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Multiplier ideals, *b*-function, and spectrum of a hypersurface singularity

Morihiko Saito

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Abstract

We prove that certain roots of the Bernstein–Sato polynomial (i.e. b-function) are jumping coefficients up to a sign, showing a partial converse of a theorem of L. Ein, R. Lazarsfeld, K. E. Smith, and D. Varolin. We also prove that certain roots are determined by a filtration on the Milnor cohomology, generalizing a theorem of B. Malgrange in the isolated singularity case. This implies a certain relation with the spectrum which is determined by the Hodge filtration, because the above filtration is related to the pole order filtration. For multiplier ideals we prove an explicit formula in the case of locally conical divisors along a stratification, generalizing a formula of Mustaţă in the case of hyperplane arrangements. We also give another proof of a formula of U. Walther on the b-function of a generic hyperplane arrangement, including the multiplicity of -1.

Introduction

Let X be a complex manifold and let D be an effective divisor on it. For a positive rational number α, the multiplier ideal $\mathcal{J}(X, \alpha D)$ is a coherent ideal of the structure sheaf $\mathcal{O}_X$ defined by the local integrability of $|g|^2/|f|^{2\alpha}$ for $g \in \mathcal{O}_X$, where f is a holomorphic function defining D locally, see [ELSV04, Laz04, Nad90]. This gives a decreasing filtration on $\mathcal{O}_X$, and essentially coincides with the filtration induced by the V-filtration of Kashiwara [Kas83] and Malgrange [Mal81] along D indexed by $\mathbb{Q}$, see [BS05]. It is also related to the spectrum $\mathrm{Sp}(f, x)$, see [Bud03, BS05].

The numbers α at which the $\mathcal{J}(X, \alpha D)$ jump are called the jumping coefficients of D. It is shown by Ein *et al.* (see [ELSV04]) that any jumping coefficients which are less than one are roots of the Bernstein–Sato polynomial $b_f(s)$ (i.e. the b-function) up to a sign. It is well known that the minimal jumping coefficient α_f coincides with the minimal root of $b_f(-s)$, see [Kol97]. For $x \in D$, we define $b_{f,x}(s)$, $\alpha_{f,x}$ by replacing X with a sufficiently small neighborhood of x. For $\alpha > 0$ with $0 < \varepsilon \ll 1$, the graded pieces are defined by

$$\mathcal{G}(X, \alpha D) = \mathcal{J}(X, (\alpha - \varepsilon)D)/\mathcal{J}(X, \alpha D) \quad (= \mathrm{Gr}_V^{\alpha}\mathcal{O}_X).$$

We say that α is a local jumping coefficient of D at x if $\mathcal{G}(X, \alpha D)_x \neq 0$. We have a partial converse to the theorem of [ELSV04] as follows (see § 3.3).

Theorem 1. *Let α be a root of $b_{f,x}(-s)$ contained in $(0, 1)$. Assume that:*

(a) *$\xi f = f$ for a holomorphic vector field ξ;*

(b) *$\alpha < \alpha_{f,y}$ for any $y \neq x$ sufficiently near x.*

Then α is a local jumping coefficient of D at x.

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Theorem 1 does not hold if either of the two conditions is not satisfied, see Remarks 3.4. Condition (b) is satisfied if $\exp(-2\pi i\beta)$ is not an eigenvalue of the Milnor monodromy of f at $y \neq x$ for any $\beta \in [\alpha_{f,x}, \alpha]$. By definition, $\mathcal{J}(X, (\alpha+1)D) = f\mathcal{J}(X, \alpha D)$ for $\alpha > 0$, and the jumping coefficients have a periodicity so that $\alpha > 0$ is a jumping coefficient if and only if $\alpha + 1$ is. However, the roots of $b_f(-s)$ do not have such a periodicity and we have to restrict to $(\alpha_{f,x}, 1)$.

As for the relation with the spectrum, Budur [Bud03] proved that, if $\alpha \in (0,1)$ and $\mathcal{G}(X, \alpha D)$ is supported on a point x of D (e.g. if condition (b) of Theorem 1 is satisfied), then the coefficient m_α of the spectrum $\mathrm{Sp}(f, x) = \sum_\beta m_\beta t^\beta$ is given by

$$m_\alpha = \dim \mathcal{G}(X, \alpha D)_x. \tag{0.1}$$

Indeed, under the above hypothesis, $\mathcal{G}(X, \alpha D)(= \mathrm{Gr}_V^\alpha \mathcal{O}_X)$ is identified with the Hodge filtration F^{n-1} on the λ-eigenspace of the Milnor monodromy $H^{n-1}(F_x, \mathbb{C})_\lambda$ for $\lambda = \exp(-2\pi i\alpha)$ where F_x denotes the Milnor fiber around x and $n = \dim X$, see [BS05]. In the isolated singularity case, (0.1) is closely related to [Loe85, Mal75, Vaq92, Var82]. We have a generalization of a result of Malgrange [Mal75] as follows (see § 4.5).

THEOREM 2. *There exists canonically a decreasing filtration $\widetilde{P}$ on $H^{n-1}(F_x, \mathbb{C})_\lambda$ stable by the Milnor monodromy and containing the Hodge filtration F, and for any rational number α such that $\lambda = \exp(-2\pi i\alpha)$, we have the following.*

(a) *If $\mathrm{Gr}_{\widetilde{P}}^p H^{n-1}(F_x, \mathbb{C})_\lambda \neq 0$ with $p = [n - \alpha]$, then α is a root of $b_{f,x}(-s)$.*

(b) *If $\alpha + i$ is not a root of $b_{f,y}(-s)$ for any $y \neq x$ and any $i \in \mathbb{N}$, then the converse of the assertion (a) holds.*

(c) *If λ is not an eigenvalue of the Milnor monodromy at $y \neq x$, then the multiplicity of the root α coincides with the degree of the minimal polynomial of the action of the monodromy on $\mathrm{Gr}_{\widetilde{P}}^p H^{n-1}(F_x, \mathbb{C})_\lambda$.*

Note that the spectrum [Ste89] is defined in the same way as in assertions (a) and (c) replacing $\widetilde{P}$ with the Hodge filtration F and the minimal polynomial with the characteristic polynomial, see § 3.5. The filtration $\widetilde{P}$ is defined by using the saturated Brieskorn lattices $\widetilde{\mathcal{G}}_f^{(-i)}$ (see (4.1.6)), and contains the Hodge filtration F, see Proposition 4.4. Replacing $\widetilde{\mathcal{G}}_f^{(0)}$ with the Brieskorn lattices $\mathcal{G}_f^{(-i)}$, we have the filtration P contained in $\widetilde{P}$, see (4.1.6). In the isolated singularity case, P coincides with the Hodge filtration F, see [SS85, Var82]. In the quasi-homogeneous isolated singularity case, this also follows from [Mal75, Ste77a] (where the Milnor cohomology is identified with the Jacobian ring, and the Hodge filtration is described by using the weighted degree of monomials). If f is a homogeneous polynomial in general, then $\widetilde{P}$ coincides with P and with the pole order filtration defined by using a local system on an open subvariety of $\mathbb{P}^{n-1}$ calculating $H^{n-1}(F_x, \mathbb{C})_\lambda$, see Proposition 4.9.

In general it is not easy to calculate $\mathcal{J}(X, \alpha D)$ explicitly except for some special cases, see [How03, How01, Mus06], etc. In this paper, we give an explicit formula for $\mathcal{J}(X, \alpha D)$ in the case D is a locally conical divisor along a stratification, i.e. D is locally defined by a weighted homogeneous function with nonnegative weights and the zero weight part, which is the limit of the (local) $\mathbb{C}^*$-action, is given by the stratum passing through the point (e.g. D is an affine cone of a divisor on $\mathbb{P}^2$ which is defined locally in classical topology by a weighted homogeneous polynomial), see § 1.2 for details. We have a (shifted) decreasing filtration $\{G_x^\alpha\}_{\alpha \in \mathbb{Q}}$ on $\mathcal{O}_{X,x}$ associated to the weights at each $x \in D$, see § 1.3. Let D^{nnc} denote the smallest closed analytic subset of D such that D is a divisor with normal crossings outside D^{nnc}, and let $D_{\mathrm{red}}^{\mathrm{sm}}$ be the smooth part of the reduced variety D_{red}.

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Theorem 3. *Let X be a complex manifold and let D be a locally conical divisor along a stratification. Then a section g of $\mathcal{O}_X$ belongs to $\mathcal{J}(X, \alpha D)$ if and only if $g_x \in \mathcal{O}_{X,x}$ belongs to $G_x^{>\alpha}$ for any $x \in D^{\mathrm{nnc}} \cup D^{\mathrm{sm}}_{\mathrm{red}}$.*

This generalizes a formula of Mustaţă [Mus06] for a hyperplane arrangement with a reduced equation. The condition for $x \in D^{\mathrm{sm}}_{\mathrm{red}}$ is equivalent to that the vanishing order of g along D is strictly greater than $\alpha m_x - 1$ where m_x is the multiplicity of D at x. A similar formula has been known for a function with nondegenerate Newton boundary, see [How03, How01, Laz04] (and also § 2.4). By induction on stratum, Theorem 3 is reduced to Theorem 2.2 whose proof uses the above analytic definition of multiplier ideal together with some commutative algebra, see Proposition 2.1, Theorem 2.2 and § 2.3.

For a divisor D on a complex manifold, let $\alpha_D = \min\{\alpha_{f,x} : x \in D\}$ where f is a holomorphic function defining D on a neighborhood of x. As a corollary of Theorem 2.2 we can deduce the following.

Proposition 1. *Assume $X = \mathbb{C}^n$ and that D is the affine cone of a divisor Z of degree d on $\mathbb{P}^{n-1}$. Let $\mathcal{I}_0$ be the ideal sheaf of $\{0\} \subset \mathbb{C}^n$. Then we have for $\alpha < \alpha_Z$*

$$\mathcal{J}(X, \alpha D) = \mathcal{I}_0^k \quad \text{with } k = [d\alpha] - n + 1.$$

In particular, j/d is a local jumping coefficient of D at 0 if $n \leqslant j < d\alpha_Z$.

Note that $\alpha_Z \leqslant 1$, and the equality holds if Z is a reduced divisor with normal crossings, e.g. if D is a generic hyperplane arrangement, see also [Mus06]. Since $\dim \mathcal{I}_0^k/\mathcal{I}_0^{k+1} = \binom{n+k-1}{n-1}$, we get from (0.1) and Proposition 1 the following.

