(b, \varphi(N)) = 1$ for all odd b less than $\varphi(N)$. Furthermore, since the order of χ is $\varphi(N)$,

$$|p_\chi(z)/\tilde{p}_\chi(z)| = |(1 - \zeta_N z)/(1 - \zeta_N^{-1} z)| = 1$$

precisely when $z = \pm 1$. It is then easy to show that $\Omega = [0, \pi]$ and $\mu_1 = 1$ and $\mu_2 = -1$. Therefore the conditions of this proposition are satisfied.

Suppose that the conductor N is equal to 2^a for $a > 2$. Then the group $(\mathbf{Z}/N\mathbf{Z})^{\cdot}$ has 4 elements of order 2, ± 1 and $\pm b$ where $b \equiv 5^{2^{a-3}} \bmod 2^a$. Thus if χ has order $m = \varphi(N)/2$,

$$|p_\chi(z)/\tilde{p}_\chi(z)| = \frac{|(1 - \zeta z)(1 - \zeta^{-b} z)|}{|(1 - \zeta^{-1} z)(1 - \zeta^b z)|}$$

where $\zeta = \zeta_{2^a}$.

A calculation shows that

$$|p_\chi(z)/\tilde{p}_\chi(z)| = 1$$

if and only if

$$p_\chi(z)/\tilde{p}_\chi(z) = \pm \zeta^{1-b}.$$

Solving for z, we obtain two roots, $z = \pm 1$ of

$$\text{(5e)} \quad (\zeta^{1-b} \pm 1)z^2 - (\zeta^{-b} + \zeta)z \mp (\zeta^{-b} + \zeta)z \pm \zeta^{1-b} + 1 = 0$$

lying on the unit circle. We therefore find that Ω consists of $[0, \pi]$ and so $\mu_1 = 1$ and $\mu_2 = -1$. Thus the conditions of Proposition 22 are satisfied in this case also. We remark that this provides infinitely many examples of characters for which (5b) holds.

We now present an alternate way of generalizing our results in Section 1 to odd characters of even order greater than 2. Let F be $\mathbf{Q}(\chi)$, that is the field containing the values of χ. Let D_F be its different. Then for a fixed element $\alpha \in D_F^{-1}$, define

$$\eta_\alpha(k) = \mathrm{Tr}(\alpha_\chi(k)) = \sum_i \alpha^{(i)}\chi^{(i)}(k),$$

where $\alpha^{(i)}$ denotes the ith conjugate of a in F. By the definition of the different, η_α has integer values for all k. Finally, set

$$p_{\eta_\alpha}(z) = \prod_{\eta_\alpha(k)>0} (1 - \zeta^k z)^{\eta_\alpha(k)}$$

$$\widetilde{p}_{\eta_\alpha}(z) = \prod_{\eta_\alpha(k)<0} (1 - \zeta^k z)^{\eta_\alpha(k)}$$

and

$$f_{\eta_\alpha} = z_2\widetilde{p}_{\eta_\alpha}(z_1) - p_{\eta_\alpha}(z_1).$$

The value of k in the products runs from 1 to N. It is easy to see that if χ is quadratic and $a = 1, f_{\eta_\alpha} = f_\chi$. As before, define the region $\Omega \subseteq [0, 2\pi]$ to be the set of θ such that

$$|p_{\eta_\alpha}(e^{i\theta})| \geqq |\widetilde{p}_{\eta_\alpha}(e^{i\theta})|.$$

We then have

PROPOSITION 23. *Let χ be an odd character of even order. Suppose that Ω consists of the interval* $[0, \pi]$ *with* $\mu_1 = 1$ *and* $\mu_2 = -1$. *Then*

$$\text{(5f)} \quad \log M(f_{\eta_\alpha}) = \frac{2}{N} \sum_j \alpha^{(j)}(2 - \overline{\chi^{(j)}}(2)/2)L'(-1, \chi^{(j)}).$$

Proof. A computation similar to the one in Section 2 shows that if $\Omega = [0, \pi]$, then

$$\text{(5g)} \quad \log M(f_{\eta_\alpha}) = \frac{1}{2\pi} \mathrm{Re} \sum_{k=1}^{N} i\eta_\alpha(k)[-Li_2(\zeta^k) + Li_2(-\zeta^k)].$$

We write out the n^{th} term of the Taylor series for

$$\sum_{k=1}^{N} \eta_\alpha(k)Li_2(\mu\zeta^k):$$

$$\text{(5h)} \quad \frac{1}{n^2} \sum_k \eta_\alpha(k)\mu^n\zeta^{kn}$$

$$= \frac{\mu^n}{n^2} \sum_j \alpha^{(j)} \sum_k \chi^{(j)}(k)\zeta^{kn}$$

$$= \frac{\mu^n}{n^2} \sum_j \alpha^{(j)} \tau(\chi^{(j)}) \overline{\chi^{(j)}(n)}.$$

Therefore setting $\mu = \pm 1$, from (5g) we obtain,

$$\text{(5i)} \quad \log M(f_{\eta\alpha}) = \frac{1}{2\pi} \operatorname{Re} \sum_j i\alpha^{(j)} \tau(\chi^{(j)})[-L_1(2, \overline{\chi^{(j)}}) + L_{-1}(2, \overline{\chi^{(j)}})]$$

$$= \frac{1}{2\pi} \operatorname{Re} \sum_j i\alpha^{(j)} \tau(\chi^{(j)})(\overline{\chi^{(j)}}(2)/2 - 2) L(2, \overline{\chi^{(j)}}).$$

From the fact that χ is odd, it is easy to see that

$$\sum_{k=1}^{N} \eta_\alpha(k) Li_2(\mu\zeta^k)$$

is purely imaginary. Therefore, we do not need to take the real part in (5i). Thus applying (1d) to (5i), we obtain (5f) and the proposition is proven.

Propositions 22 and 23 extend Corollary 2 by giving collections of polynomials in two variables whose Mahler measure can be expressed as a linear combination of derivatives of L functions with algebraic coefficients. However, f_χ and f_η do not have integer coefficients in general; the coefficients would lie in the field $\mathbf{Q}(\chi)$. One could of course multiply together all the conjugates of f_χ (or f_{η_α}) to obtain a polynomial in $\mathbf{Z}[x, y]$.

There are many directions which could be pursued. One is to try to use the generalization by Sandham **[8]** of the methods of Rogers to obtain identities similar to (3k) with trilogarithms instead of dilogarithms. The resulting relations would undoubtably be extremely complicated and further progress is intimately tied to an old unsolved problem of finding functional equations for general polylogarithms.

It would also be desirable to attack the following questions:

(i) Are there other "nice" identities for other characters besides χ_{-7}?

(ii) Do there exist polynomials $p_\chi(x, y, z)$ for each non-trivial even quadratic character χ such that

$$\log M(p_\chi) = (\text{a rational number}) \times L'(-2, \chi)?$$

(iii) Finally, is Mahler's measure the natural quantity to look at when investigating these questions?

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