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Characterisation of quasi-Anosov diffeomorphisms

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Let f be a C^1 diffeomorphism of a compact C^∞ boundary-less manifold, and let $f^\#$ be the operator on the bounded or continuous sections of the tangent bundle (with supremum norm) defined by $f^\#\eta = Tf \circ \eta \circ f^{-1}$. The main result of this paper is that f is quasi-Anosov if and only if $1 - f^\#$ is injective and has closed range.

1. Introduction

Let M be a compact C^∞ manifold without boundary, and let $\text{diff}^1(M)$ denote the C^1 diffeomorphisms of M with the C^1 topology. Write TM for the tangent bundle of M , and for $f \in \text{diff}^1(M)$ let $Tf : TM \to TM$ denote the tangent map of f . Fix a riemannian metric on M , and let $\|\cdot\|$ be the associated Finsler norm on TM . Let $\Gamma^b(TM)$ $\left(\Gamma^0(TM)\right)$ denote the Banach space of bounded (continuous) sections of TM , with supremum norm. Define $f^\# : \Gamma^i(TM) \to \Gamma^i(TM)$, $i = b, 0$, by: for every $\eta \in \Gamma^i(TM)$ and every $x \in M$, $\left(f^\#\eta\right)(x) = Tf \circ \eta\left(f^{-1}x\right)$. $f^\#$ is a bounded linear operator (on either space).

In [11], Mather showed that f is Anosov if and only if $\left(1-f^\#\right) : \Gamma^0(TM) \to \Gamma^0(TM)$ is an isomorphism. It is also known that f

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satisfies Axiom A and the strong transversality condition if (Mañé, [9]) and only if (Robbin, [12]) $(1-f^{\#}) : \Gamma^0(TM) \to \Gamma^0(TM)$ is surjective. In this paper we prove that there is an analogous result for quasi-Anosov diffeomorphisms.

DEFINITION. We say that $f \in \text{diff}^1(M)$ is *quasi-Anosov* if and only if, for every $x \in M$, and every non-zero $v \in T_xM$, $\{\|Tf^n v\| : n \in Z\}$ is unbounded.

THEOREM. *Let* $f \in \text{diff}^1(M)$. *The following are equivalent:*

(a) f *is quasi-Anosov;*

(b) $(1-f^{\#}) : \Gamma^b(TM) \to \Gamma^b(TM)$ *is injective and has closed range;*

(c) $(1-f^{\#}) : \Gamma^0(TM) \to \Gamma^0(TM)$ *is injective and has closed range.*

This is the characterisation that should have appeared in [8]. There Mañé states that the following are equivalent:

(a') f *is quasi-Anosov;*

(b') $(1-f^{\#}) : \Gamma^b(TM) \to \Gamma^b(TM)$ *is injective;*

(c') $(1-f^{\#}) : \Gamma^0(TM) \to \Gamma^0(TM)$ *has closed range.*

In Section 5 below we give an example of a diffeomorphism f on T^2 which is not quasi-Anosov and for which $(1-f^{\#}) : \Gamma^b(TM) \to \Gamma^b(TM)$ is injective. If g is the north pole - south pole diffeomorphism of S^2 (see [12]), then g satisfies Axiom A and strong transversality, and by Robbin's result $(1-g^{\#}) : \Gamma^0(TM) \to \Gamma^0(TM)$ has closed range. But the north pole - south pole diffeomorphism is not quasi-Anosov.

The name quasi-Anosov was introduced by Mañé [7] in relation to a question of Hirsch regarding invariant hyperbolic submanifolds. Mañé has established another characterisation of quasi-Anosov diffeomorphisms in [10]. Franks and Robinson [5] have given an example of a quasi-Anosov diffeomorphism that is not Anosov (and resolves Hirsch's question). Relationships between f and $(1-f^{\#})$ have appeared in the work of several other authors; see [3], [4], [6], [13].