Corollary 1. *With the notation and the assumption of Proposition 1, assume that Z is a reduced divisor with normal crossings on $\mathbb{P}^{n-1}$. Then the coefficients m_α and $m_{n-\alpha}$ of the spectrum $\mathrm{Sp}(f, 0)$ are $\binom{j-1}{n-1}$ for $\alpha = j/d < 1$.*

The assertion for $m_{n-\alpha}$ is reduced to that for m_α by the symmetry of the Hodge numbers for the nonunipotent monodromy part of the vanishing cohomology (which is identified with that of the nearby cycle sheaf in this case). Note that the formula is the same as in the case of a homogeneous polynomial with an isolated singularity, and can also be deduced from the calculation of the Hodge filtration in § 4.8.

In the case of a generic central hyperplane arrangement with a reduced equation f, the b-function is determined by Walther [Wal05] (except for the multiplicity of the root -1). Using Theorem 2 together with [Bri73, ESV92], we first see that the roots of $b_f(-s)$ are strictly smaller than two, see Proposition 5.2. Then we can give another proof of his formula together with the multiplicity of -1, using the relation between the b-function and the V-filtration in [Kas83, Mal81] together with Corollary 1, see § 5.4. Note that for any arbitrary hyperplane arrangement, -1 is the only integral root of $b_f(s)$ (see [Wal05]), and we can show that its multiplicity is $-n$ if the arrangement is not the pull-back of an arrangement in a strictly lower-dimensional vector space, see Proposition 5.3. More details will be given in a forthcoming paper on the b-functions of hyperplane arrangements.

Walther's formula shows that, without restricting to the interval $(0, 1)$, there is no relation between the spectrum and the roots of $b_f(-s)$ (in contrast to the case of a homogeneous polynomial with an isolated singularity). This comes from the difference between the Hodge and pole order filtrations on the Milnor cohomology in Theorem 2, see Proposition 4.9.

As for the spectrum of generic central hyperplane arrangement, the m_α for $\alpha \in \mathbb{Z}$ are easily calculated, see (5.6.1). Combined with Corollary 1, this gives the spectrum of a generic central hyperplane arrangement for $n = 3$, because the Euler characteristic is calculated in [CS95, OR93]. It is possible, in principle, to calculate the spectrum for a general n, using [DS04].

In § 1, we introduce locally conical divisors along a stratification. In § 2, we prove Theorem 2.2 which implies Theorem 3. In § 3, we explain the relation with b-function and spectrum, and prove Theorem 1. In § 4, we explain the relation with Brieskorn modules and Gauss–Manin systems, and prove Theorem 2. In § 5, we treat the case of a generic central hyperplane arrangement, and give another proof of Walther's theorem as an application of Theorem 2.

1. Locally conical divisors along a stratification

1.1 Conical divisors

Let Y be a complex manifold, and $(x_1,\dots,x_r)$ be the coordinate system of $\mathbb{C}^r$. Let $\mathbf{w}=(w_1,\dots,w_r) \in \mathbb{Q}^r_{>0}$ (i.e. w_i are positive rational numbers). We say that an effective divisor D on $X := Y\times\mathbb{C}^r$ is a *conical divisor* along $Y\times\{0\}$ with positive weight $\mathbf{w}=(w_1,\dots,w_r)$ if D is locally defined by a relatively weighted homogeneous function f with positive weight $\mathbf{w}$ (i.e. f is a linear combination of $x_1^{\nu_1}\cdots x_r^{\nu_n}$ with $\mathcal{O}_Y$-coefficients such that $(\nu_1,\dots,\nu_r)\in\mathbb{N}^r$ satisfies $\sum_i w_i\nu_i=1$). For a positive real number λ, we define $\phi_\lambda:\mathbb{C}^r\to\mathbb{C}^r$ by

$$\phi_\lambda(x_1,\dots,x_r)=(\lambda^{w_1}x_1,\dots,\lambda^{w_r}x_r), \tag{1.1.1}$$

and $id\times\phi_\lambda : Y\times\mathbb{C}^r\to Y\times\mathbb{C}^r$ will be denoted also by ϕ_λ. Then $\phi_\lambda^* f=\lambda f$.

1.2 Locally conical divisors along a stratification

We say that a divisor D on a complex manifold X is a *locally conical divisor* along a smooth submanifold Z if for each $z\in Z$, there exist a complex manifold Y, a conical divisor D' on $Y\times\mathbb{C}^r$ with positive weight $\mathbf{w}$ along $Y\times\{0\}$, an open subset U' of $Y\times\mathbb{C}^r$, and an open neighborhoods U of z in X together with an isomorphism $U\simeq U'$ inducing isomorphisms $U\cap D\simeq U'\cap D'$, $U\cap Z\simeq U'\cap(Y\times\{0\})$ (in particular, z corresponds to a point of $Y\times\{0\}$). Note that the weight $\mathbf{w}$ is not necessarily unique in general.

Let D be an effective divisor on a complex manifold X. Let D^{nnc} be the smallest closed analytic subset of D such that $D\setminus D^{\mathrm{nnc}}$ is a divisor with normal crossings on $X\setminus D^{\mathrm{nnc}}$. We say that D is a *locally conical divisor along a stratification* $\{S_i\}$ of D^{nnc}, if D is a locally conical divisor along S_i for each i.

1.3 Shifted w-filtrations along strata

With the notation of § 1.1, let $x=(y,0)\in Y\times\{0\}$ and $g\in\mathcal{O}_{X,x}$. We have the expansion

$$g=\sum_\beta g_\beta \tag{1.3.1}$$

such that g_β is a linear combination of $x_1^{\nu_1}\cdots x_r^{\nu_n}$ with $\mathcal{O}_Y$-coefficients satisfying

$$\sum_i w_i(\nu_i+1)=\beta. \tag{1.3.2}$$

We define a decreasing filtration G of ideals of $\mathcal{O}_{X,x}$ such that G^α is generated by $x_1^{\nu_1}\cdots x_r^{\nu_n}$ with $\sum_i w_i(\nu_i+1)\geqslant\alpha$ (i.e. $g\in G^\alpha$ if and only if g_β vanishes for $\beta<\alpha$). This is called the shifted $\mathbf{w}$-filtration.

If D is a locally conical divisor along Z or a stratification $\{S_i\}$ as in § 1.2, we have the shifted $\mathbf{w}$-filtration G_x on $\mathcal{O}_{X,x}$ for each $x\in Z$ or D^{nnc}. This is not necessarily unique in general.

If $x\in D^{\mathrm{sm}}_{\mathrm{red}}(:=D\setminus\mathrm{Sing}D_{\mathrm{red}})$, let h be a holomorphic function defining D_{red} on a neighborhood of x and let m_x be the multiplicity of D at x. Then for $\alpha>0$, we have $G^\alpha_x=h^{i-1}\mathcal{O}_{X,x}$ where i is the minimal integer such that $i\geqslant m_x\alpha$.

Let $G_x^{>\alpha} = \bigcup_{\beta>\alpha} G_x^\beta$ in general.

Remarks 1.4. (i) If $f = \sum_i u_i x_i^{a_i}$ with $u_i(0) \neq 0$ and $a_i \in \mathbb{Z}_{>0}$, then $D := f^{-1}(0)$ is locally conical along the origin, because $f = \sum_i y_i^{a_i}$ with $y_i = u_i^{-1/a_i} x_i$. This implies, for example, that a divisor D on $\mathbb{C}^3$ is locally conical along a stratification, if it is defined by $f = x^a y^b + y^a z^b + z^a x^b$ with $a, b > 1$.

(ii) It is possible that the moduli of singularity really vary along a stratum, e.g. $f = (x^3 + y^3 + z^3)u + xyzv$.

2. Calculation of multiplier ideals

The following is the key to the proof of Theorem 3.

Proposition 2.1. *With the notation and the assumptions of § 1.1, let $X = Y \times \mathbb{C}^r$, $X' = X \setminus Y \times \{0\}$ with the inclusion $j : X' \to X$. Put $D' = D \cap X'$. If $g \in \mathcal{O}_{X,x}$ belongs to $j_*\mathcal{J}(X', \alpha D')$, then each g_β in (1.3.1) belongs to $j_*\mathcal{J}(X', \alpha D')$.*

Proof. Since $\mathcal{J}(X', \alpha D')$ is extended to a coherent sheaf on X, $j_*\mathcal{J}(X', \alpha D') \cap \mathcal{O}_X$ is coherent. (Indeed, the assertion is reduced to the case where the complement of the image of j is a divisor, using a Cech complex. Then any section of $j_*\mathcal{J}(X', \alpha D') \cap \mathcal{O}_X$ defines a section of $\mathcal{O}_X/\mathcal{J}(X, \alpha D)$, which is supported on the divisor, and hence is annihilated by a sufficiently high power of a function defining the divisor. So the intersection with $j_*\mathcal{J}(X', \alpha D')$ can be replaced by the intersection with the algebraic localization of a coherent extension of $\mathcal{J}(X', \alpha D')$ which is quasi-coherent.)

For $x = (y, 0) \in Y \times \{0\}$, let

$$M = (j_*\mathcal{J}(X', \alpha D') \cap \mathcal{O}_X)_x, \quad N = \mathcal{O}_{X,x}.$$

We denote G_x by G in this section to simplify the notation. For $\beta \in \mathbb{Q}$, let

$$G^\beta \widehat{M} = \varprojlim\nolimits_\gamma G^\beta M / G^\gamma M, \quad G^\beta \widehat{N} = \varprojlim\nolimits_\gamma G^\beta N / G^\gamma N,$$

and $\widehat{M}, \widehat{N}$ be their inductive limit for β, respectively. By the Mittag–Leffler condition, we have the injectivity of $G^\beta \widehat{M} \to G^{\beta'} \widehat{M}$ for $\beta > \beta'$ so that we get the filtration G of $\widehat{M}$ (similarly for $\widehat{N}$), see also [Sai88a]. By the Artin–Rees lemma, $G^\beta \widehat{M}, G^\beta \widehat{N}$ coincide with the I-adic completion of $G^\beta M, G^\beta N$ by the ideal I of $Y \times \{0\}$, because the filtration G is induced by G on N which is essentially equivalent to the I-adic filtration (i.e. there are positive rational numbers α, β such that $G^{i\alpha} \subset I^i \subset G^{i\beta}$ for $i \gg 0$).

