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# A REMARK ON COMPLETELY REDUCIBLE NEAR-RINGS

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A characterization of the completely reducible (zerosymmetric right) near-rings which are the direct sum of their socle of type 2 and of their 2-radical is given.

## 1. Completely reducible near-rings

Throughout this note $N$ indicates a zerosymmetric right near-ring; terminology and notation are those of [5]. In particular a non-zero left ideal $L$ of $N$ will be called *minimal* if it does not contain proper left ideals of $N$; *simple* if the $N$-group $L$ has no proper ideals; *of type* 2 if the $N$-group $L$ is monogenic and has no proper $N$-subgroups.

It is well known that if the near-ring $N$ is *completely reducible* - that is, it is the sum of its minimal left ideals - then every minimal left ideal of $N$ is a simple one, so that $N$ is actually the direct sum of (a few of) its simple left ideals.

Assume that $N$ has at least one left ideal of type 2; then the *socle of type* 2 - that is the sum $\zeta_2(N)$ of all left ideals of type 2 of $N$ - is a non-zero left ideal of $N$ and therefore is a direct summand of $N$ (see [1], Lemma 1.3). Moreover the following result holds.

**PROPOSITION 1.** *A completely reducible near-ring $N$ is the direct sum of its socle of type 2 and of a left ideal which is contained in the*

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*annihilator of the N-group* $\zeta_2(N)$ .

**Proof.** The near-ring $N$ can be written as the direct sum of $\zeta_2(N)$ and of a left ideal which is the direct sum of simple left ideals $L_h$ $(h \in H)$ of $N$ which are not of type 2 : $N = \zeta_2(N) \oplus \{\bigoplus_{h \in H} L_h\}$ .

Now, for every $h \in H$ , $L_h$ is contained in the annihilator $(0 : \zeta_2(N))$ of $\zeta_2(N)$ .

Indeed let $L$ be a simple left ideal and $x$ an element of $\zeta_2(N)$ such that the $N$-subgroup $Lx$ is different from zero; then $Lx$ is simple (being $N$-isomorphic to $L$ ). On the other side, $Lx$ is contained in $\zeta_2(N)$ , since $\zeta_2(N)$ is a left ideal. Thus $Lx$ is $N$-isomorphic to a left ideal of type 2 (see [1], Proposition 2.2) and $L$ too must be of type 2 . □

Looking next to *the annihilator* $(0 : \zeta_2(N))$ , it can be remarked that it *is the intersection of the annihilators of the left ideals* $L$ *of type* 2 *of* $N$ ; in fact the inclusion $(0 : \zeta_2(N)) \subseteq \bigcap(0 : L)$ holds trivially, and the other is a consequence of $\zeta_2(N)$ being the direct sum of (a few of) the left ideals of type 2 and of the distributive property of direct sums.

Hence $(0 : \zeta_2(N))$ *contains* the intersection of the annihilators of all the $N$-groups of type 2 , that is *the* 2*-radical* $J_2(N)$ *of* $N$ .

These assertions (which are true for every zerosymmetric near-ring) can be strengthened when $N$ is completely reducible, because every group of type 2 over such a near-ring is $N$-isomorphic to a left ideal of type 2 of $N$ (see [5], Corollary 3.11). Therefore

**PROPOSITION 2.** *If* $N$ *is a completely reducible near-ring, the annihilator of* $\zeta_2(N)$ *is the* 2*-radical of* $N$ .

Then Proposition 1 can be rewritten as follows

**PROPOSITION 3.** *A completely reducible near-ring* $N$ *is the direct sum of its socle of type* 2 *and of a left ideal which is contained in the*

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*2-radical* $J_2(N)$ *of* $N$ .

## 2. $\zeta_2$-decomposable near-rings

The last result leads to study the intersection between $J_2(N)$ and $\zeta_2(N)$ , in order to establish conditions under which $N$ is the direct sum of $\zeta_2(N)$ and $J_2(N)$ . It seems useful to examine the question in a slightly more general context; call $\zeta_2$-*decomposable* a near-ring $N$ which has its socle of type 2 as a direct summand, and recall that $J_2(N)$ is contained in $(0 : \zeta_2(N))$ .

Generally - as shown by the Counterexample 4 of Section 3 - if $N$ is $\zeta_2$-decomposable but not completely reducible, $J_2(N)$ does not coincide with $(0 : \zeta_2(N))$ ; however they coincide restrictedly to $\zeta_2(N)$ . In order to see this, let us prove

**LEMMA 4.** *Let* $N$ *be a* $\zeta_2$-*decomposable near-ring. For each left ideal* $C$ *of* $N$ *such that* $J_2(N) \subseteq C \subseteq (0 : \zeta_2(N))$ , *the intersection* $\zeta_2(N) \cap C$ *is the direct sum of all the left ideals of* $N$ *which are nilpotent and of type* 2 .