2. Proof that (a) implies (b)

Let f be quasi-Anosov. For every non-zero $v \in TM$ there exists $n(v) \in Z$ such that $\|Tf^{n(v)}v\| > 2$. By continuity, there is an open neighbourhood V of v in TM such that $\|Tf^{n(v)}w\| > 2$ for every $w \in V$. Let $S = \{v \in TM : \frac{1}{2} \le \|v\| \le 1\}$. By compactness of S there exists $N \in Z^+$ with the following property: for every $v \in S$ there exists $n \in Z$ with $|n| \le N$ such that

(1) $$\|Tf^n v\| > 2 .$$

Let $\eta \in \Gamma^b(TM)$ with $\|\eta\| = 1$, and let $\zeta = (1-f^{\#})\eta$. Then, for all $x \in M$,

$$\eta(fx) = Tf\eta(x) + \zeta(fx)$$

and

$$\eta(f^{-1}x) = Tf^{-1}\eta(x) - Tf^{-1}\zeta(x) .$$

Using these repeatedly gives that, for all $n \in Z\backslash\{0\}$,

(2) $$\eta(f^n x) = \begin{cases} Tf^n\eta(x) + \sum_{k=1}^{n} Tf^{k-1}\zeta(f^{n-k+1}x) & \text{if } n \ge 1 , \\ Tf^n\eta(x) - \sum_{k=1}^{-n} Tf^{-k}\zeta(f^{n+k}x) & \text{if } n \le -1 . \end{cases}$$

Let $\varepsilon > 0$ and such that

(3) $$\sup\{\|Tf^n v\| : |n| \le N, v \in TM, \|v\| < \varepsilon\} < 1/N .$$

Suppose $\|\zeta\| < \varepsilon$. Let x be a point of M such that $\|\eta(x)\| > \frac{1}{2}$. Then by (1), (2), and (3) there exists $n \in Z$, $|n| \le N$, such that

$$\|\eta(f^n x)\| > 2 - N.(1/N) = 1 ,$$

contradicting $\|\eta\| = 1$. Hence $\|(1-f^{\#})\eta\| \ge \varepsilon$. This means that $(1-f^{\#}) : \Gamma^b(TM) \to \Gamma^b(TM)$ is injective and has closed range (see [1] for example).

3. Proof that *(b)* implies *(c)*

If *(b)* is true, then there exists $\varepsilon > 0$ such that every $\eta \in \Gamma^b(TM)$ with $\|\eta\| = 1$ satisfies $\|(1-f^{\#})\eta\| \geq \varepsilon$ (see [1]). Since $\Gamma^0(TM) \subset \Gamma^b(TM)$, every $\eta \in \Gamma^0(TM)$ with $\|\eta\| = 1$ satisfies $\|(1-f^{\#})\eta\| \geq \varepsilon$, so *(c)* is true.

4. Proof that *(c)* implies *(a)*

Let f satisfy *(c)*. Then there exists $\varepsilon > 0$ such that every $\eta \in \Gamma^0(TM)$ with $\|\eta\| = 1$ satisfies $\|(1-f^{\#})\eta\| \geq \varepsilon$.

We first show that this implies that the nonperiodic points are dense in M. The argument is due to Mather [11] Let P_n denote the closed set of points of M of period n. We will show that, for each positive integer n, P_n has no interior point. It then follows that $\cup P_n$ is nowhere dense, by Baire's theorem.

Suppose for some n that P_n does contain an interior point. Let n be the least such integer. Then $\bigcup_{k<n} P_k$ is nowhere dense and $(\text{int } P_n) \setminus \bigcup_{k<n} P_k \neq \emptyset$. Let $x_0 \in (\text{int } P_n) \setminus \bigcup_{k<n} P_k$. Then there is a neighbourhood U of x_0 such that $U \subset P_n$ and $f^k(U) \cap U = \emptyset$ for $1 \leq k \leq n-1$. Let $\zeta_0 \in \Gamma^0(TM)$, have support in U, and satisfy $\zeta_0(x_0) \neq 0$. Let $\zeta = \sum_{k=0}^{n-1} f^{\#k}\zeta_0$. Then $\zeta(x_0) = \zeta_0(x_0)$, so $\zeta \neq 0$; and $f^{\#}\zeta = \zeta$. This contradicts the fact that $(1-f^{\#}) : \Gamma^0(TM) \to \Gamma^0(TM)$ is injective.

Now assume that f is not quasi-Anosov. We will show that this implies that there exists $\eta_2 \in \Gamma^b(TM)$ with finite support, $\|\eta_2\| = 1$, and $\|(1-f^{\#})\eta_2\| < \varepsilon/2$. Then we will smooth η_2 to obtain $\eta \in \Gamma^0(TM)$ with $\|\eta\| = 1$ and $\|(1-f^{\#})\eta\| < \varepsilon$, which is impossible.