For $\lambda \in \mathbb{C}^*$, we see that M, $G^\beta N$, and hence $G^\beta M$ are stable by the action of ϕ_λ^*. So the filtration G on $\widehat{M}$ splits canonically (because G on G^β/G^γ does). Thus, for $g \in G^\beta M$, we have $g = g_\beta + g'$ where g_β is as in (1.3.1) and $g' \in G^{>\beta}\widehat{M} \cap N$ because $g_\beta \in N$. So the assertion is reduced to

$$G^{>\beta} M = G^{>\beta} \widehat{M} \cap N, \tag{2.1.1}$$

because this implies $g_\beta \in M$ so that we can proceed by induction on β replacing g with g'.

For the proof of (2.1.1), consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G^{>\beta}M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N/G^{>\beta}M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (G^{>\beta}M)^\wedge \arrow[r] & \widehat{N} \arrow[r] & (N/G^{>\beta}M)^\wedge \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is the I-adic completion of the top row. By the above argument, we have $G^{>\beta}\widehat{M} = (G^{>\beta}M)^\wedge$, and the vertical morphisms are injective by Krull's intersection theorem. So (2.1.1) follows. This completes the proof of Proposition 2.1. ☐

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THEOREM 2.2. *With the above notation and assumption, we have*

$$\mathcal{J}(X,\alpha D)_x = (j_*\mathcal{J}(X',\alpha D'))_x \cap G^{>\alpha}\mathcal{O}_{X,x} \quad \text{for } x \in Y \times \{0\}, \tag{2.2.1}$$

where $G^{>\alpha}$ is as in § 1.3.

Proof. We first show that $g \in \mathcal{J}(X,\alpha D)$ if $g \in (j_*\mathcal{J}(X',\alpha D'))_x \cap G^{>\alpha}\mathcal{O}_{X,x}$. We have the expansion $g = \sum_\beta g_\beta$ as in (1.3.1). By Proposition 2.1 we may assume $g = g_\beta$ with $\beta > \alpha$, because $(j_*\mathcal{J}(X',\alpha D') \cap \mathcal{O}_X)/\mathcal{J}(X,\alpha D)$ is annihilated by a sufficiently high power of the ideal of $Y \times \{0\}$ in the notation of Proposition 2.1 so that the assertion is clear if $g \in G_x^\beta$ for β sufficiently large. We have a relatively compact open subset U of X' together with $\lambda \in (0,1)$ such that $\bigcup_{j \geqslant 0} \phi_{\lambda^j} U$ contains $U_x \setminus Y \times \{0\}$ where U_x is a sufficiently small open neighborhood of x in X on which g is defined. (For example, use a function defined by $\rho(x) = \sum_{i \leqslant r} |x_i|^{1/w_i}$ so that $\rho(\phi_\lambda x) = \lambda\rho(x)$.)

Let $\omega = dx_1 \wedge \cdots \wedge dx_r \wedge dy_1 \wedge \cdots \wedge dy_{n-r}$, where $(y_1, \ldots, y_{n-r})$ is a local coordinate system of Y. Then $\phi_{\lambda^j}^*(g_\beta\omega) = \lambda^{j\beta} g_\beta \omega$, and

$$\begin{aligned}\sum_{j \geqslant 0} \int_{U \setminus D} \phi_{\lambda^j}^*((|g_\beta|^2/|f|^{2\alpha})\omega \wedge \overline{\omega}) &= \sum_{j \geqslant 0} \lambda^{2(\beta-\alpha)j} \int_{U \setminus D} (|g_\beta|^2/|f|^{2\alpha})\omega \wedge \overline{\omega} \\ &= (1-\lambda^{2(\beta-\alpha)})^{-1} \int_{U \setminus D} (|g_\beta|^2/|f|^{2\alpha})\omega \wedge \overline{\omega}.\end{aligned}$$

So the assertion follows.

Similarly, we see that $g \notin \mathcal{J}(X,\alpha D)$ if $g \in (j_*\mathcal{J}(X',\alpha D'))_x$ and $g \notin G_x^{>\alpha}$. Here we may assume $g \in G_x^\alpha$ by replacing α with a smaller number if necessary. Then we may assume further that $g = g_\alpha$ using the above argument. So the assertion follows by considering a sufficiently small open subset U of X' such that the $\phi_{\lambda^j} U$ $(j \in \mathbb{N})$ are disjoint. This completes the proof of Theorem 2.2. □

2.3 Proof of Theorem 3

The assertion is well known outside D^{nnc}, i.e. if D is a divisor with normal crossings, see e.g. [Bud03, BS05]. We proceed by induction on stratum. Since the assertion is local, we may assume $X = Y \times \mathbb{C}^r$ with $Y = S_i$ and D is defined by a relatively homogeneous function f with positive weight $\mathbf{w} = (w_1, \ldots, w_r)$ as in § 1.1. Then the assertion follows from Theorem 2.2 by induction on stratum.

2.4 Nondegenerate Newton boundary case

Assume $f \in \mathcal{O}_{X,x}$ has a nondegenerate Newton boundary [Kou76, Var76]. Then we have a formula similar to Theorem 3 by [How03, How01, Laz04]. There is a shifted Newton filtration G on $\mathcal{O}_{X,0}$ such that $G^\alpha\mathcal{O}_{X,0}$ is generated over $\mathcal{O}_{X,0}$ by the monomials $x_1^{\nu_1} \cdots x_n^{\nu_n}$ satisfying

$$\sum_i w_{\sigma,i}(\nu_i + 1) \geqslant \alpha \tag{2.4.1}$$

for any $(n-1)$-dimensional faces σ of the Newton polyhedron, where the $w_{\sigma,i}$ are positive rational numbers such that σ is contained in the hyperplane defined by $\sum_i w_{\sigma,i}\nu_i = 1$. Then

$$\mathcal{J}(X,\alpha D)_x = G^{>\alpha}\mathcal{O}_{X,0} \quad \text{for } \alpha < 1. \tag{2.4.2}$$

This is proved in [How03] in the polynomial case. The proof in the analytic case should be essentially same. (It would also be possible to use an argument similar to the proof of Theorem 2.2 together with the torus embedding constructed in [Var76], because the nondegeneracy corresponds to the condition that the restriction of the proper transform of the hypersurface to each stratum, which is isomorphic to a torus, is nonsingular.) In the isolated singularity case with nondegenerate Newton boundary, this is related to [Sai88a] using § 3.2 and Proposition 4.7.

3. Relation with the b-function and spectrum

3.1 b-function

Let X be a complex manifold of dimension n, and f be a nonconstant holomorphic function on X. Let

$$M = \mathcal{D}_X[s]f^s.$$

It is identified with a $\mathcal{D}_X[s]$-submodule of

$$\mathcal{B}_f := \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t],$$

generated by $1 \otimes 1$ (which is identified with f^s), where $s = -\partial_t t$, see [Kas77, Mal81]. Here $\mathcal{B}_f$ is the direct image of $\mathcal{O}_X$ by the graph embedding $i_f : X \to X \times \mathbb{C}$ as a $\mathcal{D}$-module, and the action of $\mathcal{D}_{X\times\mathbb{C}}$ on $\mathcal{B}_f$ is defined by identifying $1 \otimes 1$ with the delta function $\delta(t - f)$. More precisely, for a vector field ξ on X and the coordinate t of $\mathbb{C}$, we have

$$\begin{aligned} \xi(g \otimes \partial_t^j) &= (\xi g) \otimes \partial_t^j - (\xi f) g \otimes \partial_t^{j+1}, \\ t(g \otimes \partial_t^j) &= fg \otimes \partial_t^j - jg \otimes \partial_t^{j-1}, \end{aligned} \tag{3.1.1}$$

and the actions of $h \in \mathcal{O}_X$ and ∂_t^i are natural, see also [BS05].

The b-function $b_f(s)$ is the minimal polynomial of the action of s on M/tM. Since M/tM is holonomic, the b-function exists if X is (relatively) compact or X, f are algebraic. By Kashiwara [Kas83] and Malgrange [Mal81], $\mathcal{B}_f$ has the filtration V together with a canonical isomorphism of perverse sheaves

$$\mathrm{DR}_X(\mathrm{Gr}_V^\alpha \mathcal{B}_f) = \psi_{f,\lambda}\mathbb{C}_X[n-1] \quad \text{for } \alpha > 0,\ \lambda = \exp(-2\pi i\alpha) \tag{3.1.2}$$

such that $\exp(-2\pi i \partial_t t)$ on the left-hand side corresponds to the monodromy T on the right-hand side. Here DR_X denotes the de Rham functor (which induces an equivalence of categories between regular holonomic $\mathcal{D}$-modules and perverse sheaves) and $\psi_{f,\lambda}\mathbb{C}_X[n-1]$ is the λ-eigenspace of the nearby cycle (perverse) sheaf $\psi_f\mathbb{C}_X[n-1]$ for the semisimple part of the monodromy T, see [BBD82, Del73a, Del73b].

3.2 Relation of the multiplier ideals with the V-filtration

By [BS05] we have

$$\mathcal{J}(X, \alpha D) = V^\alpha \mathcal{O}_X \quad \text{if } \alpha \text{ is not a jumping coefficient}, \tag{3.2.1}$$

where the filtration V on $\mathcal{O}_X$ is induced by the V-filtration on $\mathcal{B}_f (= \mathcal{O}_X \otimes \mathbb{C}[\partial_t])$ in § 3.1. If α is a jumping coefficient (or actually, for any α), we have for $0 < \varepsilon \ll 1$

$$\mathcal{J}(X, \alpha D) = V^{\alpha+\varepsilon}\mathcal{O}_X, \quad V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)D). \tag{3.2.2}$$

This implies another proof of a theorem of Ein *et al.* (see [ELSV04]) that any jumping coefficients which are less than one are roots of the b-function up to a sign.

