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Character Degree Graphs of Solvable Groups of Fitting Height 2

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Abstract. Given a finite group G, we attach to the character degrees of G a graph whose vertex set is the set of primes dividing the degrees of irreducible characters of G, and with an edge between p and q if pq divides the degree of some irreducible character of G. In this paper, we describe which graphs occur when G is a solvable group of Fitting height 2.

1 Introduction

Let G be a finite group. We write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ for the set of (irreducible) character degrees of G, and take $\rho(G)$ to be the set of primes that divide degrees in $\mathrm{cd}(G)$. The degree graph $\Delta(G)$ is the graph with vertex set $\rho(G)$. There is an edge between p and q if pq divides some degree $a \in \mathrm{cd}(G)$. These graphs have been studied in a number of places. For basic information on these graphs, we suggest [1, Theorem 14], [2, Section 30], and [8, Sections 18 and 19]. We study the relationship between the group structure of G and the graph structure of $\Delta(G)$. In particular, we continue the investigation of the relationship between the Fitting height of G and $\Delta(G)$.

In [4], we said that a graph Γ occurring as the degree graph of a solvable group had bounded Fitting height if there was an upper bound on the Fitting heights of the groups G such that $\Delta(G) = \Gamma$. We proved that Γ had bounded Fitting height if and only if Γ had at most one vertex that was adjacent to all the other vertices in Γ. In [6, 8], this study went further into specific bounds for certain families of graphs.

In this paper, we want to look at the lower bound on the Fitting height of G when $\Delta(G) = \Gamma$. We know that if G is nilpotent (*i.e.,* has Fitting height 1), then $\Delta(G)$ is a complete graph. Thus, if $\Delta(G)$ is not a complete graph, then G must have Fitting height at least 2. In this paper, we study the graphs that arise when G has Fitting height 2. In fact, we will classify which graphs can occur in this case.

We now state the main theorem of this note.

Theorem A *Let Γ be a graph with n vertices. There exists a solvable group G of Fitting height 2 with $\Delta(G) = \Gamma$ if and only if the vertices of degree less than $n-1$ can be partitioned into two subsets, each of which induces a complete subgraph of Γ and one of which contains only vertices of degree $n-2$.*

The next corollary restates Theorem A in a manner that is practical to check. (Notice that Corollary B is weaker than Theorem A.)

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Corollary B *If G is a solvable group with $n = |\rho(G)|$ and $\Delta(G)$ contains two vertices of degree less than $n - 2$ that are not adjacent, then the Fitting height of G is at least 3.*

Proof In Theorem A, we prove that if G is solvable of Fitting height 2, then all the vertices with degree less than $n-2$ form a complete graph, and so they will be adjacent to each other. Since this does not happen, G must have Fitting height at least 3. ■

Notice that if G is solvable and $\Delta(G)$ is disconnected where each connected component has at least two vertices, then G must have Fitting height at least 3. In fact, this was proved in [5, Theorem 5.3].

The distance between two connected vertices in a graph is the number of edges in a path between the two vertices that contains the fewest number of edges. We say that the diameter of a graph is the largest distance between connected vertices in the graph. This means that the diameter of a graph is the largest diameter of a connected component. Pálfy has proved that if G is solvable and $\Delta(G)$ is disconnected, then the diameter of $\Delta(G)$ is at most 1 (see [8, Corollary 18.8]). Thus, diameters bigger than 1 only occur when $\Delta(G)$ is connected. In [7], we found a solvable group G so that $\Delta(G)$ has diameter 3. Before that, it had been conjectured that the diameter of $\Delta(G)$ is at most 2 when G is solvable. Based on Theorem A, we can prove this is true when G is solvable of Fitting height 2.

Corollary C *If G is a solvable group of Fitting height 2, then $\Delta(G)$ has diameter at most 2.*

Proof If $\Delta(G)$ has diameter 3, then we can find primes p and q with the distance between p and q equal to 3. Let $n = |\rho(G)|$. It is easy to see that p is not adjacent to q or any neighbor of q, so its degree is less than $n - 2$. Similarly, q has degree less than $n - 2$. Now, $\Delta(G)$ has two nonadjacent vertices of degree less than $n - 2$, so by Corollary B, G must have Fitting height at least 3, which is a contradiction. ■

2 Examples

In this section, we prove the backward direction of Theorem A by constructing examples of solvable groups with Fitting height 2 that have the desired graphs. We begin by considering the construction of groups with Fitting height 2 and having a disconnected graph.