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If f is not quasi-Anosov, there exists $x_1 \in M$ and non-zero $v \in T_{x_1}M$ such that $\{\|Tf^n v\| : n \in Z\}$ is bounded. We may suppose that $\sup\{\|Tf^n v\| : n \in Z\} = 1$. By replacing v by $Tf^k v$ if necessary, we may suppose that $\|v\| > \frac{1}{2}$.

Let ε_1 satisfy $0 < \varepsilon_1 < \min(1, \varepsilon/8)$. Let $n_1 \in Z^+$ be such that $(1-\varepsilon_1)^{n_1} < \varepsilon/8$. Choose a non-periodic point x so close to x_1 and $w \in T_x M$ so close to v that $\|w\| > \frac{1}{2}$ and $\|Tf^n w\| < 2$ for $|n| \le 1 + n_1$.

Define $\eta_1 \in \Gamma^b(TM)$ by

$$\eta_1(y) = \begin{cases} (1-\varepsilon_1)^{|n|} Tf^n w & \text{for } y = f^n x \text{ and } |n| \le n_1 , \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\eta_1\| \ge \|\eta_1(x)\| = \|w\| > \frac{1}{2}$. For $-n_1 + 1 \le n \le n_1$,

$$\begin{aligned} \left\|(1-f^{\#})\eta_1(f^n x)\right\| &= \left\|(1-\varepsilon_1)^{|n|} Tf^n w - (1-\varepsilon_1)^{|n-1|} Tf^n w\right\| \\ &\le \varepsilon_1 \|Tf^n w\| \\ &< \varepsilon/4 . \end{aligned}$$

Further,

$$\left\|(1-f^{\#})\eta_1(f^{-n_1}x)\right\| = \left\|(1-\varepsilon_1)^{n_1} Tf^{-n_1} w\right\| < \varepsilon/4 ,$$

$$\left\|(1-f^{\#})\eta_1(f^{n_1+1}x)\right\| = \left\|(1-\varepsilon_1)^{n_1} Tf^{n_1+1} w\right\| < \varepsilon/4 ;$$

and for all other points $y \in M$, $\left\|(1-f^{\#})\eta_1(y)\right\| = 0$. Hence

$$\left\|(1-f^{\#})\eta_1\right\| < \varepsilon/4 < \tfrac{1}{2}\varepsilon\|\eta_1\| .$$

Now let $\eta_2 = \eta_1/\|\eta_1\|$. Then η_2 has the required properties.

We now smooth η_2 to get $\eta \in \Gamma^0(TM)$. The riemannian metric determines a metric on M which we denote by d. Let

$$U_r = \{y \in M : d(x, y) < r\} .$$

Choose $r > 0$ so small that the following conditions are satisfied:

the sets $f^n(U_r)$, $|n| \le n_1 + 1$, are pairwise disjoint;

for each n with $|n| \le n_1 + 1$, $f^n(U_r)$ is contained in a normal neighbourhood of $f^n(x)$.

Let $p, q \in M$ and $v \in T_pM$, let $\tau_{pq}v \in T_qM$ denote the parallel translation of v along the geodesic joining p to q.

By making r smaller if necessary, we may ensure that the following condition is also satisfied:

for all n with $-n_1 \le n \le n_1+1$, all $y \in U_r$ and all $v \in T_{f^{n-1}x}M$,

$$\left\| (Tf)_{f^{n-1}x} v - \tau_{f^n y, f^n x} (Tf)_{f^n y} \tau_{f^{n-1}x, f^{n-1}y} v \right\| < \tfrac{1}{2}\varepsilon \|v\| .$$

Let $\phi : \mathrm{R} \to \mathrm{R}$ be C^∞ with $\phi(0) = 1$, $0 \le \phi(x) \le 1$ for all $x \in \mathrm{R}$, and with $\operatorname{supp}(\phi)$ contained in $(-r, r)$. Define $\eta \in \Gamma^0(TM)$ by

$$\eta(f^n y) = \phi(d(x, y)) \cdot \tau_{f^n x, f^n y} \eta_2(f^n x) \quad \text{if } y \in U_r \text{ and } |n| \le n_1 ,$$

$$\eta(y) = 0 \quad \text{if } y \notin \bigcup_{|n| \le n_1} f^n(U_r) .$$

Then $\|\eta\| = 1$. If $y \notin \bigcup_{n=-n_1}^{n_1+1} f^n(U_r)$, $\|(1-f^\#)\eta(y)\| = 0$. If $y \in U_r$ and $-n_1 \le n \le n_1+1$, then