3.3 Proof of Theorem 1

By § 3.2 we can essentially replace $\mathcal{J}(X, \alpha D)$ with $V^\alpha \mathcal{O}_X$. By condition (a), we have $\xi f = f$ so that $\xi f^s = s f^s$, and hence

$$M := \mathcal{D}_X[s]f^s = \mathcal{D}_X f^s \subset \mathcal{B}_f. \tag{3.3.1}$$

By condition (b), $M/V^{>\alpha}M$ is supported on $\{x\}$, and is generated over $\mathbb{C}[\partial] := \mathbb{C}[\partial_1, \dots, \partial_n]$ by

$$(\mathcal{O}_X/V^{>\alpha}\mathcal{O}_X) \otimes 1 \subset \mathcal{B}_f/V^{>\alpha}\mathcal{B}_f, \tag{3.3.2}$$

where $\partial_j = \partial/\partial x_j$. Consider the filtered morphism induced by (3.3.2)

$$(\mathcal{O}_X/V^{>\alpha}\mathcal{O}_X, V) \otimes_{\mathbb{C}} \mathbb{C}[\partial] \to (\mathcal{B}_f/V^{>\alpha}\mathcal{B}_f, V). \tag{3.3.3}$$

This is strictly injective, i.e. it induces injective morphisms of the graded pieces. Indeed, for $\beta \leqslant \alpha$, the $\mathrm{Gr}_V^\beta \mathcal{O}_{X,x}$ are finite-dimensional vector spaces, and are annihilated by the maximal ideal of $\mathcal{O}_{X,x}$, see [Bud03] and [Sai88b, Lemma 3.2.6]. Moreover, V on $\mathcal{B}_f/V^{>\alpha}\mathcal{B}_f$ is a filtration as $\mathcal{D}_X$-modules and the morphism of graded pieces induced by (3.3.2) is injective by the definition of the induced filtration. So we get the strict injectivity of (3.3.3), because any holonomic $\mathcal{D}_X$-module supported on $\{x\}$ is isomorphic to a direct sum of $\mathbb{C}[\partial]$, and is freely generated over $\mathbb{C}[\partial]$ by its annihilator of the maximal ideal of $\mathcal{O}_{X,x}$.

By the above argument, the image of (3.3.3) is $M/V^{>\alpha}M$, and the strict injectivity implies that there is no $\beta \leqslant \alpha$ such that $\mathrm{Gr}_V^\beta M \neq 0$ but $\mathrm{Gr}_V^\beta \mathcal{O}_X = 0$. So the assertion follows.

Remarks 3.4. (i) The assertion of Theorem 1 does not hold if either of the two conditions is not satisfied. For example, consider $f = x^5 + y^4 + x^3y^2$ or $f = x^5 + y^4 + x^3y^2z$ where condition (a) or (b) is not satisfied, respectively. Here $\alpha = \frac{4}{5} + \frac{3}{4} - 1 = \frac{11}{20} > \alpha_f = \frac{1}{5} + \frac{1}{4} = \frac{9}{20}$. Note that their jumping coefficients coincide with those for $f = x^5 + y^4$ by [How03], and $\frac{11}{20}$ is not a jumping coefficient.

(ii) In the case of $f = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)z$, we see that $\frac{5}{7}$ is not a jumping coefficient by an argument in [Mus06] (because there is no hypersurface of degree two on $\mathbb{P}^2$ containing all the points of Z whose multiplicity is three), but it is a root of $b_f(-s)$ as shown in § 5.5 below. In this case, condition (a) in Theorem 1 with positive weights is satisfied, but condition (b) is not.

(iii) The first assertion of Theorem 1 trivially follows from [Mal81], if any rational number β in (α_f, α) such that $\exp(-2\pi i\beta)$ is an eigenvalue of the Milnor monodromy is a jumping coefficient. This condition for any $\alpha \in (\alpha_f, 1)$ is satisfied by the generic central hyperplane arrangement, but not necessarily by nongeneric arrangements, e.g. if f is as in remark (ii) above.

3.5 Spectrum

With the notation of § 3.1, let F_x denote the Milnor fiber around $x \in D := f^{-1}(0)$. The spectrum $\mathrm{Sp}(f, x) = \sum_{\alpha\in\mathbb{Q}} m_\alpha t^\alpha$ is defined by

$$m_\alpha = \sum_j (-1)^{j-n+1} \dim \mathrm{Gr}_F^p \widetilde{H}^j(F_x, \mathbb{C})_\lambda \quad \text{with } p = [n-\alpha],\ \lambda = \exp(-2\pi i\alpha),$$

where $\widetilde{H}^j(F_x, \mathbb{C})_\lambda$ is the λ-eigenspace of the reduced cohomology for the semisimple part T_s of the Milnor monodromy T, and F is the Hodge filtration, see [Ste77b, Ste89].

In this paper we define a mixed Hodge structure [Del70b, Del71, Del74] on the Milnor cohomology $H^j(F_x, \mathbb{C})$ by using the pull-back of the nearby cycle sheaf $\psi_f\mathbb{C}_X[n-1]$ by the inclusion $i_x : \{x\} \to X$ in the derived category of mixed Hodge modules. This pull-back is defined by iterating the pull-back by $i_j : X^j \to X^{j-1}$, where $X^j = \{z_i = 0 : i \leqslant j\} \subset X$ with $z_1, \ldots, z_n$ local coordinates around x. (Here we may assume that X is a polydisk around x.) The pull-back by i_j is defined by using the mapping cone of $\partial_j := \partial/\partial z_j : \mathrm{Gr}_{V_j}^1 \to \mathrm{Gr}_{V_j}^0$ where V_j is the V-filtration of Kashiwara and Malgrange along $\{z_j = 0\}$ and the Hodge filtration F on $\mathrm{Gr}_{V_j}^1$ is shifted by one so that ∂_j preserves F. (We can prove (0.1) using this, because $\mathcal{G}(X, \alpha D)$ is annihilated by the maximal ideal under the assumption of (0.1).)

The following lemma will be used in Proposition 5.3 to determine the multiplicity of the root -1 of the b-function of a hyperplane arrangement.

Lemma 3.6. *With the notation of § 3.1, assume $\mathrm{Gr}_{n-1+k}^W H^{n-1}(F_x, \mathbb{C})_\lambda \neq 0$ for a positive integer k, where W is the weight filtration. Then $N^k \neq 0$ on $\psi_{f,\lambda}\mathbb{C}_X$, where N is the logarithm of the unipotent part of the monodromy.*

Proof. We have the weight filtration W on the perverse sheaf $\psi_{f,\lambda}\mathbb{C}_X[n-1]$ (see [BBD82]) such that we have isomorphisms for $j>0$

$$N^j : \mathrm{Gr}^W_{n-1+j}\psi_{f,\lambda}\mathbb{C}_X[n-1] \to \mathrm{Gr}^W_{n-1-j}\psi_{f,\lambda}\mathbb{C}_X[n-1].$$

This gives the weight filtration of a mixed Hodge module, see [Sai88b, Sai90]. Furthermore, the mixed Hodge structure on $H^{n-1}(F_x,\mathbb{C})$ is given by applying the pull-back functor $H^j i_x^*$ to $\psi_{f,\lambda}\mathbb{C}_X[n-1]$ as in § 3.5. The functor $H^0 i_x^*$ preserves the condition that $\mathrm{Gr}^W_i = 0$ for $i>r$ where r is any fixed integer, see [Sai90]. So the hypothesis implies that $\mathrm{Gr}^W_{n-1+m}\psi_{f,\lambda}\mathbb{C}_X[n-1] \neq 0$ for some $m \geqslant k$, and the assertion follows. □

4. Brieskorn modules and Gauss–Manin systems

4.1 Let f be a non-constant holomorphic function on a complex manifold X of dimension $n \geqslant 2$, and $x \in D := f^{-1}(0)$. With the notation of § 3.1, the Gauss–Manin system is defined by

$$\mathcal{G}_f = H^0\mathrm{DR}_X\mathcal{B}_{f,x}\,(=\omega_{X,x}\otimes_{\mathcal{D}_{X,x}}\mathcal{B}_{f,x}), \tag{4.1.1}$$

where ω_X is the sheaf of the differential forms of degree n. Here we consider only the cohomology of degree 0 because we assume essentially isolated singularity conditions when we consider Gauss–Manin systems in this paper. This is a regular holonomic $\mathbb{C}\{t\}\langle\partial_t\rangle$-module where $\mathbb{C}\{t\}\langle\partial_t\rangle = \mathcal{D}_{S,0}$ with S an open disk. It is known that $\mathcal{G}_f$ is a finite free $\mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]$-module of rank μ_{n-1} where μ_j is the rank of the jth cohomology of the Milnor fiber around x, see e.g. [BS04].

The Brieskorn module is defined by

$$\mathcal{H}''_f = \Omega^n_{X,x}/df\wedge d\Omega^{n-2}_{X,x}.$$

It is a $\mathbb{C}\{t\}\langle\partial_t^{-1}\rangle$-module, where $\partial_t^{-1}\omega$ is defined by $df\wedge\eta$ with $d\eta=\omega$. There is a canonical morphism

$$\mathcal{H}''_f \to \mathcal{G}_f, \tag{4.1.2}$$

compatible with the action of $\mathbb{C}\{t\}\langle\partial_t^{-1}\rangle$ so that $\mathcal{G}_f$ is identified with the localization of $\mathcal{H}''_f$ by ∂_t^{-1}.

Let V be the filtration of Kashiwara and Malgrange on $\mathcal{G}_f$ indexed by $\mathbb{Q}$ so that

$$\partial_t t - \alpha \text{ is nilpotent on } \mathrm{Gr}^\alpha_V\mathcal{G}_f.$$

It is known that $V^\alpha\mathcal{G}_f$ for $\alpha>0$ is naturally identified with the Deligne extension of the restriction to a punctured disk of a coherent extension of $\mathcal{G}_f$ such that the eigenvalues of the residue of the connection are contained in $[\alpha-1,\alpha)$. In particular, we have isomorphisms for $\alpha\in(0,1]$ and $\lambda=\exp(-2\pi i\alpha)$

$$H^{n-1}(F_x,\mathbb{C})_\lambda \simeq \mathrm{Gr}^\alpha_V\mathcal{G}_f, \tag{4.1.3}$$

where the left-hand side is the λ-eigenspace of the Milnor cohomology for the Milnor monodromy. We define the Brieskorn lattices of $\mathcal{G}_f$ by

$$\mathcal{G}_f^{(i)} = \partial_t^{-i}\mathcal{G}_f^{(0)} \quad (i\in\mathbb{Z}) \quad \text{with } \mathcal{G}_f^{(0)} = \mathrm{Im}(\mathcal{H}''_f \to \mathcal{G}_f),$$

where the last morphism $\mathcal{H}''_f\to\mathcal{G}_f$ is as in (4.1.2). Note that $\mathcal{G}_f^{(i)}\subset\mathcal{G}_f^{(i-1)}$ because $\mathcal{G}_f^{(0)}$ is stable by the action of ∂_t^{-1}. Let $\widetilde{\mathcal{G}}_f^{(i)}$ be the saturations of $\mathcal{G}_f^{(i)}$, i.e.

