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SMOOTH PARTITIONS OF UNITY ON BANACH SPACES

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ABSTRACT. It is shown that if a Banach space X admits a C^k-smooth bump function, and X^* is Asplund, then X admits C^k-smooth partitions of unity.

1**. Introduction.** The existence of C^k-smooth partitions of unity on Banach spaces is important for the approximation of continuous maps by C^k-smooth functions [BF]. Recall that a Banach space X is said to admit *C^k-smooth partitions of unity*, if for any open cover $\{U_\alpha\}_{a\in I}$ of X, there exists a family, $\{f_\alpha\}_{\alpha\in I}$, of real-valued, C^k-smooth maps such that:

(i) The sets, support$(f_\alpha) \equiv \overline{\{x \in X : f_\alpha(x) \neq 0\}}$, are locally finite (each $x \in X$ has a neighbourhood intersecting only finitely many support (f_α)),

(ii) $0 \leq f_\alpha \leq 1$ for all α, and $\sum_\alpha f_\alpha = 1$,

(iii) For each $\alpha \in I$, we have support$(f_\alpha) \subset U_\alpha$.

Clearly, the existence of C^k-smooth partitions of unity on X implies the existence of a C^k-smooth, real-valued function with bounded, non-empty support (a *C^k-smooth bump function* for short), and so the latter condition is a necessary one for X to admit such smooth partitions. It is an open problem whether or not any Banach space which admits a C^k-smooth bump function admits C^k-smooth partitions of unity. This problem has a positive solution under various additional assumptions on X. For example, it is known to be true for weakly countably determined X [GTWZ], and for X in which X^* is weakly compactly generated [M1]. Further recent results on smooth partitions of unity can be found in [M2], [F], [V], and [DGZ2]. In this note we show that the above problem has a positive solution for Banach spaces X for which X^* is Asplund. This includes quasi-reflexive spaces. We also indicate how slight modifications of our main construction yields the results of [GTWZ] and [M1].

2**. Notation.** X shall denote a real Banach space, X^* its dual, and X^{**} its second dual. Operations involving the weak and weak star topologies will be prefixed by w and w^*, respectively. The closed linear span of a set $Y \subset X$ is written $\overline{\mathrm{sp}(Y)}$. The closed unit ball of X is denoted B_X, the unit sphere S_X, and similar expressions for the dual spaces. For a set Γ, the Banach space $c_0(\Gamma)$ is defined by

$$c_0(\Gamma) = \Big\{f \colon \Gamma \longrightarrow \mathbb{R} : \forall \epsilon > 0, \big\{\alpha \in \Gamma : |f(\alpha)| \geq \epsilon\big\} \text{ is finite}\Big\},$$

equipped with the supremum norm. X is said to be an *Asplund space* if every separable

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subspace has separable dual. X has the *Radon-Nikodym Property (RNP)*, if for each non-empty, bounded set $A \subset X$, and $\epsilon > 0$, there exists $\Lambda \in X^*$ and $\alpha \in \mathbb{R}$ such that $A \cap \{x \in X : \Lambda(x) > \alpha\}$ is non-empty with diameter less than ϵ. We note that X^* has the RNP iff X is Asplund (see, *e.g.*, [P]). If X does not contain an isomorphic copy of l_1, we write $l_1 \underset{\neq}{\subset} X$. Smoothness is meant in the Fréchet sense, unless otherwise stated. We call a Banach space C^k-*smooth* if it admits a C^k-smooth, real-valued function with non-empty, bounded support.

A Banach space X is said to admit a *separable projectional resolution of the identity (SPRI)* if for the limit ordinal $\mu = \operatorname{dens}(X)$, there exist continuous linear projections, $\{T_\alpha\}_{\alpha\in\Gamma}$, where $\Gamma = [\aleph_0, \mu]$, such that if $R_\alpha = \frac{T_{\alpha+1}-T_\alpha}{\|T_{\alpha+1}\|+\|T_\alpha\|}$,

(i) $T_\alpha T_\beta = T_{\min(\alpha,\beta)}$
(ii) $(T_{\alpha+1} - T_\alpha)(X)$ is separable for all $\alpha \in \Gamma$
(iii) For all $x \in X$ and $\epsilon > 0$, $\{\alpha \in \Gamma : \|R_\alpha(x)\| \geq \epsilon\}$ is finite
(iv) For all $x \in X$, $x \in \overline{\operatorname{sp}\{R_\alpha(x) : \alpha < \mu\}}$

We shall require the following two fundamental results.