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$$\begin{aligned}
&\|(1-f^{\#})\eta(f^n y)\| \\
&= \left\|\eta(f^n y)-(Tf)_{f^{n-1}y}\eta(f^{n-1}y)\right\| \\
&\le \left\|\eta(f^n y)-\tau_{f^n x, f^n y}(Tf)_{f^{n-1}x}\tau_{f^{n-1}y, f^{n-1}x}\eta(f^{n-1}y)\right\| \\
&\quad + \left\|\tau_{f^n x, f^n y}(Tf)_{f^{n-1}x}\tau_{f^{n-1}y, f^{n-1}x}\eta(f^{n-1}y)-(Tf)_{f^{n-1}y}\eta(f^{n-1}y)\right\| \\
&= \left\|\phi(d(x,y))\cdot\tau_{f^n x, f^n y}\eta_2(f^n x)-\tau_{f^n x, f^n y}(Tf)_{f^{n-1}x}\phi(d(x,y))\cdot\eta_2(f^{n-1}x)\right\| \\
&\quad + \Big\|\tau_{f^n x, f^n y}(Tf)_{f^{n-1}x}\phi(d(x,y))\cdot\eta_2(f^{n-1}x)-(Tf)_{f^{n-1}y}\phi(d(x,y)) \\
&\qquad\qquad \cdot\tau_{f^{n-1}x, f^{n-1}y}\eta_2(f^{n-1}x)\Big\| \\
&= |\phi(d(x,y))|\left\|\eta_2(f^n x)-(Tf)_{f^{n-1}x}\eta_2(f^{n-1}x)\right\| \\
&\quad + |\phi(d(x,y))|\Big\|(Tf)_{f^{n-1}x}\eta_2(f^{n-1}x) \\
&\qquad -\tau_{f^n y, f^n x}(Tf)_{f^{n-1}y}\tau_{f^{n-1}x, f^{n-1}y}\eta_2(f^{n-1}x)\Big\|
\end{aligned}$$

since parallel translation preserves the norm. Therefore

$$\|(1-f^{\#})\eta(f^n y)\| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon\left\|\eta_2(f^{n-1}x)\right\| \le \varepsilon .$$

So we have $\eta \in \Gamma^0(TM)$ with $\|\eta\| = 1$ and $\|(1-f^{\#})\eta\| < \varepsilon$, giving the contradiction.

5. Examples

Here we construct an example of a diffeomorphism f on the 2-torus $M = T^2$ which is not quasi-Anosov and for which $(1-f^{\#}) : \Gamma^i(TM) \to \Gamma^i(TM)$ is injective, $i = 0, b$.

Let $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$g_0 = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} ,$$

and let g be the induced diffeomorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let

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$\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the covering map, and let $p = \pi(0)$. We will perturb g in a neighbourhood of p to give a diffeomorphism f of T^2 with the following properties:

(i) $f(p) = p$;

(ii) $T_p f$ equals minus the identity;

(iii) for every $q \in T^2 \setminus \{p\}$ and for every non-zero $v \in T_q M$, $\{\|Tf^n v\| : n \in \mathbb{Z}\}$ is unbounded.

By (i) and (ii), for every $v \in T_p M$, $\{\|Tf^n v\| : n \in \mathbb{Z}\}$ is bounded; so f is not quasi-Anosov. Now suppose $\eta \in \Gamma^b(TM)$ and $(1-f^{\#})\eta = 0$. Then $0 = (1-f^{\#})\eta(p) = 2\eta(p)$ by (i) and (ii), so $\eta(p) = 0$; and for any $q \in M$, $0 = (1-f^{\#})\eta(q)$, which gives $\eta(f^n q) = Tf^n \eta(q)$ for any $n \in \mathbb{Z}$. Hence $\{\|Tf^n \eta(q)\| : n \in \mathbb{Z}\}$ is bounded, and therefore $\eta(q) = 0$ by (iii). Thus $(1-f^{\#})$ is injective on $\Gamma^b(TM)$, and so also on $\Gamma^0(TM)$.