$$\widetilde{\mathcal{G}}_f^{(i)} = \sum_{k\geqslant 0}(\partial_t t)^k\mathcal{G}_f^{(i)} = \sum_{k\geqslant 0}(t\partial_t)^k\mathcal{G}_f^{(i)}.$$

They have the induced filtration V. By [Kas77, Mal74], we have

$$\widetilde{\mathcal{G}}_f^{(0)} \subset V^{>0}\mathcal{G}_f. \tag{4.1.4}$$

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This implies $\partial_t t\widetilde{\mathcal{G}}_f^{(i)} = \widetilde{\mathcal{G}}_f^{(i)}$ for $i \geqslant 0$, and

$$\widetilde{\mathcal{G}}_f^{(i)} = \partial_t^{-i}\widetilde{\mathcal{G}}_f^{(0)} = t^i\widetilde{\mathcal{G}}_f^{(0)} \quad \text{for } i \in \mathbb{N}.$$

For $\alpha > 0$ and $i \in \mathbb{N}$, we have

$$t^i : \mathrm{Gr}_V^{\alpha}\widetilde{\mathcal{G}}_f^{(-i)} \xrightarrow{\sim} \mathrm{Gr}_V^{\alpha+i}\widetilde{\mathcal{G}}_f^{(0)} \subset \mathrm{Gr}_V^{\alpha+i}\mathcal{G}_f, \tag{4.1.5}$$

because the action of $t^i\partial_t^i = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - i + 1)$ on $\mathrm{Gr}_V^{\alpha+i}\widetilde{\mathcal{G}}_f^{(0)}$ for $\alpha > 0$ is injective and hence surjective.

Using the isomorphism (4.1.3) for $\alpha \in (0,1]$ and $\lambda = \exp(-2\pi i\alpha)$, we define decreasing filtrations P and $\widetilde{P}$ on the Milnor cohomology by

$$\begin{aligned} P^{n-1-i}H^{n-1}(F_x,\mathbb{C})_\lambda &\simeq \mathrm{Gr}_V^{\alpha}\mathcal{G}_f^{(-i)},\\ \widetilde{P}^{n-1-i}H^{n-1}(F_x,\mathbb{C})_\lambda &\simeq \mathrm{Gr}_V^{\alpha}\widetilde{\mathcal{G}}_f^{(-i)}. \end{aligned} \tag{4.1.6}$$

Note that $P^{n-1-i} = \widetilde{P}^{n-1-i} = 0$ for $i < 0$ by (4.1.4), and $\widetilde{P}^{n-1-i}$ is stable by the Milnor monodromy because $\widetilde{\mathcal{G}}_f^{(-i)}$ is stable by the action of $t\partial_t$.

If $\xi f = f$ for a vector field ξ, then $\mathcal{G}_f^{(-i)} = \widetilde{\mathcal{G}}_f^{(-i)}$ and $P = \widetilde{P}$. In the isolated singularity case, P coincides with the Hodge filtration, see [SS85, Var82]. Note that $\mathcal{G}_f^{(1)}$ is the image of $df \wedge \Omega_{X,x}^{n-1}$ and $\mathcal{G}_f^{(0)}/\mathcal{G}_f^{(1)}$ is a quotient of

$$\Omega_f := \omega_{X,x}/df \wedge \Omega_{X,x}^{n-1},$$

because $d(\eta \otimes 1) = d\eta \otimes 1 - (df \wedge \eta) \otimes \partial_t$ where d is the differential of $\mathrm{DR}_X\mathcal{B}_f$, see (3.1.1). If D has an isolated singularity, then it is well known that (4.1.2) is injective, and $\mathcal{G}_f^{(0)}/\mathcal{G}_f^{(1)} = \Omega_f$, see [Pha79, SS85].

Proposition 4.2. *The filtration V on the Gauss–Manin system $\mathcal{G}_f$ coincides with the filtration induced by the filtration V of Kashiwara and Malgrange on $\mathcal{B}_f$ via the isomorphism (4.1.1) using any trivialization of $\omega_{X,x}$. We have the canonical isomorphisms for $\alpha \in \mathbb{Q}$*

$$V^{\alpha}\mathcal{G}_f = H^0\mathrm{DR}_X(V^{\alpha}\mathcal{B}_{f,x}), \quad \mathrm{Gr}_V^{\alpha}\mathcal{G}_f = H^0\mathrm{DR}_X(\mathrm{Gr}_V^{\alpha}\mathcal{B}_{f,x}). \tag{4.2.1}$$

Proof. For $-\infty < \alpha < \beta < \gamma < +\infty$, we have a long exact sequence

$$\xrightarrow{\partial} H^i\mathrm{DR}_X(V^{\beta}/V^{\gamma})\mathcal{B}_{f,x} \to H^i\mathrm{DR}_X(V^{\alpha}/V^{\gamma})\mathcal{B}_{f,x} \to H^i\mathrm{DR}_X(V^{\alpha}/V^{\beta})\mathcal{B}_{f,x} \xrightarrow{\partial},$$

compatible with the action of $\partial_t t$ so that the connecting morphisms ∂ vanish, where $(V^{\alpha}/V^{\beta})\mathcal{B}_{f,x} = V^{\alpha}\mathcal{B}_{f,x}/V^{\beta}\mathcal{B}_{f,x}$. We have the finiteness of $H^i\mathrm{DR}_X(V^{\alpha}\mathcal{B}_{f,x})$ over $\mathbb{C}\{t\}$ by [Sai88b, Proposition 3.4.8], and the connecting morphisms ∂ vanish also for $\gamma = +\infty$ (where $V^{+\infty} = 0$) using the completion, see [Sai88b]. So the first isomorphism follows, and the second isomorphism then follows using the vanishing of ∂. □

Proposition 4.3. *With the notation of § 3.1 we have for any $\alpha \in \mathbb{Q}$*

$$\mathrm{Im}(H^0\mathrm{DR}_X(V^{\alpha}M_x) \to H^0\mathrm{DR}_X(\mathcal{B}_{f,x})) = V^{\alpha}\widetilde{\mathcal{G}}_f^{(0)},$$

$$\mathrm{Im}(H^0\mathrm{DR}_X(\mathrm{Gr}_V^{\alpha}M_x) \to H^0\mathrm{DR}_X(\mathrm{Gr}_V^{\alpha}\mathcal{B}_{f,x})) = \mathrm{Gr}_V^{\alpha}\widetilde{\mathcal{G}}_f^{(0)}.$$

Proof. The canonical morphism $H^0\mathrm{DR}_X(V^{\alpha}M_x) \to H^0\mathrm{DR}_X(\mathrm{Gr}_V^{\alpha}M_x)$ is surjective, because $H^1\mathrm{DR}_X(V^{>\alpha}M_x) = 0$. So the second isomorphism is reduced to the first using Proposition 4.2, and it is enough to show the first isomorphism. The right-hand side is the intersection of the images of $H^0\mathrm{DR}_X(M_x)$ and $H^0\mathrm{DR}_X(V^{\alpha}\mathcal{B}_{f,x})$, because $\widetilde{\mathcal{G}}_f^{(0)}$ is the image of $H^0\mathrm{DR}_X(M_x)$. By the

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commutative diagram

$$\begin{tikzcd} H^0\mathrm{DR}_X(V^\alpha M) \arrow[r] \arrow[d] & H^0\mathrm{DR}_X(V^\alpha\mathcal{B}_f) \arrow[r] \arrow[d] & H^0\mathrm{DR}_X V^\alpha(\mathcal{B}_f/M) \arrow[r] \arrow[d] & 0 \\ H^0\mathrm{DR}_X(M) \arrow[r] & H^0\mathrm{DR}_X(\mathcal{B}_f) \arrow[r] & H^0\mathrm{DR}_X(\mathcal{B}_f/M) \arrow[r] & 0 \end{tikzcd}$$

the assertion is reduced to the injectivity of the last vertical morphism, but this is easily proved by using the action of $\partial_t t$ together with a long exact sequence as in the proof of Proposition 4.2. So the assertion follows. □

Proposition 4.4. *With the notation of § 4.1 the Hodge filtration F on the Milnor cohomology is contained in $\widetilde{P}$.*

Proof. The Hodge filtration F on the Milnor cohomology is defined by using the construction in § 3.5. For any regular holonomic $\mathcal{D}_{X^{j-1}}$-module N, we have canonical morphisms of complexes

$$C(\partial_j:\mathrm{Gr}^1_{V_j}N\to\mathrm{Gr}^0_{V_j}N)\leftarrow C(\partial_j:V^1_jN\to V^0_jN)\to C(\partial_j:N\to N),$$

which are quasi-isomorphisms at least after taking the de Rham functor on X^j. Iterating this, we get a canonical isomorphism in the derived category

$$i_x^*\mathrm{Gr}^\alpha_V\mathcal{B}_f=\mathrm{DR}_X(\mathrm{Gr}^\alpha_V\mathcal{B}_{f,x}),$$

where the left-hand side is defined as in § 3.5. If N underlies a mixed Hodge module so that it has the Hodge filtration F, then the Hodge filtration F on $\mathrm{Gr}^\alpha_{V_j}N$ is induced by F on N, and the canonical surjection $V^0_jN\to\mathrm{Gr}^0_{V_j}N$ is strictly compatible with F. This implies that the Hodge filtration F on $H^0i_x^*\mathrm{Gr}^\alpha_V\mathcal{B}_f$ is contained in the filtration on $H^0\mathrm{DR}_X(\mathrm{Gr}^\alpha_V\mathcal{B}_{f,x})=\mathrm{Gr}^\alpha_V\mathcal{G}_f$ induced by F on $\mathrm{Gr}^\alpha_V\mathcal{B}_{f,x}$ (up to an appropriate shift) via the above isomorphism. Moreover, the latter filtration on $\mathrm{Gr}^\alpha_V\mathcal{G}_f$ is contained in $\widetilde{P}$ by (4.1.5) and Proposition 4.3. Indeed,

$$F_p\mathcal{B}_{f,x}=\bigoplus_{0\leqslant i\leqslant p}\mathcal{O}_X\otimes\partial_t^i,$$

and the image of $F_pV^\alpha\mathcal{B}_{f,x}$ in $H^0\mathrm{DR}_X(\mathcal{B}_{f,x})=\mathcal{G}_f$ is contained in $\widetilde{\mathcal{G}}^{(-p)}\cap V^\alpha\mathcal{G}_f$ by $\widetilde{\mathcal{G}}_f^{(-i)}\subset\widetilde{\mathcal{G}}_f^{(-i-1)}$. So the assertion follows. □