THEOREM 2.1 [FG]. *If X is an Asplund space, then X^* admits a separable projectional resolution of the identity. Further, there are a set Γ and a bounded, linear injection $T: X^* \longrightarrow c_0(\Gamma)$.*

THEOREM 2.2 [T]. *A Banach space X admits C^k-smooth partitions of unity iff there exist a set Γ and a coordinatewise smooth homeomorphic embedding of X into $c_0(\Gamma)$.*

3. Main Results.

THEOREM 3.1. *Let X be a C^k-smooth Banach space for which X^* is Asplund. Then X admits C^k-smooth partitions of unity.*

PROOF. We shall construct the map required by Theorem 2.2. We follow the construction in [M1]. Since X admits a C^k-smooth bump function, X is Asplund (see [DGZ1]), and by Theorem 2.1 there exists a SPRI, $\{T_\alpha\}_{\alpha\in\Gamma}$, on X^*. We write $\sigma = T_{\alpha+1} - T_\alpha$, and $\sigma^* = T^*_{\alpha+1} - T^*_\alpha$. Also, since X admits a C^k-smooth bump function, by composing such a function with suitable smooth bump functions in $C^\infty(\mathbb{R}, \mathbb{R})$, we can construct maps $\phi_n \in C^k(X, [0, 1])$, such that for some constant $c \in (0, 1)$, we have

$$\phi_n(x) = \begin{cases} 0 & \text{for } \|x\| < c/n \\ 1 & \text{for } \|x\| \geq 1/n \end{cases} .$$

In a similar fashion, we can define a map $\xi \in C^k(X, [0, 1])$ with $\xi \equiv 0$ on $X \backslash B_X$, and $\xi > 0$ in a neighbourhood of the origin.

Because $\sigma_\alpha(X^*)$ is separable, the unit ball B_α of $\sigma^*_\alpha(X^{**})$ is w^*-metrizable and w^*-separable. Let $\{z^\alpha_n\}^\infty_{n=1}$ be w^*-dense in B_α, and for each k, let $B_{1/k}(z^\alpha_n)$ be a (relative) w^*-ball of radius $1/k$ about z^α_n. Using the w^*-density of B_X in B_α ([H], Lemma 16F), choose $y^\alpha_{n,k} \in B_X \cap B_{1/k}(z^\alpha_n)$. Then we have $\overline{\mathrm{sp}\{y^\alpha_{n,k} : n,k \in \mathbb{N}\}}^{w^*} \supset \sigma^*_\alpha(X^{**})$. Now, from the properties of a SPRI, we have that for any $x^{**} \in X^{**}$, $x^{**} \in \overline{\mathrm{sp}\{\sigma^*_\alpha(x^{**}) : \alpha \in \Gamma\}}^{w^*}$, and so in particular, for $x \in X$, $x \in \overline{\mathrm{sp}\{\sigma^*_\alpha(X^{**}) : \alpha \in \Gamma\}}^{w^*} \subset \overline{\mathrm{sp}\{y^\alpha_{n,k} : \alpha \in \Gamma, n,k \in \mathbb{N}\}}^{w^*}$. Since the $y^\alpha_{n,k}$ and x are in X, and w^*-continuous functionals on X^{**}, are w-continuous on X, we have, for every $x \in X$, $x \in \overline{\mathrm{sp}\{y^\alpha_{n,k} : \alpha \in \Gamma, n,k \in \mathbb{N}\}}^{w} = \overline{\mathrm{sp}\{y^\alpha_{n,k} : \alpha \in \Gamma, n,k \in \mathbb{N}\}}$, the last equality following from Mazur's Theorem. To ease notation, we relabel the countable indices of $\{y^\alpha_{n,k}\}^\infty_{n,k=1}$, and write simply $\{y^\alpha_n\}^\infty_{n=1}$, which shall be used in the sequel.