We now set about constructing f. Let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of coordinate axes that takes the x-axis (y-axis) into the contracting (expanding) eigenspace of g_0. Call these new axes the x_1- and y_1-axes. Let g_1 represent g_0 with respect to these new coordinates, that is, $g_1 = \theta g_0 \theta^{-1}$. Then $g_1 = \operatorname{diag}(-\lambda^{-1}, -\lambda)$, where $\lambda > 1$. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{T}^2$ be the covering map with respect to the new coordinates, that is, $\pi_1 = \pi\theta^{-1}$. From now on we work with the new coordinates in $\mathbb{R}^2$.

Let ϕ_t be the flow of

(4) $$\frac{dx_1}{dt} = -\mu x_1 , \quad \frac{dy_1}{dt} = \mu y_1 ,$$

where $\mu = \log \lambda$. Then $g_1 = -\phi_1$. Let

$$B(0, k) = \left\{(x_1, y_1) : \left(x_1^2+y_1^2\right)^{\frac{1}{2}} \le k\right\} .$$

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choose $k_1 > 0$, $k_2 > 0$ such that the following condition is satisfied:

(A) for all t with $|t| \le 1$, $\phi_t B(0, k_1) \subset \text{int } B(0, \tfrac{1}{2})$, and $\phi_t B(0, k_2) \subset \text{int } B(0, k_1)$.

Let $0 < k_4 < k_3 < k_2$ and let $\rho : R \to R$ be C^∞ with these properties: $\rho(x) = 1$ for $x \le k_4$; $\rho(x) = 0$ for $x \ge k_3$; $\rho'(x) \le 0$ for all x . Let $r = \left(x_1^2+y_1^2\right)^{\frac{1}{2}}$. The differential equations

(5)
$$\frac{dx_1}{dt} = -\mu x_1\left(x_1^2+y_1^2\right)\rho(r) - \mu x_1\left(1-\rho(r)\right) ,$$
$$\frac{dy_1}{dt} = \mu y_1\left(x_1^2+y_1^2\right)\rho(r) + \mu y_1\left(1-\rho(r)\right) ,$$

define a flow ψ_t on R^2 . The flow lines for both (4) and (5) are the curves $x_1 y_1 = c$, c constant. Equations (4) and (5) are identical in $R^2 \backslash B(0, k_2)$. Because of condition (A), we have:

(B) in $R^2 \backslash B(0, k_1)$, $\psi_1 = \phi_1 \left(= -g_1\right)$ and $\psi_{-1} = \phi_{-1} \left(= -g_1^{-1}\right)$.

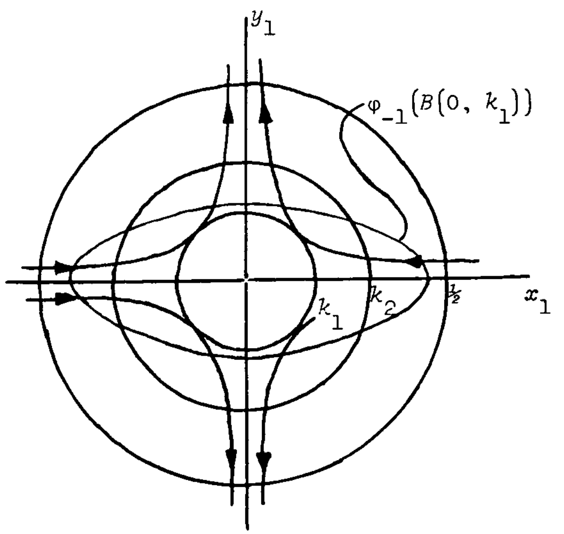

FIGURE 1

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Note also that (A) ensures the following property:

(C) suppose that $(x_1, y_1) \in B(0, k_1)$ and that $y_1 \neq 0$. Then there exists $n \in Z^+$ such that $\psi_1^{n-1}(x_1, y_1) \in B(0, k_1)$, and $\psi_1^n(x_1, y_1) \in B(0, \frac{1}{2}) \backslash B(0, k_1)$. By (B),

$\psi_1 \circ \psi_1^n(x_1, y_1) = -g_1 \circ \psi_1^n(x_1, y_1)$.

There is a similar statement to (C) for ψ_{-1}.