4.5 Proof of Theorem 2

Since the de Rham complex DR_X is the Koszul complex for $\partial_1,\dots,\partial_n$ (trivializing ω_X by $dx_1\wedge\cdots\wedge dx_n$) and M_x is generated over $\mathcal{D}_X[s]$ by $1\otimes 1$ (or f^s), we see that the image of $H^0\mathrm{DR}_XM_x$ in $\mathcal{G}_f$ coincides with $\widetilde{\mathcal{G}}_f^{(0)}$. By (4.1.3) the filtration $\widetilde{P}$ in (4.1.6) is identified with a filtration on $\mathrm{Gr}^\alpha_V\mathcal{G}_f$ for $\alpha\in(0,1]$, which is also denoted by $\widetilde{P}$. Then, by (4.1.5), we have for $i\in\mathbb{N}$

$$t^i:\widetilde{P}^{n-1-i}\mathrm{Gr}^\alpha_V\mathcal{G}_f\xrightarrow{\sim}\mathrm{Gr}^{\alpha+i}_V\widetilde{\mathcal{G}}_f^{(0)}\subset\mathrm{Gr}^{\alpha+i}_V\mathcal{G}_f.$$

Let $\widetilde{P}$ denote also the filtration on $\mathrm{Gr}^\alpha_V\mathcal{B}_f$ for $\alpha\in(0,1]$ such that for $i\in\mathbb{N}$

$$t^i:\widetilde{P}^{n-1-i}\mathrm{Gr}^\alpha_V\mathcal{B}_f\xrightarrow{\sim}\mathrm{Gr}^{\alpha+i}_VM\subset\mathrm{Gr}^{\alpha+i}_V\mathcal{B}_f.$$

Then

$$t^i:\mathrm{Gr}^{n-1-i}_{\widetilde{P}}\mathrm{Gr}^\alpha_V\mathcal{B}_f\xrightarrow{\sim}\mathrm{Gr}^{\alpha+i}_V(M/tM).$$

By Proposition 4.3 the filtration $\widetilde{P}$ on $\mathrm{Gr}^\alpha_V\mathcal{B}_f$ induces $\widetilde{P}$ on $\mathrm{Gr}^\alpha_V\mathcal{G}_f$ taking the de Rham functor DR_X. So the assertion (a) follows from the spectral sequence

$$E_1^{p,q}=H^{p+q}\mathrm{DR}_X\mathrm{Gr}^p_{\widetilde{P}}\mathrm{Gr}^\alpha_V\mathcal{B}_{f,x}\Rightarrow H^{p+q}\mathrm{DR}_X\mathrm{Gr}^\alpha_V\mathcal{B}_{f,x},\tag{4.5.1}$$

because $E_1^{p,q}\neq 0$ if $E_\infty^{p,q}\neq 0$.

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If $\alpha + i$ is not a root of $b_{f,y}(-s)$ for any $y \neq x$ and $i \geqslant i_0$ where i_0 is a nonnegative integer, then $E_1^{p,q} = 0$ for $p + q > 0$ or $p + q < 0$ and $p \leqslant p_0 := n - 1 - i_0$. Indeed, a $\mathcal{D}_X$-module supported on a point is a direct sum of $\mathbb{C}[\partial]$ in the notation of § 3.3, and $\mathrm{DR}_X(\mathbb{C}[\partial]) = \mathbb{C}$. So we have $E_1^{p,q} = E_\infty^{p,q}$ for $p \leqslant p_0$, and the assertion (b) follows. If the assumption of assertion (c) is satisfied, then $E_1^{p,q} = 0$ for $p + q \neq 0$ and $E_1^{p,q} = E_\infty^{p,q}$ for any p, q by a similar argument. So the assertion follows.

PROPOSITION 4.6. *With the notation of §§ 3.1 and 4.1, assume that f is a weighted homogeneous polynomial of strictly positive weights $(w_1, \ldots, w_n)$. Let $\xi = \sum_i w_i x_i \partial_i$ so that $\xi f = f$ and, hence, $P = \widetilde{P}$. Let Ω_f^β be the β-eigenspace for the Lie derivation by ξ. Let α be a rational number such that $\exp(-2\pi i\alpha)$ is not an eigenvalue of the Milnor monodromy at $y \neq x$. Then α is a root of $b_f(-s)$ if and only if the image of Ω_f^α in $\mathcal{G}_f^{(0)}/\mathcal{G}_f^{(1)}$ does not vanish. Its multiplicity is one if α is a root.*

Proof. Since $\xi f = f$, we have $t\mathcal{G}_f^{(0)} \subset \partial_t^{-1}\mathcal{G}_f^{(0)}$ and, hence, $\widetilde{\mathcal{G}}_f^{(0)} = \mathcal{G}_f^{(0)}$ by the definition of the action of ∂_t^{-1}. Let $\omega_{X,x}^\beta$ be the β-eigenspace for the action of ξ. Then the action of $\partial_t t$ on the image of $\omega_{X,x}^\beta \otimes 1 \otimes 1$ in $\mathcal{G}_f$ is given by the multiplication by β. Indeed, if we take $g dx_1 \wedge \cdots \wedge dx_n \in \omega_{X,x}^\beta$ so that $\xi g = (\beta - \sum_i w_i) g$, then we have, by (3.1.1),

$$\Big(\sum_i w_i \partial_i x_i\Big)(g \otimes 1) = \beta g \otimes 1 - \partial_t t(g \otimes 1),$$

where the left-hand side vanishes in the cohomology of the de Rham complex which is identified with the Koszul complex for $\partial_1, \ldots, \partial_n$ as above, see also [Sai88a].

This implies that $\mathrm{Gr}_V^\beta \mathcal{G}_f^{(0)}$ is generated by the image of $\omega_{X,x}^\beta$ so that the action of $\partial_t t$ on $\mathrm{Gr}_V^\beta \mathcal{G}_f^{(0)}$ is semisimple (using the algebraic Gauss–Manin system if necessary). Then the assertions follow from Theorem 2. □

In the isolated singularity case, we have the following.

PROPOSITION 4.7. *With the notation of § 4.1, assume that D has an isolated singularity at x. Let V denote the filtration on $\omega_{X,x} = \Omega_{X,x}^n$ induced by the filtration V on $\mathcal{O}_{X,x}$ using any trivialization of $\omega_{X,x}$. Then the natural projection $\omega_{X,x} \to \mathcal{H}_f''$ is strictly compatible with the filtration V^α for $\alpha \leqslant 1$.*

Proof. Let V' denote the filtration on $\omega_{X,x}$ induced by the filtration V on $\mathcal{H}_f''$ using the projection $\omega_{X,x} \to \mathcal{H}_f''$. Since the filtration V on $\mathcal{B}_f$ induces the filtration V on the Gauss–Manin system by Proposition 4.2, we have $V^\alpha \omega_{X,x} \subset V'^\alpha \omega_{X,x}$. Then we get the equality for $\alpha \leqslant 1$ by calculating the dimension of their graded pieces for $\alpha < 1$, because they both give the coefficient m_α of the spectrum for $\alpha < 1$. So the assertion follows. □

4.8 Hodge and pole order filtrations

Assume that X is affine space $\mathbb{C}^n$ and that D is the affine cone of a divisor Z of degree d on $Y := \mathbb{P}^{n-1}$. Then there is a cyclic covering $\pi : \widetilde{Y} \to Y$ of degree d which is ramified along Z. Put $U = Y \setminus Z$, $\widetilde{U} = \pi^{-1}(U)$. Then $\widetilde{U}$ is identified with the Milnor fiber F_0 of a function f defining the affine cone D of Z, and the geometric Milnor monodromy corresponds to a generator of the covering transformation group of $\widetilde{U} \to U$, see [DS06, § 1.8].

For $k = 1, \ldots, d$, let $L^{(k/d)}$ be the direct factor of $\pi_* \mathbb{C}_{\widetilde{U}}$ on which the action of the Milnor monodromy is the multiplication by $\exp(-2\pi ik/d)$ so that $H^j(U, L^{(k/d)}) = H^j(F_0, \mathbb{C})_\lambda$ where $\lambda = \exp(-2\pi ik/d)$. Note that $L^{(k/d)}$ is a local system of rank one on U, and its monodromy around a smooth point x of Z_{red} is the multiplication by $\exp(2\pi ikm_x/d)$ where m_x is the multiplicity of Z at x. (This can be shown by blowing up along the origin of $\mathbb{C}^n$ and considering the nearby cycles for the pull-back of f, see also [DS04].)

Let $\mathcal{L}^{(k/d)}$ be the meromorphic extension of $L^{(k/d)} \otimes_{\mathbb{C}} \mathcal{O}_U$. This is a regular holonomic $\mathcal{D}_Y$-module on which the action of a function h defining Z is bijective. We see that $\mathcal{L}^{(k/d)}$ is locally isomorphic to a free $\mathcal{O}_Y(*Z)$-module generated by a multivalued function $h_j^{-k/d}$ where $h_j = x_j^{-d} f$ on $\{x_j \neq 0\} \subset \mathbb{P}^{n-1}$. Note that the $\mathcal{O}_Y$-submodule generated locally by $h_j^{-k/d} (= x_j^k f^{-k/d})$ is isomorphic to $\mathcal{O}_Y(k)$, because the relation $g_j h_j^{-k/d} = g_i h_i^{-k/d}$ means that $g_j x_j^k = g_i x_i^k$, i.e. $\{g_j\}$ defines a section of $\mathcal{O}_Y(k)$.

The pole order filtration $P_i\mathcal{L}^{(k/d)}$ is defined to be the locally free $\mathcal{O}_Y$-submodule of $\mathcal{L}^{(k/d)}$ generated by $h_j^{-i-(k/d)}$ on $\{x_j \neq 0\}$ for $i \in \mathbb{N}$, and $P_i\mathcal{L}^{(k/d)} = 0$ for $i < 0$. Then $P_i\mathcal{L}^{(k/d)}$ is isomorphic to $\mathcal{O}_Y(id+k)$ by the above argument. On the other hand, there is the Hodge filtration F on $\mathcal{L}^{(k/d)}$ such that $F_i\mathcal{L}^{(k/d)} = P_i\mathcal{L}^{(k/d)}$ outside $\mathrm{Sing} Z_{\mathrm{red}}$ for any i by the theory of mixed Hodge modules. Then we have $F_i\mathcal{L}^{(k/d)} \subset P_i\mathcal{L}^{(k/d)}$ on Y because $P_i\mathcal{L}^{(k/d)}$ is locally free and $\mathrm{Sing} Z_{\mathrm{red}}$ has codimension at least two in Y.