LEMMA 3.2. *Let X satisfy the hypothesis of Theorem 3.1. Then, with notation as above, we have $\{\|R^*_\alpha(x)\|\}_{\alpha\in\Gamma} \in c_0(\Gamma)$. Further, for each $x \in X$ and $\epsilon > 0$, there are a finite set $K \subset \Gamma$, and a neighbourhood N of x with $\|R^*_\alpha(y)\| \le \epsilon$ for all $y \in N$ and $\alpha \notin K$.*

PROOF. We claim that for each $x \in X$, and $\epsilon > 0$, there is a finite set $K \subset \Gamma$, so that $\|R^*_\alpha(x)\| < \epsilon$ for all $\alpha \notin K$. If not, then there exists an $x \in X$, an $\epsilon > 0$, and a sequence $\alpha_n \in \Gamma$ with $\|R^*_{\alpha_n}(x)\| \ge 2\epsilon$. Hence, there are $z_{\alpha_n} \in B_{X^*}$ with $(R^*_{\alpha_n}x)(z_{\alpha_n}) = (R_{\alpha_n})(z_{\alpha_n})(x) > \epsilon$. Set $k_{\alpha_n} = (R_{\alpha_n})(z_{\alpha_n}) \in B_{X^*}$, and $Z = \overline{\mathrm{sp}\{\cup^\infty_{n=1} R_{\alpha_n}(X^*)\}}$. Now Z is a separable space with the RNP, which is Asplund since X^* is assumed Asplund. It follows from Theorem A [EW] that (B_Z, w) is Polish. Since $l_1 \subsetneq Z$, $\{k_{\alpha_n}\}^\infty_{n=1} \in B_Z$ contains a w-Cauchy subsequence [R], which we also write as $\{k_{\alpha_n}\}^\infty_{n=1}$. Thus, there exists a k_0 with $k_{\alpha_n} \xrightarrow{w} k_0$. Because $k_{\alpha_n} \in R_{\alpha_n}(X^*)$, by the properties of a SPRI, for all $\alpha \in \Gamma$ and $\phi \in X^{**}$, $\lim_n R^*_\alpha(\phi)(k_{\alpha_n}) = 0$. Thus, $0 = \lim_n R^*_\alpha(\phi)(k_{\alpha_n}) = R^*_\alpha(\phi)(k_0)$. As noted above, for each $x \in X$, $x \in \overline{\mathrm{sp}\{R^*_\alpha(x) : \alpha \in \Gamma\}}^{w^*}$, and therefore $k_0 = 0$. But $k_{\alpha_n}(x) > \epsilon$ for all n, and so $k_0(x) \ge \epsilon$. This contradiction establishes the claim, and hence statement one of the lemma. If the second statement is not true, then there are an $x_0 \in X$ and $\epsilon > 0$ so that for $K = \{\alpha \in \Gamma : \|R^*_\alpha(x_0)\| \ge \epsilon/2\}$ there is an x with $\|x - x_0\| < \epsilon/2$, and $\alpha \notin K$ with $\|R^*_\alpha(x)\| \ge 2\epsilon$. Choose $z \in B_{X^*}$ such that $R^*_\alpha(x)(z) = (R_\alpha)(z)(x) > \epsilon$, and setting $k = (R_\alpha)(z)$, we then have $k(x) - k(x_0) > \epsilon - \epsilon/2 = \epsilon/2$. On the other hand, $k(x - x_0) \le \|k\|\,\|x - x_0\| < \epsilon/2$, a contradiction. ∎