Define $f : T^2 \to T^2$ by

$$f(\pi_1(x_1, y_1)) = \pi_1 \circ (-\psi_1)(x_1, y_1) \quad \text{if } (x_1, y_1) \in B(0, \tfrac{1}{2}) ,$$
$$f(q) = g(q) \quad \text{if } q \notin \pi_1 \circ B(0, \tfrac{1}{2}) .$$

Condition (B) shows that f is a diffeomorphism. Condition (C) is needed to give the following property for f, which we will use later:

(D) let $q \in T^2$. Then either $f^n(q) \to p$ as $n \to \infty$, or there exists a sequence $\{n_k\} \to \infty$ such that $f \circ f^{n_k}(q) = g \circ f^{n_k}(q)$.

There is a similar statement to (D) for f^{-1}.

We must establish conditions (i)-(iii) for f. Condition (i) is immediate. For (ii), use the local coordinates, given by π_1. In these coordinates, T_pf is given by $-D\psi_1(0, 0)$. From (5), $D\psi_1(0, 0)$ is the identity.

It remains to establish (iii). The covering map π_1 can be used to give a chart at any point of $M = T^2$. All charts used henceforth will be the charts given by π_1. Let $q \in M\backslash\{p\}$, $w \in T_qM\backslash\{0\}$, and suppose w is represented by $[u, v]$ in terms of the chart. We will show that if $|v| \geq |u|$, then $\{\|(Tf)^n w\| : n \in Z^+\}$ is unbounded. A similar argument yields that if $|v| \leq |u|$, then $\{\|(Tf)^n w\| : n \in Z^-\}$ is unbounded.

In local coordinates, Tf is represented either by $-D\psi_1$ or by

Dg_1 . We examine $D\psi_1$ first. Let $F(x_1, y_1)$ be the right hand side of (5), and let $[u_0, v_0]$ be a vector in $\mathbb{R}^2$. The vector $D\psi_t(x_1, y_1)[u_0, v_0]$ satisfies the variational equation

$$(6) \qquad \frac{d}{dt} D\psi_t(x_1, y_1)[u_0, v_0] = DF(\psi_t(x_1, y_1))D\psi_t(x_1, y_1)[u_0, v_0] .$$

A short calculation yields

$$\begin{aligned}(-u, v)DF(x_1, y_1)[u, v] &= \mu\left[\left(3u^2+v^2\right)x_1^2+4uvx_1y_1+\left(u^2+3v^2\right)y_1^2\right]\rho(r) \\ &\quad + \mu\left(u^2+v^2\right)[1-\rho(r)] + \mu r^{-1}\rho'(r)\left(ux_1+vy_1\right)^2\left\{-1+x_1^2+y_1^2\right\} \\ &\geq 0 ,\end{aligned}$$

with equality only when $x_1 = y_1 = 0$ or $u = v = 0$. This means that, for any $t \in \mathbb{R}$ and any $(x_1, y_1) \in \mathbb{R}^2\backslash\{(0, 0)\}$, the vector field determined by (6) in the punctured (u, v)-plane $\left((0, 0) \text{ removed}\right)$ is nowhere tangent to the family of hyperbolas $v^2 - u^2 = c$, $c \in \mathbb{R}$, and is directed as shown in Figure 2.

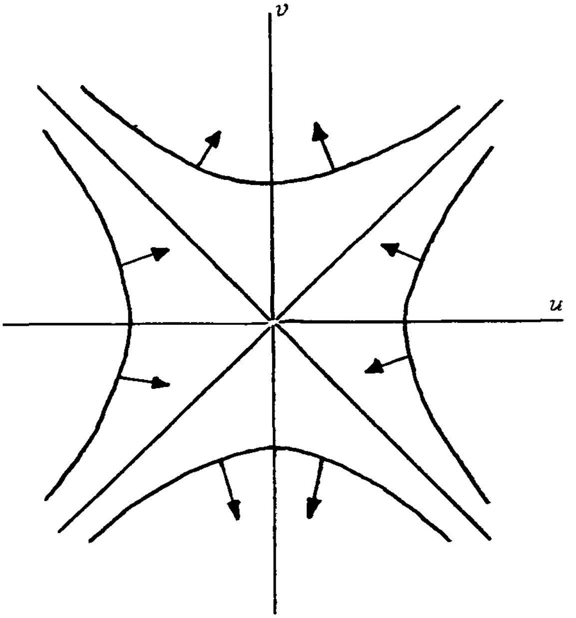

FIGURE 2

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This yields the following property of $D\psi_1$.