Lemma 2.1 *Let $p, q_1, \dots, q_n$ be distinct primes so that p is odd. Then there is a solvable group G of Fitting height 2 so that $\Delta(G)$ has two connected components: $\{p\}$ and $\{q_1, \dots, q_n\}$.*

Our work here is based on the construction found in [9, Example 3.4].

Proof Now, p is relatively prime to $q_1 \cdots q_n$, so p is a unit modulo $q_1 \cdots q_n$. In particular, there exists a positive integer m so that $p^m \equiv 1 \pmod{q_1 \cdots q_n}$, and hence,

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$q_1 \cdots q_n$ divides $p^m - 1$. Let E be the extra-special p-group of order p^{2m+1} and exponent p. It is well known that E has an automorphism σ of order $q_1 \cdots q_n$ that centralizes the center of E. We take G to be the semi-direct product of $\langle\sigma\rangle$ acting on E. One can show that $\operatorname{cd}(G) = \{1, p^m, q_1 \cdots q_n\}$ (see [9, Example 3.4]), and thus, $\Delta(G)$ has two connected components: $\{p\}$ and $\{q_1, \ldots, q_n\}$. ■

The following theorem is the backwards direction of Theorem A. Notice that the group G constructed is a direct product of groups with disconnected graphs with groups whose graphs have a single vertex.

Theorem 2.2 *Let Γ be a graph with n vertices and suppose that the vertices with degree less than $n - 1$ can be partitioned into two subsets U and V that both yield complete subgraphs of Γ and every vertex in V has degree $n - 2$. Then there is a solvable group G with Fitting height 2 so that $\Delta(G) = \Gamma$.*

Proof We begin by labeling the vertices in $U = \{u_1, \ldots, u_r\}$. If $v \in V$, then v has degree $n - 2$, so there is exactly one vertex not adjacent to v. Since v is adjacent to the rest of V, it follows that v is not adjacent to some vertex in U. It follows that if we let V_i be the set of vertices in V that are not adjacent to u_i, then $V_1, \ldots, V_r$ will partition V. Let m be the number of vertices with degree $n - 1$.

We now find $p_1, \ldots, p_r$ to be distinct odd primes, and we choose $\pi_1, \ldots, \pi_r$ to be sets of primes so that $|\pi_i| = |V_i|$ and so that these sets are pairwise disjoint and contain none of the primes $p_1, \ldots, p_r$. Finally, we choose distinct primes $s_1, \ldots, s_m$ to be disjoint from $\{p_1, \ldots, p_r\} \cup \pi_1 \cup \cdots \cup \pi_r$. Using Lemma 2.1, we can find for each $i = 1, \ldots, r$, a solvable group N_i of Fitting height 2 so that $\Delta(N_i)$ has two connected components: $\{p_i\}$ and π_i.

For $j = 1, \ldots, m$, we can use Dirichlet's theorem to find a prime t_j so that s_j divides $t_j - 1$. We let S_j be the semi-direct product of a cyclic group of order s_j acting on a cyclic group of order t_j. It is well known that $\operatorname{cd}(S_j) = \{1, s_j\}$, and S_j has Fitting height 2.

We take

$$G = N_1 \times \cdots \times N_r \times S_1 \times \cdots \times S_m.$$

It is not difficult to see that there is a graph automorphism between Γ and $\Delta(G)$ so that u_i corresponds to p_i, the set V_i corresponds to π_i, and the vertices of degree $n-1$ in Γ correspond to the s_j's. As G is a direct product of groups of Fitting height 2, it will also have Fitting height 2. ■

3 $\Delta(G)$ When G Has Fitting Height 2

In this section, we prove the forward direction of Theorem A. In particular, we show that all the graphs that arise as $\Delta(G)$ when G has Fitting height 2 are included among the graphs for the groups constructed in Section 2. If n is a positive integer, then $\pi(n)$ is the set of primes that divide n.