Next, for each α, since $\sigma_\alpha(X^*)$ is separable, choose $\{f^\alpha_j\}^\infty_{j=1} \subset \sigma_\alpha(S_{X^*})$ separating on $\sigma^*_\alpha(X^{**})$, and set $\tilde f^\alpha_j = \frac{f^\alpha_j}{\|T_{\alpha+1}\|+\|T_\alpha\|}$. Note that since $f^\alpha_j = \sigma_\alpha(x^*_j)$, for some $x^*_j \in S_{X^*}$, we have that, $f^\alpha_j(x) = \sigma_\alpha(x^*_j)(x) = \sigma^*_\alpha(x)(x^*_j) \le \|\sigma^*_\alpha(x)\|\,\|x^*_j\| \le \|\sigma^*_\alpha(x)\|$. It follows that $\tilde f^\alpha_j(x) \le \|R^*_\alpha(x)\|$, and so $\sum^\infty_{j=1} 2^{-j}\tilde f^\alpha_j(x) \le \|R^*_\alpha(x)\|$. Also, from Theorem 2.1, let $T\colon X^{**} \longrightarrow c_0(\Gamma_1)$ be a continuous, linear injection, for some set Γ_1.

Next we enumerate $\mathbb{Q} = \{r_j\}^\infty_{j=1}$, $\cup^\infty_{n=1}\mathbb{Q}^n = \{\rho_j\}^\infty_{j=1}$, and $\cup^\infty_{n=1}\mathbb{N}^n = \{\nu_j\}^\infty_{j=1}$. Then for $\rho_j = (r_{j_1},\ldots,r_{j_m})$, $\nu_k = (n_{k_1},\ldots,n_{k_m})$, and $a = (\alpha_1,\ldots,\alpha_m) \in \Gamma^m$, we define $P_{\rho_j,\nu_k,a}$ to be a projection onto the one dimensional subspace $\mathrm{sp}\{\sum^m_{i=1} r_{j_i} y^{\alpha_i}_{n_{j_i}}\}$. Let F be the collection of finite subsets of Γ, and if $a = (\alpha_1,\ldots,\alpha_m) \in \Gamma^m$, we define $d(a)$ to be the distinct elements of the set $\{\alpha_1,\ldots,\alpha_m\}$. We also define $S = \{(\rho_j,\nu_k,a) \in \mathbb{Q}^m \times \mathbb{N}^m \times \Gamma^m\}^\infty_{m=1}$. Let I be the disjoint union of $S \times \mathbb{N}$, F, $\mathbb{N}$, a copy $\mathbb{N}'$ of $\mathbb{N}$, and Γ_1. We define a map

$\zeta\colon X \longrightarrow c_0(I)$ by

$$\zeta(x)(\iota) = \begin{cases} 2^{-j-k-l}\phi_l\Big(P_{\rho_j,\nu_k,a}(x)\Big) \prod_{\alpha\in d(a)}\Big\{\sum_{m=1}^{\infty} 2^{-m}\Big(\tilde{f}_m^{\alpha}(x)\Big)^2\Big\} \\ \qquad \iota = \Big(\big(\rho_j,\nu_k,a\big),l\Big) \in S\times\mathbb{N} \\ \prod_{\alpha\in K}\Big\{\sum_{m=1}^{\infty} 2^{-m}\Big(\tilde{f}_m^{\alpha}(x)\Big)^2\Big\}\iota = K \in F \\ 2^{-n}\xi(x/n) \quad \iota = n\in\mathbb{N} \\ 2^{-n'}\phi_{n'}(x) \quad \iota = n' \in \mathbb{N}' \\ T(x)(\gamma) \quad \iota = \gamma\in\Gamma_1. \end{cases}$$

We now show that line one maps into $c_0(I)$. Let $\epsilon > 0$, and fix $x_0 \in X$. From Lemma 3.2 we have that $\{\|R_\alpha^*(x_0)\|\}_{\alpha\in\Gamma} \in c_0(\Gamma)$, and therefore there are only finitely many $\alpha \in \Gamma$, say N, for which $\|R_\alpha^*(x_0)\| \geq 1$. Letting $M = \max_{\alpha\in\Gamma}\{\|R_\alpha^*(x_0)\|\}$, we then have for $a \in \Gamma^m$

$$\prod_{\alpha\in d(a)}\Big\{\sum_{m=1}^{\infty} 2^{-m}\Big(\tilde{f}_m^{\alpha}(x_0)\Big)^2\Big\} \leq \prod_{\alpha\in d(a)} \|R_\alpha^*(x_0)\|^2 \leq M^{2N}.$$

