On Absolute Lambert Sums

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Let

(1) $$\sum_{n=1}^{\infty} a_n$$

be a series for which the Abelian generator,

(2) $$f(s) = \sum_{n=1}^{\infty} a_n e^{-ns},$$

converges whenever $s > 0$. Then the same is true of the Lambertian generator,

(3) $$g(s) = \sum_{n=1}^{\infty} ns\, a_n e^{-ns} / (1 - e^{-ns}),$$

and *vice versa*.

Since the A-summability of (1), viz., the existence of a finite limit $f(+0)$, is equivalent to the convergence of the improper integral

$$\int_{+0} f'(s)\, ds, \qquad (f' = df/ds),$$

J. M. Whittaker [2] has defined the series (1) to be absolutely A-summable if the integral is absolutely convergent. This requirement, i.e., the condition

(4) $$\int_{+0} |\, df(s)\, | < \infty,$$

will be referred to as the $|A|$-summability of (1). Correspondingly, since the L-summability of (1), being defined by the requirement of a finite limit $g(+0)$ for the Lambertian generator (3), is equivalent to the convergence of the improper integral

$$\int_{+0} g'(s)\, ds,$$

let (1) be called $|L|$-summable if

(5) $$\int_{+0} |\, dg(s)\, | < \infty.$$

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If $M \to N$ means that every M-summable series is N-summable, it is clear that

$$|A| \to A \text{ and } |L| \to L.$$

But it turns out that the four summation methods can be ordered in a single chain of curious structure, as follows:

$$|L| \to |A| \to L \to A. \tag{6}$$

None of the three implications (6) is evident (in fact, two of them prove to lie deeper than the prime number theorem), and the chain (6) becomes false when either of the pairs $|L|$, L or $|A|$, A is replaced by the Cesàro pair $|C, k|$, (C, k) of any common order.

The third of the three implications (6) is due to Hardy and Littlewood [1]. The second was proved in [3]. The first will be verified below along the lines of the Hardy-Littlewood proof of the third.

Corresponding to the latter proof, the starting point will be the following elementary identity, derived by Hardy and Littlewood (*loc. cit.*) from an application of Möbius' inversion:

$$f(s) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{\mu(m)}{m} \int_{ns}^{(n+1)s} \frac{g(t)}{t} dt \qquad (s > 0). \tag{7}$$

(f, g are defined by the series (2), (3), which are supposed to converge for $s > 0$, and $\mu(m)$ is Möbius' factor.) What is then needed for the application of (7) is

$$\sum_{n=1}^{\infty} |\beta(n)| / n < \infty, \tag{8}$$

where

$$\beta(n) = \sum_{m=1}^{n} \mu(m) / m. \tag{9}$$

Remark. Hardy and Littlewood refer to the case $a = 2$ of the estimate

$$\beta(n) = O(1) / \log^{a} n \tag{10}$$

which, according to de la Vallée Poussin's refinement of the prime number theorem, is true for every a. But what is actually needed is

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precisely (8), which is less than (10) for any $a > 1$. The prime number theorem itself is less than (10) for any $a > 0$, since it is equivalent to

(11) $$\beta(n) = o(1);$$

and it is insufficient for (8), since (11) is. Incidentally, (8) *of itself*, *i.e.*, (8) *without* (9) *and*

(12) $$\sum_{n=1}^{\infty} \mu(n)/n^s = \prod_p (1 - p^{-s}), \qquad (\sigma > 1),$$

is insufficient for (11) (and even for

(13) $$\beta(n) = O(1),$$

although (13) is elementary).

In order to prove the first of the implications (6), suppose first merely that the series (2), (3) converge for $s > 0$. Then (7) is valid for $s > 0$. Furthermore, (7) can be differentiated term-by-term if $s > 0$ (this is clear from (13) and from the fact that, in view of (3), the function $g(x)$ tends exponentially to 0 as $x \to \infty$). This gives

$$f'(s) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \mu(m)/m\{[n+1]g([n+1]s)/([n+1]s) - ng(ns)/(ns)\},$$

where $s > 0$. Accordingly, if both terms of the difference { } are reduced and the notation (9) is inserted,

$$f'(s) = \sum_{n=1}^{\infty} \beta(n)\{g([n+1]s) - g(ns)\}/s.$$

Consequently, $$\int_{+0}^{\infty} \left| f'(s) \right| ds \leqq \sum_{n=1}^{\infty} \left| \beta(n)/n \right| h_n \leqq \infty,$$

where $$h_n = \int_0^{\infty} \left| g([n+1]s) - g(ns) \right| / (s/n)\, ds \leqq \infty.$$

Hence it is seen from (8) that (4) is true if $h_1, h_2, \ldots$ is a bounded sequence. Since the substitution $s \to s/n$ transforms the integral h_n into $h(1/n)$, where

$$h(\epsilon) = \int_0^{\infty} \left| g(s + \epsilon s) - g(s) \right| / (\epsilon s)\, ds,$$

it follows that (4) is true if $h(\epsilon)$ is bounded for $0 < \epsilon \leqq 1$. But $h(\epsilon)$ can be written in the form

$$h(\epsilon) = \int_0^{\infty} \left| \int_s^{s+\epsilon s} g'(t)\, dt \right| / (\epsilon s)\, ds,$$

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and is therefore majorised by

(14) $$q(\epsilon) = \int_0^\infty \int_s^{s+\epsilon s} p(t)\, dt \,/\, (\epsilon s)\, ds,$$

where

(15) $$p(t) = |\, g'(t)\, | \qquad (t > 0).$$

Hence (4) is true if

(16) $$q(\epsilon) < \text{Const.} \qquad \text{for } 0 < \epsilon \leqq 1.$$

It follows that the proof of the first of the implications (6) will be complete if it is verified that (16) is fulfilled whenever the (measurable) function $p(t)$ occurring in (14) satisfies

(17) $$\int_0^\infty p(t)\, dt < \infty, \text{ where } p \geqq 0.$$

For, on the one hand, the $|\, A\, |$ -summability of (1) is defined by (4) and its $|\, L\, |$ -summability by (5), and, on the other hand, (5) is equivalent to (17). In fact, the truth of the latter equivalence follows by observing that, in view of (15), the assumption (17) can be written in the form

(18) $$\int_{+0}^\infty \left| g'(t) \right| dt < \infty,$$

and that (18) is equivalent to the local condition (5) (simply because the derivative of (3) tends to 0 exponentially as $s \to \infty$).

Accordingly, all that remains to be ascertained is that the assumption (17) and the definition (14) imply the existence of a constant satisfying (16).

Clearly

$$\int_s^{s+\epsilon s} p(t)\, dt = s \int_1^{1+\epsilon} p(st)\, dt.$$

If this is inserted in (14), an application of Fubini's theorem gives

$$\epsilon q(\epsilon) = \int_1^{1+\epsilon} \left(\int_0^\infty p(st)\, dt \right) ds$$

(whether the values of the integrals, in which everything is non-

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negative, are finite or not). But the interior integral occurring in the last formula is identical with C/s, if C denotes the value of the integral (17). Consequently

$$\epsilon q(\epsilon) = \int_1^{1+\epsilon} (C/s)\, ds = C \log (1 + \epsilon).$$

This implies (16), since $\log (1 + \epsilon) \sim \epsilon$ as $\epsilon \to 0$.

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