The Hodge and pole order filtrations are closely related respectively to the spectrum and the b-function of f. Indeed, the Hodge filtration F on $\mathcal{L}^{(k/d)}$ induces the Hodge filtration on the Milnor cohomology by taking the de Rham cohomology. Similarly the pole order filtration P on the Milnor cohomology is defined by using the de Rham cohomology. Here the filtration is shifted by the degree of the differential forms, and the associated decreasing filtration is used. Then we have Theorem 2 together with the following.

Proposition 4.9. *With the above notation and assumptions, the above pole order filtration P coincides with the filtration $\widetilde{P} = P$ in (4.1.6). Moreover, for $\alpha = k/d \in (0,1)$ and $\lambda = \exp(-2\pi i\alpha)$, the above P^{n-1-i} is identified with the image of*

$$\mathrm{Gr}_V^{\alpha+i}\mathcal{G}_f^{(0)} \subset \mathrm{Gr}_V^{\alpha+i}\mathcal{G}_f \simeq \mathrm{Gr}_V^{\alpha}\mathcal{G}_f \simeq H^{n-1}(F_0,\mathbb{C})_\lambda,$$

where the middle isomorphism can be induced by both ∂_t^k and t^{-k}, and the last morphism is induced by (4.1.3).

Proof. This follows from the arguments in [DS06], using the local generator $h_j^{-k/d}$ in § 4.8 to define an isomorphism generalizing [DS06, Lemma 1.2]. Note that $\mathcal{S}^i (0 < i < d)$ in [DS06] is identified with $\mathcal{L}^{((d-i)/d)} \otimes \mathcal{O}_Y(-Z)$ in this paper, and $\Omega^j[f^{-1}]_k^{(\xi)}$ (the degree k part of the image of the interior product ι_ξ) is identified with the vector space of meromorphic sections of $\pi_*\pi^*\Omega_Y^j$ over $Y \setminus Z$ on which the Lie derivation L_ξ acts as the multiplication by k. Here $\pi : \mathbb{C}^n \setminus \{0\} \to Y$ denotes the canonical projection. Then we get the desired isomorphism by using the restriction to $\{x_i = 1\} \subset \mathbb{C}^n$ for any i. The above identification of the filtrations is compatible with (4.1.6) because $t^i\mathrm{Gr}_V^\alpha \partial_t^i \mathcal{G}_f^{(0)} = t^i\partial_t^i \mathrm{Gr}_V^{\alpha+i}\mathcal{G}_f^{(0)} = \mathrm{Gr}_V^{\alpha+i}\mathcal{G}_f^{(0)}$. □

Remark 4.10. If Z is smooth, the two filtrations F_i and P_i on $\mathcal{L}^{(k/d)}$ coincide for any i, and this explains the coincidence of the spectrum and the roots of $b_f(-s)$ (forgetting the multiplicity) in this case. However, if Z is a reduced divisor with normal crossings, these two filtrations coincide only for $i = 0$, and not for $i > 0$ because the Hodge filtration is defined by using the *sum* of the pole orders along the irreducible components, see [Del70a]. This explains the fact that the spectrum and the roots of $b_f(-s)$ coincide (forgetting the multiplicity) only if they are restricted to the interval $(0,1]$ in this case.

Remark 4.11. Let D be as in § 4.8 so that f is a homogeneous polynomial of degree d. Then the Brieskorn lattice $\mathcal{G}_f^{(0)}$ has a monomial basis (ω_j) over $\mathbb{C}\{t\}$ such that each ω_j is represented by

$$x^\nu dx_1 \wedge \cdots \wedge dx_n \quad \text{with } \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n.$$

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Restricting to a Milnor fiber, this gives a basis $([\omega_j])$ of the Milnor cohomology. For the last assertion, it is enough to assume that (ω_j) gives a basis of $\mathcal{G}_f^{(0)}[t^{-1}]$ over $\mathbb{C}\{t\}[t^{-1}]$, and the minimality condition in [Wal05, Theorem 4.8], corresponds to that it gives a basis of $\mathcal{G}_f^{(0)}$ over $\mathbb{C}\{t\}$. Using this basis, the pole order filtration P on the Milnor cohomology is expressed as

$$P^{n-1-i} = \sum_{\deg(\omega_j)/d \leqslant i+1} \mathbb{C}[\omega_j],$$

where $\deg(x^\nu dx_1 \wedge \cdots \wedge dx_n) = |\nu| + n$. This follows from the well-known formula

$$\partial_t t(x^\nu dx_1 \wedge \cdots \wedge dx_n) = (\deg(x^\nu dx_1 \wedge \cdots \wedge dx_n)/d) x^\nu dx_1 \wedge \cdots \wedge dx_n.$$

(See the proof of Proposition 4.6.) So Theorem 2(a) may be viewed as a generalization of [Wal05, Theorem 4.8].

5. Case of hyperplane arrangements

5.1 Cohomology of twisted de Rham complexes

With the notation and the assumptions of § 4.8, assume further that D is the affine cone of a projective hyperplane arrangement Z in $Y = \mathbb{P}^{n-1}$, i.e. D is a central hyperplane arrangement. Then, by [Bri73, ESV92, STV95], the cohomology of the local systems on $U = Y \setminus Z$ in § 4.8 can be calculated as follows.

Let $Z_i\,(1 \leqslant i \leqslant d)$ be the irreducible components of Z where $d = \deg Z$, and $x_1, \ldots, x_n$ be coordinates of $\mathbb{C}^n$ such that $Z_d = \{x_n = 0\}$. Then the complement Y' of Z_d in Y is identified with $\mathbb{C}^{n-1}$. Let g_i be a polynomial of degree 1 on Y' defining $Z_i \cap Y'$. Put

$$\omega_i = dg_i/g_i\,(1 \leqslant i \leqslant d-1), \quad h = g_1 \cdots g_{d-1}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in \mathbb{C}^{d-1}$, let

$$h^\alpha = g_1^{\alpha_1} \cdots g_{d-1}^{\alpha_{d-1}},$$

and $\mathcal{O}_{Y'}h^\alpha$ be a free $\mathcal{O}_{Y'}$-module of rank one on Y' with formal generator h^α. There is a regular singular integrable connection ∇ such that for $u \in \mathcal{O}_{Y'}$

$$\nabla(uh^\alpha) = (du)h^\alpha + u\omega h^\alpha \quad \text{with } \omega = \sum_{1 \leqslant i \leqslant d-1} \alpha_i \omega_i.$$

Let $\mathcal{A}^p_{h,\alpha}$ be the $\mathbb{C}$-vector subspace of $\Gamma(U, \Omega^p_U h^\alpha)$ generated by $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} h^\alpha$ for any $i_1 < \cdots < i_p$. Then $\mathcal{A}^\bullet_{h,\alpha}$ with differential $\omega\wedge$ is a subcomplex of $\Gamma(U, \Omega^\bullet_U h^\alpha)$. Put $\alpha_d = -\sum_{1 \leqslant i \leqslant d-1} \alpha_i$. By [Bri73, ESV92, STV95], we have the canonical quasi-isomorphism

$$\mathcal{A}^\bullet_{h,\alpha} \xrightarrow{\sim} \Gamma(U, \Omega^\bullet_U h^\alpha), \tag{5.1.1}$$

if the following condition holds for any dense edge L of Z:

$$\alpha_L := \sum_{Z_i \supset L} \alpha_i \notin \mathbb{N} \setminus \{0\}. \tag{5.1.2}$$

Here an edge is an intersection of Z_i over a subset of $\{1, \ldots, d\}$, and an edge is called dense if and only if the hyperplanes containing the edge are identified with an indecomposable central arrangement (where an arrangement in $\mathbb{C}^n$ is called decomposable if and only if there is a decomposition $\mathbb{C}^n = \mathbb{C}^{n'} \times \mathbb{C}^{n''}$ such that the arrangement is the union of the pull-backs of arrangements on $\mathbb{C}^{n'}$ and $\mathbb{C}^{n''}$, see [STV95] for details.) In the case of a constant local system, this is due to [Bri73]. In a general case, it is shown in [ESV92] and is improved in [STV95]. Note that if Z is a divisor with normal crossings (i.e. if Z is generic), then condition (5.1.2) is equivalent to $\alpha_i \notin \mathbb{N} \setminus \{0\}$ for any $i \in [1, d]$ (because the dense edges consist of the Z_i in this case), and [ESV92] is sufficient in this case.

As a corollary, we obtain the following.

PROPOSITION 5.2. *Let D be a central hyperplane arrangement in $\mathbb{C}^n$ defined by a reduced polynomial f of degree d. Let Z be the projective arrangement in $\mathbb{P}^{n-1}$ corresponding to D. For an edge L of Z, let m_L be the number of hyperplanes $Z_i \supset L$. Assume that all of the roots of $b_{h,y}(-s)$ are strictly less than two for any $y \in Z$ where h is a reduced local equation of Z at y. Assume, moreover, that there is a hyperplane, denoted by Z_d, such that $\mathrm{GCD}(m_L, d) = 1$ for any dense edge L of Z contained in Z_d. Then the roots of $b_{f,0}(-s)$ are strictly less than two.*

Proof. We apply the above argument to the case $\alpha_i = -k/d (0 \leqslant i < d)$ for each $k \in [0, d-1]$. Then for $k \in [1, d-1]$, we see that $h^{\alpha} = h^{-k/d}$ is a section of $P_0\mathcal{L}^{(k/d)}$ which has a zero of order k along the divisor at infinity Z_d, and condition (5.1.1) is satisfied for any dense edge L of Z. (Indeed, $\alpha_L \notin \mathbb{Z}$ if $L \subset Z_d$, and $\alpha_L \leqslant 0$ otherwise.) Moreover, the meromorphic extension of $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ to Y has at most a pole of order one along each $Z_i (1 \leqslant i \leqslant d)$. Thus, $\mathcal{A}^{n-1}_{h,\alpha}$ is contained in $\Omega^{n-1}_Y \otimes_{\mathcal{O}} P_i\mathcal{L}^{(k/d)}$ with $i = 1$ if $0 < k < d$ and $i = 0$ if $k = 0$ by the definition of P_i. So the assertion follows from Theorem 2 together with (5.1.1) using [Bri73, ESV92, STV95]. □

The first assertion of the proposition below is due to [Wal05].