Throughout this section, G will be a solvable group of Fitting height 2 with Fitting subgroup F. The first lemma shows that every vertex in $\Delta(G)$ lies in one of two complete subgraphs.

Lemma 3.1 *If G is a solvable group of Fitting height 2 and F is the Fitting subgroup of G, then $\rho(G) = \rho(F) \cup \pi(|G:F|)$ and $\rho(F)$ and $\pi(|G:F|)$ induce complete subgraphs of $\Delta(G)$. In particular, if $\Delta(G)$ is not a complete graph, then $\rho(F)$ contains a prime that is not adjacent to some prime in $\pi(|G:F|)$.*

Proof If $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(F)$ is an irreducible constituent of χ_F, then $\pi(\theta(1)) \subseteq \rho(F)$ and $\pi(\chi(1)/\theta(1)) \subseteq \pi(|G:F|)$. It follows that $\rho(G) \subseteq \rho(F) \cup \pi(|G:F|)$.

By the discussion on [8, p. 254], there is a character degree in $\mathrm{cd}(G)$ which is divisible by all the primes in $\pi(|G:F|)$. This implies that $\pi(|G:F|) \subseteq \rho(G)$ and $\pi(|G:F|)$ induces a complete subgraph of $\Delta(G)$. Since F is a direct product of its Sylow subgroups, there is a degree in $\mathrm{Irr}(F)$ and hence in $\mathrm{Irr}(G)$ that is divisible by all the primes in $\rho(F)$. This implies $\rho(F) \subseteq \rho(G)$ and $\rho(F)$ induces a complete subgraph of $\Delta(G)$.

We now assume $\Delta(G)$ is not a complete graph, so there exists a prime $p \in \rho(G)$ that is not adjacent to some prime in $\rho(G)$. If $p \in \rho(F)$, then since p is adjacent to the other primes in $\rho(F)$, it must not be adjacent to some prime in $\pi(|G:F|)$. If $p \in \pi(|G:F|)$, then there is a prime $q \in \rho(G)$ that is not adjacent to p. Since p is adjacent to all the primes in $\pi(|G:F|)$, it follows that $q \in \rho(F)$, and q is a prime in $\rho(F)$ that is not adjacent to some prime in $\pi(|G:F|)$. ■

We now look at the structure of G in terms of a prime $p \in \rho(F)$ that is not adjacent to some other prime in $\rho(G)$. Using Lemma 3.1, it is not difficult to see that ρ must be a subset of $\pi(|G:F|)$.

Theorem 3.2 *Let G be a solvable group of Fitting height 2 with Fitting subgroup F. Let p be a prime in $\rho(F)$ such that p is not adjacent in $\Delta(G)$ to some prime in $\rho(G) - \{p\}$. Let ρ be the primes in $\rho(G) - \{p\}$ that are not adjacent to p. Then G has a normal nonabelian Sylow p-subgroup P, and if H is a Hall ρ-subgroup of G, then PH is normal in G. Furthermore, the graph $\Delta(PH)$ has two connected components $\{p\}$ and ρ.*

Proof By Lemma 3.1, we know that $\rho(G) = \rho(F) \cup \pi(|G:F|)$ and $\rho(F)$ and $\pi(|G:F|)$ induce complete subgraphs of $\Delta(G)$. It follows that every prime in $\rho(F) \cap \pi(|G:F|)$ is adjacent in $\Delta(G)$ to every other prime in $\rho(G)$. Hence, p does not lie in $\pi(|G:F|)$, and so p does not divide $|G:F|$. Let P be a Sylow p-subgroup of G, then it follows that $P \subseteq F$. Since F is nilpotent, P is characteristic in F, and P is normal in G. Because p is in $\rho(F)$, we see that P is not abelian.

The primes in ρ are not adjacent to p, so the intersection $\rho \cap \rho(F)$ must be empty. Let H be a Hall ρ-subgroup of G. Since G/F is nilpotent, $M = FH$ is normal in G. We know $\Delta(M)$ is a subgraph of $\Delta(G)$ and $\rho(M) = \rho(F) \cup \rho$. In particular, ρ is the set of primes in $\rho(M)$ that are not adjacent to p. Let Q be the Hall $\{p\} \cup \rho$-complement of F, and note that $M = PHQ$.