Hence, if $\max\{j,k,l\} > M^{2N}/\epsilon$, then line one maps into $c_0(I)$ since $\phi_l \in [0,1]$. If $\max\{j,k,l\} \leq M^{2N}/\epsilon$, we proceed as follows. If $d(a) = \{\alpha_1,\ldots,\alpha_m\} \in F^m$ is such that $\|R_{\alpha_{j_0}}^*(x_0)\| < \sqrt{\epsilon}/M^N$ for some j_0, then

$$\prod_{\alpha\in d(a)}\Big\{\sum_{m=1}^{\infty} 2^{-m}\Big(\tilde{f}_m^{\alpha}(x_0)\Big)^2\Big\} \leq \prod_{\substack{\alpha\neq\alpha_{j_0}\\ \alpha\in d(a)}} \|R_\alpha^*(x_0)\|^2\frac{\epsilon}{M^{2N}} < \epsilon.$$

Again, since $\{\|R_\alpha^*(x_0)\|\}_{\alpha\in\Gamma} \in c_0(\Gamma)$, there is a finite set $K \in F$ with $\|R_\alpha^*(x_0)\| \geq \sqrt{\epsilon}/M^N$ only for $\alpha \in K$. It follows that line one in the definition of ζ maps into $c_0(I)$. That the other lines in the definition of ζ map into $c_0(I)$ follows similarly. Hence, ζ maps into $c_0(I)$. That ζ maps continuously into $c_0(I)$ is established by noting that the lines defining ζ are equicontinuous using the second statement in Lemma 3.2. Also, ζ is injective since T is, and is coordinatewise C^k-smooth since each line in the definition of ζ is C^k-smooth.

Finally, we show that ζ^{-1} is continuous using the method of [GTWZ]. Because ζ is injective and continuous, we will establish the continuity of ζ^{-1} if we can show that $\zeta(x_p) \longrightarrow \zeta(x_0)$ implies $x_p \longrightarrow x_0$ for $\{x_0\}\cup\{x_p\}_{p=1}^{\infty} \subset X$. This will be proven by showing that $\zeta(x_p) \longrightarrow \zeta(x_0)$ implies that $\{x_0\}\cup\{x_p\}_{p=1}^{\infty}$ is relatively norm compact. The fact that $\{x_0\}\cup\{x_p\}_{p=1}^{\infty}$ is bounded follows from line three in the definition of ζ. Indeed, choose $n \in \mathbb{N}$ so that $\xi(x_0/n) > 0$. Then $\zeta(x_p)(n) \longrightarrow \zeta(x_0)(n) \neq 0$, and the definition of ξ now implies that for only finitely many p do we have $\zeta(x_p)(n) = 0$, the remaining x_p lying in nB_X. We next show that $\{x_0\}\cup\{x_p\}_{p=1}^{\infty}$ is totally bounded.

Let $\epsilon > 0$, and first suppose $x_0 = 0$. Choose $n_0' > c/\epsilon$. Then line four implies that $\phi_{n_0'}(x_p) \longrightarrow 0$, and hence there exists an N with $p > N$ implying $\|x_p\| < \epsilon$. Therefore, the tail of $\{x_p\}_{p=1}^{\infty}$ lies within ϵ of a one dimensional subspace, and so is totally bounded. This gives us that $\{x_0\}\cup\{x_p\}_{p=1}^{\infty}$ is relatively norm compact, and we are done.