PROPOSITION 5.3. *With the above notation, -1 is the only integral root of $b_f(s)$ (see [Wal05]), and its multiplicity is n, assuming that the arrangement is not the pull-back of an arrangement in a strictly lower-dimensional vector space.*

Proof. The assertion is well known in the normal crossing case. In particular, it holds on the smooth part of Z. By induction on stratum, we may assume that the assertion holds for any $y \in \mathbb{C}^n \setminus \{0\}$. Note that the b-function of a global defining equation of a central hyperplane arrangement is equal to that of a local equation at zero, using the $\mathbb{C}^*$-action. We can apply § 5.1 with $\alpha_i = 0$ for any i, and (5.1.1) holds by [Bri73] where $\omega\wedge = 0$. In particular, $H^{n-1}(\mathbb{P}^{n-1}\setminus Z, \mathbb{C})$ is nonzero and is generated by logarithmic forms on an embedded resolution of $(\mathbb{P}^{n-1}, Z)$, see [ESV92]. Then $\mathrm{Gr}^p_F H^{n-1}(F_0, \mathbb{C})_1 = 0$ for $p \neq n-1$, and hence -1 is the only integral root by Theorem 2. Moreover, $\mathrm{Gr}^W_i H^{n-1}(F_0, \mathbb{C})_1 = 0$ for $i \neq 2n-2$ by the Hodge symmetry of $\mathrm{Gr}^W_i H^{n-1}(F_0, \mathbb{C})_1$. Thus $\mathrm{Gr}^W_{2n-2} H^{n-1}(F_0, \mathbb{C})_1 \neq 0$. (This also follows from [DS04] in the case Z is a divisor with normal crossings.) So the assertion holds from Lemma 3.6 together with [Kas83, Mal81]. □

5.4 *b*-function of a generic hyperplane arrangement

The b-function $b_f(s)$ of a generic central hyperplane arrangement with a reduced equation is determined as follows by Walther [Wal05].

(W) The roots of $b_f(s)$ are $-j/d$ for $n \leqslant j \leqslant 2d-2$, and the multiplicity of a root α is 1 for $\alpha \neq -1$ and is n for $\alpha = -1$, assuming $d > n$.

(The last assertion on the multiplicity of -1 was not proved in [Wal05]) Here generic means that a central hyperplane arrangement has normal crossings outside the origin. In particular, the arrangement is not the pull-back of an arrangement in a strictly lower-dimensional vector space since $d > n$. Using the arguments in this paper, we can give another proof of his theorem as follows.

By Proposition 5.2, using [Bri73, ESV92] in the normal crossing case, we first get

$$\text{The roots of } b_f(-s) \text{ is strictly smaller than two.} \tag{5.4.1}$$

The assertion on the integral roots follows from Proposition 5.3. For the non-integral roots, the multiplicity is always one by Proposition 4.6. Moreover, Theorem 2 and (5.4.1) imply that $1 + k/d$ is a root of $b_f(-s)$ for $1 \leqslant k \leqslant d-2$, and is not a root for $k = d-1$. Indeed, by Corollary 1, m_α for $\alpha = k/d$ is strictly smaller than $\binom{d-2}{n-1}$ if $k \leqslant d-2$, and they coincide if $k = d-1$.

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Since $\dim H^{n-1}(F_0,\mathbb{Q})_\lambda = \binom{d-2}{n-1}$ for $\lambda = \exp(-2\pi ik/d)$ with $1 \leqslant k \leqslant d-1$ (see [CS95, OR93]), these imply

$$F^{n-1}H^{n-1}(F_0,\mathbb{C})_\lambda \neq H^{n-1}(F_0,\mathbb{C})_\lambda \quad \text{if } 1 \leqslant k \leqslant d-2,$$
$$F^{n-1}H^{n-1}(F_0,\mathbb{C})_\lambda = H^{n-1}(F_0,\mathbb{C})_\lambda \quad \text{if } k = d-1.$$

Note that $P^{n-1}H^{n-1}(F_0,\mathbb{C}) = F^{n-1}H^{n-1}(F_0,\mathbb{C})$ because $P_0\mathcal{L}^{(k/d)} = F_0\mathcal{L}^{(k/d)}$ (since Z is a divisor with normal crossings). We have, moreover, $P^{n-2}H^{n-1}(F_0,\mathbb{C})_\lambda = H^{n-1}(F_0,\mathbb{C})_\lambda$ by (5.4.1). So the assertion follows from Theorem 2.

5.5 Example of a nongeneric hyperplane arrangement

With the notation of § 5.1, assume $n=3$, $d=7$, and $h=(x^2-y^2)(x^2-1)(y^2-1)$ so that f is as in Remarks 3.4(ii). Then $\frac{5}{7}$ is a root of $b_f(-s)$ (although it is not a jumping coefficient).

Indeed, let $Z' = \{x^2-y^2=0\}$, $Z'' = \{(x^2-1)(y^2-1)=0\}$. Then we can apply the argument in § 5.1 to the case where $\alpha_i = -\frac{5}{7}$ if $Z_i \subset Z'$, and $\alpha_i = \frac{2}{7}$ if $Z_i \subset Z''$. In this case we have

$$\dim \mathcal{A}^1_{h,\alpha} = 6, \quad \dim \mathcal{A}^2_{h,\alpha} = 9, \quad \dim H^2(\mathcal{A}^\bullet_{h,\alpha}) = \chi(U) = 4.$$

Since $\Omega^2_Y \otimes_{\mathcal{O}} P_0\mathcal{L}^{(5/7)} \simeq \mathcal{O}_Y(2)$ where $Y = \mathbb{P}^2$, we see that $g(x,y)h^{-5/7}dx \wedge dy$ can be extended to a section of $\Omega^2_Y \otimes_{\mathcal{O}} P_0\mathcal{L}^{(5/7)}$ if $g(x,y)$ is a polynomial of degree at most two. Moreover, $g(x,y)h^{-5/7}dx \wedge dy$ is contained in $\mathcal{A}^2_{h,\alpha}$ if $g(x,y)$ is a linear combination of $(x-\varepsilon)(y-\varepsilon')$ with $\varepsilon,\varepsilon' = \pm 1$, i.e. if $g(x,y)$ has order at most one for both x and y. Indeed, h^α is naturally extended to a section of $P_0\mathcal{L}^{(5/7)}$ having a simple zero along Z'' and the divisor at infinity, and $d(x+\varepsilon)/(x+\varepsilon) \wedge d(y+\varepsilon')/(y+\varepsilon')$ has a simple pole along the divisor at infinity.

Let V be the vector subspace of $\mathcal{A}^2_{h,\alpha}$ consisting of such elements. We see that the dimension of the image of $d\mathcal{A}^1_{h,\alpha}$ in $\mathcal{A}^2_{h,\alpha}/V$ is at least two, calculating the differential d of $\mathcal{A}^\bullet_{h,\alpha}$ which is defined by $\omega\wedge$, see § 5.1. So we get $\dim V \cap d\mathcal{A}^1_{h,\alpha} < \dim V = 4$ because $\dim d\mathcal{A}^1_{h,\alpha} = 5$. Thus, the image of V in $H^2(U, L^{(5/7)})$ does not vanish by (5.1.1) (using [ESV92]), and hence $\frac{5}{7}$ is a root of $b_f(-s)$ by Theorem 2 together with Proposition 4.9.

5.6 Spectrum of a generic hyperplane arrangement

The calculation in § 5.4 implies that the coefficient m_α of $\mathrm{Sp}(f,0)$ for $\alpha \in \mathbb{Z}$ is given by

$$m_{n-i} = (-1)^{n-1-i}\binom{d-1}{i} \quad \text{for } 1 \leqslant i \leqslant n-1. \tag{5.6.1}$$

Assume $\alpha \notin \mathbb{Z}$. Then m_α is calculated by Corollary 1 if $\alpha < 1$ or $\alpha > n-1$. For $1 < \alpha < n-1$, it is possible to calculate m_α using [DS04] together with the (twisted) weight spectral sequence because the dimension of the cohomology of the twisted forms $\Omega^j_{\mathbb{P}^i}(r)$ on the projective space $\mathbb{P}^i$ can be calculated by using the Bott vanishing theorem and the Euler sequence.

Remark 5.7. For hyperplane arrangements, it is conjectured by Mustata [Mus06] that the jumping coefficients depend only on the combinatorial data (i.e. the dimensions of various intersections of irreducible components) of the hyperplane arrangement. This assertion can be reduced to that for the spectrum, and will be proved in a forthcoming joint paper with Budur and Mustata. Using [BS05] together with Hodge theory, it is easy to show that they remain unchanged under a deformation with the combinatorial data fixed, see also [Ran97]. However, the parameter space of hyperplane arrangements with fixed combinatorial data is not connected as shown in [Ryb98]. In the case of a cone of a curve of higher degree in $\mathbb{P}^2$, a similar fact is known as Zariski's example, see e.g. [Esn82].

For hyperplane arrangements, it is possible to show the non-connectivity of the parameter space by using the following.

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(A) Let $p_i = a_i + \lambda b_i (i = 1, 2, 3)$ be three points on $\mathbb{C}^2$ with a linear motion parametrized by $\lambda \in \mathbb{C}$. Then there are, in general, two values of λ for which the three points are on one line.

Indeed, this implies that, for a certain family of line arrangements in $\mathbb{C}^2$ with fixed combinatorial data whose parameter space is one-dimensional, it is possible only for two points of the parameter space to add one line to the corresponding line arrangement so that the obtained line arrangement has certain restricted combinatorial data. For example, consider the union of

$$\{xy(x^2 - 1)(y^2 - 1)(x - y)(x - y - 1) = 0\} \subset \mathbb{C}^2,$$

with three lines L_1, L_2, L_3 such that L_1 passes $(1, 0)$, L_2 is parallel to $\{x = y\}$, L_3 passes $(0, 0)$ and

$$L_1 \cap L_2 \subset \{y = 1\}, \quad L_2 \cap L_3 \subset \{x = -1\}, \quad L_1 \cap L_3 \subset \{y = -1\}.$$

The parameter space of such arrangements is one-dimensional if L_3 is deleted. So we can apply the above argument to the three points $(0, 0)$, $L_1 \cap \{y = -1\}$, and $L_2 \cap \{x = -1\}$.

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Morihiko Saito msaito@kurims.kyoto-u.ac.jp
RIMS Kyoto University, Kyoto 606-8502, Japan

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