We can find a character $\theta \in \mathrm{Irr}(P)$ with $\theta(1) > 1$. For any character $\lambda \in \mathrm{Irr}(Q)$, let T be the stabilizer of $\theta \times \lambda$ in M. We know that $F \subseteq T$, and $|M:T|$ divides every degree in $\mathrm{cd}(M|\theta \times \lambda)$. (We define $\mathrm{cd}(M|\theta \times \lambda)$ to be the degrees of the irreducible constituents of $(\theta \times \lambda)^M$.) On the other hand, p divides every degree in $\mathrm{cd}(M|\theta \times \lambda)$, so it follows that no prime in ρ will divide any degree in $\mathrm{cd}(M|\theta \times \lambda)$, and in particular, no prime in ρ divides $|M:T|$. Since $|M:F|$ is a ρ-number, we conclude that $T = M$. Therefore, every character in $\mathrm{Irr}(Q)$ is invariant in M, and thus, every character in $\mathrm{Irr}(Q)$ is stabilized by H. Because $|H|$ and $|Q|$ are coprime, this implies H centralizes Q. As P necessarily centralizes Q, we deduce that $M = PH \times Q$. Now, PH is a characteristic subgroup of M, and hence, PH is normal in G. We now see that $\Delta(PH)$ is a subgraph of $\Delta(G)$ with $\rho(PH) = \{p\} \cup \rho$. Thus, $\Delta(PH)$ is a disconnected graph with connected components $\{p\}$ and ρ. ∎

Finally, we look at all of the primes in $\rho(F)$ that are not adjacent to some prime in $\rho(G) - \{p\}$.

Corollary 3.3 *Let G be a solvable group of Fitting height 2, and let F be the Fitting subgroup of G. Let $n = |\rho(G)|$, and let π_1 be the primes in $\rho(F)$ of degree less than $n-1$. Label the primes in π_1 as $\{p_1, \dots, p_r\}$. For each i, let ρ_i be the primes in $\rho(G)$ not adjacent to p_1. Then there exists a normal subgroup N of G so that $N = N_1 \times \cdots \times N_r$ where for each i the group N_i is a Hall $\{p_i\} \cup \rho_i$-subgroup of G.*

Proof For $i = 1, \dots, r$, let P_i be a Sylow p_i-subgroup and H_i be a Hall ρ_i-subgroup of G. We set $N_i = P_iH_i$, and by Theorem 3.2, N_i is the normal Hall $\{p_i\} \cup \rho_i$-subgroup of G. We claim that if $i \neq j$, then $\rho_i \cap \rho_j$ is empty. Suppose that this is not the case. In particular, suppose we have a prime $q \in \rho_i \cap \rho_j$. Let Q be a Sylow q-subgroup of G. We know from Theorem 3.2, that $\Delta(N_i)$ has two connected components. By [5, Lemma 4.1], we know that any solvable group with Fitting height 2 and a disconnected graph has a unique nonabelian normal Sylow subgroup, and this Sylow subgroup has an abelian quotient. It follows that N_i/P_i is abelian, and so P_iQ is a normal subgroup of N_i. Since P_iQ is a Hall subgroup, it is characteristic in N_i, and thus, normal in G. Similarly, P_jQ is a normal subgroup of G. This implies that $M = P_iP_jQ = (P_iQ)(P_jQ)$ is a normal subgroup of G. It is not difficult to see that $\rho(M) = \{p_i, p_j, q\}$. Since q is not adjacent to p_i or p_j, it will follow that $\Delta(M)$ has two connected components $\{p_i, p_j\}$ and $\{q\}$. Now, M is a solvable group with Fitting height 2 where $\Delta(M)$ is disconnected and M has two normal nonabelian Sylow subgroups (P_i and P_j). As we mentioned earlier, this violates [5, Lemma 4.1]. Therefore, if $i \neq j$, then $\rho_i \cap \rho_j$ is empty, and hence, $N_i \cap N_j = 1$. Setting $N = N_1N_2 \cdots N_r$, we obtain $N = N_1 \times \cdots \times N_r$. ∎