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Let $\epsilon > 0$, and suppose $x_0 \neq 0$. Because $x_0 \in \overline{\mathrm{sp}\{y_n^\alpha : \alpha \in \Gamma, n \in \mathbb{N}\}}$, there is an $l_0 > 1/\epsilon$ and a triple $(\rho_j, \nu_k, a) \in S$ with $\|(I - P_{\rho_j,\nu_k,a})(x_0)\| < c/l_0$. From the definition of ϕ_n and the fact that the f_m^α are separating, we have from lines one and two that $\zeta(x_0)(\rho_j, \nu_k, a) = 0$ and $\zeta(x_0)\big(d(a)\big) = 2\eta$, for some $\eta > 0$. Therefore, there is an N such that $p > N$ implies both $\zeta(x_p)(\rho_j, \nu_k, a) < \eta 2^{-j-k-l_0-1}$ and $\zeta(x_p)\big(d(a)\big) > \eta$. From these inequalities, we have $\phi_{l_0}\big((I - P_{\rho_j,\nu_k,a})(x_p)\big) < 1/2$ for $p > N$, and so $\|(I - P_{\rho_j,\nu_k,a})(x_p)\| < 1/l_0 < \epsilon$ for all $p > N$. This establishes that for $p > N$, we have $\mathrm{dist}\big(x_p, P_{\rho_j,\nu_k,a}(X)\big) < \epsilon$, and since $P_{\rho_j,\nu_k,a}(X)$ is one dimensional, this gives us that $\{x_0\} \cup \{x_p\}_{p=1}^\infty$ is totally bounded, and thus relatively norm compact. The result now follows from Theorem 2.2. ∎

Recall that X is said to be *quasi-reflexive* if X has finite codimension in X^{**} under the canonical embedding. The classical example of a non-reflexive, quasi-reflexive Banach space is the James space. We have the following.

COROLLARY 3.3. *Let X be a C^k-smooth, quasi-reflexive Banach space. Then X admits C^k-smooth partitions of unity.*

PROOF. Since closed subspaces of quasi-reflexive spaces are quasi-reflexive, and separable, quasi-reflexive spaces have separable dual, we have that X quasi-reflexive implies X Asplund. Further, X is quasi-reflexive iff X^* is quasi-reflexive, and so by Theorem 3.1 we are done. ∎

REMARKS. 1. If X^* is Asplund, then X has the RNP, and so for $k > 1$ the existence of a C^k-Gâteaux smooth bump function on X implies that X is superreflexive [MPVZ].

2. The hypothesis that X^* is Asplund was used only in establishing Lemma 3.2, and for the existence of a linear injection $T: X \longrightarrow C_0(\Gamma)$, for some Γ. By changing the assumptions on X or X^* and modifying the proof of Lemma 3.2, we can recover the results of [GTWZ] and [M1].

Indeed, if X is C^k-smooth and weakly countably determined, then X is a weakly countably determined Asplund space, hence it admits a linear injection $T: X \longrightarrow C_0(\Gamma)$, and the maps $R_\alpha: X^* \longrightarrow X^*$ in Theorem 3.1 can then be chosen to be $w^* - w^*$-continuous. Hence, for $\phi \in X^{**}$ and $\alpha \in \Gamma$, $R_\alpha^*(\phi) \in X$, and so if k_0 is a w^*-limit point of $\{k_{\alpha_n}\}_{n=1}^\infty$, we have $R_\alpha^*(\phi)(k_{\alpha_n}) \longrightarrow R_\alpha^*(\phi)(k_0)$, and Lemma 3.2 can be completed as before. This gives the result in [GTWZ].

If X is C^k-smooth and X^* is weakly compactly generated with $K \subset X^*$ weakly compact such that $\overline{\mathrm{sp}(K)} = X^*$, then it admits a linear injection $T: X \longrightarrow C_0(\Gamma)$, and it can be shown (see [M1]) that the separating functionals, $f_j^\alpha \in \sigma_\alpha(S_{X^*})$ in Theorem 3.1, can be chosen to lie in $N(\alpha)\sigma_a(K)$, where $N(\alpha) \in \mathbb{R}$ are such that $N(\alpha)\sigma_a(K) \subset K$. It follows from this that the functionals k_{α_n} from Lemma 3.2 can be chosen to lie in K, and hence $k_{\alpha_n} \xrightarrow{w} k_0$, for some $k_0 \in K$, and Lemma 3.2 follows. This gives the result in [M1].

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