(E) Suppose $v_0^2 + u_0^2 > 0$ and $v_0^2 - u_0^2 = c_0^2 \geq 0$. Let $[-u_1, -v_1] = D\psi_1(x_1, y_1)[u_0, v_0]$, and let $v_1^2 - u_1^2 = (-v_1)^2 - (-u_1)^2 = c_1^2$. Then $c_1^2 > c_0^2$.

The action of Dg_1 is similar, and we get some additional information.

(F) Suppose $v_0^2 + u_0^2 > 0$, and $v_0^2 - u_0^2 = c_0^2 \geq 0$. Let $[u_1, v_1] = Dg_1[u_0, v_0] = \left[-\lambda^{-1}u_0, -\lambda v_0\right]$, and let $v_1^2 - u_1^2 = c_1^2$. Then $c_1^2 = \left(\lambda^2 - \lambda^{-2}\right)u_0^2 + \lambda^2 c_0^2$, so that $c_1^2 \geq \lambda^2 c_0^2$ (recall that $\lambda > 1$). Equality holds only when $u_0 = 0$; but then $c_0 > 0$, so always $c_1^2 > c_0^2$.

Let $q \in M \backslash \{p\}$, and let $w \in T_q M$ and be represented in the chart by $[u_0, v_0]$ with $|v_0| \geq |u_0|$; that is, with $v^2 - u_0^2 = c_0^2$. For $n \geq 0$, let $Tf^n w$ be represented in the chart by $[u_n, v_n]$, and let $v_n^2 - u_n^2 = c_n^2$. By (E), (F), the sequence $\left\{c_n^2\right\}_{n=0}^{\infty}$ is monotone increasing.

Suppose first that $f^n(q)$ does not tend to p as $n \to \infty$. By (D), there exists a sequence $\{n_k\} \to \infty$ such that $(Tf)_{f^{n_k}(q)} = (Tg)_{f^{n_k}(q)}$ for all k . Property (F) then gives that $c_n^2 \to \infty$ as $n \to \infty$. Hence $v_n^2 \to \infty$ as $n \to \infty$, so that $\left\{\|Tf^n w\| : n \in \mathbb{Z}^+\right\}$ is unbounded.

Now suppose that $f^n(q) \to p$ as $n \to \infty$. Then there exists $n_1 \geq 0$ such that $f^n(q)$ lies on $\pi_1\left\{(x_1, 0) : |x_1| \leq k_4\right\}$ whenever $n \geq n_1$. Let $\left(x_{n_1}, 0\right)$, $\left|x_{n_1}\right| \leq k_4$, be the point such that $\pi_1\left(x_{n_1}, 0\right) = f^{n_1}(q)$.

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Inside $B(0, k_4)$ equations (5) become

$$\frac{dx_1}{dt} = -\mu x_1^3 , \quad \frac{dy_1}{dt} = \mu y_1^3 .$$

Hence for $t \geq 0$, $\psi_t(x_{n_1}, 0) = \left(x_{n_1}\left(1+2\mu x_{n_1}^2 t\right)^{-\frac{1}{2}}, 0\right)$. The variational equation for the derivative $D\psi_t(x_{n_1}, 0)$ for $t \geq 0$ becomes

$$(7) \quad \frac{d}{dt} D\psi_t(x_{n_1}, 0) = \begin{pmatrix} -3\mu x_{n_1}^2\left(1+2\mu x_{n_1}^2 t\right)^{-1} & 0 \\ 0 & \mu x_{n_1}^2\left(1+2\mu x_{n_1}^2 t\right)^{-1} \end{pmatrix} D\psi_t(x_{n_1}, 0) .$$

Let $|u_t', v_t'| = D\psi_t(x_{n_1}, 0)\left[u_{n_1}, v_{n_1}\right]$. From (7), for all $t \geq 0$,

$$\log|v_t'| - \log|v_{n_1}| = \tfrac{1}{2} \log\left(1+2\mu x_{n_1}^2 t\right) \to \infty \text{ as } t \to \infty .$$

Therefore $|v_t'| \to \infty$ as $t \to \infty$. Finally $|v_n| = |v_n'|$ for all $n \geq n_1$, so in this case too, we have $\left\{\|Tf^n w\| : n \in Z^+\right\}$ unbounded.

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