Before proceeding to the proof of the forward direction of Theorem A, we consider consequences of Theorem 3.3. Assume the notation of Corollary 3.3. Suppose $\Delta(G)$ has no vertex that is adjacent to all the other vertices in $\Delta(G)$. It follows that $\rho(G) = \bigcup_{i=1}^{r}(\{p_i\} \cup \rho_i)$, and thus, by Itô's theorem [3, Corollary 12.34], N has normal abelian complement Q in G. In particular, $G = N_1 \times \cdots \times N_r \times Q$, and by Theorem 3.2, $\Delta(N_i)$ is a disconnected graph. We cannot obtain a similar conclusion

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without the hypothesis that G has Fitting height 2, since the group constructed in [7] has no vertex adjacent to all the other vertices in $\Delta(G)$ but the group clearly cannot be written as a direct product as above.

Under the hypotheses of Corollary 3.3, it is tempting to conjecture that $G = N_1 \times \cdots \times N_r \times Q$ where each $\rho(N_i) = \{p_i\} \cup \rho_i$ and $\rho(Q)$ is the set of primes that are adjacent to all the other vertices in $\Delta(G)$. Unfortunately, this is not true. Let p be a prime, let i be an integer, and let a divide $p^i - 1$ and b divide $p^i + 1$. Noritzsch [9, Examples 5.8] constructed a group G of Fitting height 2 where $\mathrm{cd}(G) = \{1, p^i a, ab\}$. Clearly, G cannot be a direct product as above. It is not difficult to choose our parameters so that a and b will be relatively prime.

The following is a more detailed statement of the forward direction of Theorem A. Thus, this theorem proves the forward direction of Theorem A.

Theorem 3.4 *Let G be a solvable group of Fitting height 2, let F be the Fitting subgroup of G, let π_1 be the primes in $\rho(F)$ that are not adjacent to some prime in $\pi(|G:F|)$, and let π_2 be the primes in $\pi(|G:F|)$ that are not adjacent to some prime in $\rho(F)$. If $n = |\rho(G)|$, then the vertices of degree less than $n-1$ in $\Delta(G)$ are partitioned into the sets π_1 and π_2, each of which yields a complete subgraph in $\Delta(G)$ and every prime in π_2 has degree $n-2$.*

Proof By Lemma 3.1, $\rho(G) = \rho(F) \cup \pi(|G:F|)$ and each of $\rho(F)$ and $\pi(|G:F|)$ yields a complete subgraph of $\Delta(G)$. It follows that any prime in $\rho(F)$ whose degree in $\Delta(G)$ is less than $n-1$ will be nonadjacent to some prime in $\pi(|G:F|)$ and thus lie in π_1. Similarly, any prime in $\pi(|G:F|)$ with degree less than $n-1$ in $\Delta(G)$ will not be adjacent to some prime in $\rho(F)$, and thus, will lie in π_2.

Since every prime in $\rho(G)$ is in $\rho(F)$ or $\pi(|G:F|)$, and any prime in both π_1 and π_2 will be in both $\rho(F)$ and $\pi(|G:F|)$, it is easy to see that any prime in both $\rho(F)$ and $\pi(|G:F|)$ will have degree $n-1$, so π_1 and π_2 partition the vertices in $\Delta(G)$ having degree less than $n-1$. Since $\pi_1 \subseteq \rho(F)$ and $\pi_2 \subseteq \pi(|G:F|)$, they must yield complete subgraphs of $\Delta(G)$. Finally, we know that each prime in π_2 is adjacent to all the other primes in $\pi(|G:F|)$.

Label the primes in π_1 as $\{p_1, \dots, p_r\}$, and define ρ_i to be the primes in $\rho(G)$ that are not adjacent to p_i. Observe that $\pi_2 = \rho_1 \cup \cdots \cup \rho_r$. Let N_i be a Hall $\{p_i\} \cup \rho_i$-subgroup of G. By Corollary 3.3, there is a normal subgroup N so that $N = N_1 \times \cdots \times N_r$. By Theorem 3.2, each $\Delta(N_i)$ has two connected components $\{p_i\}$ and ρ_i. This implies that each prime in ρ_i is adjacent in $\Delta(N)$ (and hence $\Delta(G)$) to every prime in π_1 except p_i. It follows that every prime in π_2 must be adjacent to all but one of the primes in $\rho(F)$, so each prime in π_2 will have degree $n-2$. ■

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