Proof. Let $\zeta_2(N) \cap C = M$ . All the nilpotent left ideals of $N$ are contained in $J_2(N)$ and therefore in $C$ ; among them, those of type 2 are necessarily contained in $\zeta_2(N)$ ; hence all the left ideals of $N$ which are nilpotent and of type 2 are contained in $M$ . So it will be enough to show that there are nilpotent left ideals of type 2 of $N$ whose sum is the whole $M$ .

Now, the $N$-group $M$ is the sum of its simple ideals $L_i$ since it is an ideal of the completely reducible $N$-group $\zeta_2(N)$ (see [5], Proposition 2.48). For each $L_i$ it results $L_i^2 = (0)$ because

$$L_i \subseteq M \subseteq C \subseteq (0 : \zeta_2(N)) \subseteq (0 : L_i) .$$

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Furthermore, each $L_i$ is a left ideal of $N$ (since $M$ is a direct summand of the $N$-group $\zeta_2(N)$ and therefore of $N$) and is of type 2, as it is a simple left ideal of $N$ contained in $\zeta_2(N)$ (see [1], Proposition 2.2). □

Lemma 4 can be read as follows: *in a $\zeta_2$-decomposable near-ring $N$ the left ideal $M = \zeta_2(N) \cap (0 : \zeta_2(N))$ coincides with $\zeta_2(N) \cap J_2(N)$. Besides, $M$ is nilpotent of class 2 and is zero if and only if $N$ has no nilpotent left ideal of type 2.*

From this statement the announced characterization follows.

PROPOSITION 5. *A completely reducible near-ring $N$ is the direct sum of $\zeta_2(N)$ and $J_2(N)$ if and only if $N$ has no nilpotent left ideal of type 2.*

## 3. Examples

There are several classes of near-rings satisfying the condition expressed by Proposition 5.

EXAMPLE 1. Let $N$ be a completely reducible near-ring with right identity; then it is easily seen that $N$ is the direct sum of a finite number of simple left ideals, so that the intersection of all the maximal left ideals of $N$ is zero (see [5], Theorem 2.50). Such an intersection coincides with $J_{\frac{1}{2}}(N)$, for in a near-ring with right identity the maximal left ideals coincide with the $0$-modular ones. On the other side $J_{\frac{1}{2}}(N)$ contains every left nil ideal of $N$ (see [5], Theorem 5.37) and therefore $N$ has no nilpotent left ideal of type 2. This proves

PROPOSITION 6. *A completely reducible near-ring $N$ with right identity is the direct sum of $\zeta_2(N)$ and $J_2(N)$.*

EXAMPLE 2. Let $N$ be a distributive near-ring completely reducible as a left $N$-group. Then $J_2(N) = J_1(N) = J_0(N)$ is a nilpotent ideal which is the direct sum of the annihilator $A(N)$ of $N$ and of the sum of the nilpotent left ideals of type 2 of $N$ (see [2], Theorem 6.1). Therefore

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**PROPOSITION 7.** *A completely reducible distributive near-ring* $N$ *is the direct sum of* $\zeta_2(N)$ *and* $J_2(N)$ *if (and only if)* $J_2(N)$ *is the annihilator of* $N$ .

Observe that if the completely reducible near-ring $N$ is distributive, then $J_2(N)$ is nilpotent. This is also true of the 2-radical of a near-ring sum of its left ideals which are $N$-simple as $N$-groups (see [4], Theorem 2), but if $N$ is a general completely reducible near-ring, $J_2(N)$ may be non nilpotent. This can be seen in the near-rings of Example 1 or in the following

**EXAMPLE 3.** Let $N$ be the (external) direct sum of a field $F$ and of the near-ring $L$ built over the symmetric group $S_3$ denoted by (1) in [5], p. 410.

Since the near-ring $L$ has no proper left ideal, the only left ideals of $N = F \oplus L$ are $F$ (which is the socle of type 2 of $N$ ) and $L$ , which is the 2-radical of $N$ and non nilpotent because it contains idempotent elements.

Finally we show that there exist ($\zeta_2$-decomposable) near-rings $N$ such that $J_2(N)$ is properly contained in $(0 : \zeta_2(N))$ .

**EXAMPLE 4.** Consider the dihedral group of order 12 , $D_{12} = \{a, b \mid 6a = 2b = 0, a+b = b+5a\}$ and let $N$ be the distributive and commutative near-ring built over $D_{12}$ denoted by $N_4$ in [4]. The only ideal of type 2 of $N$ is $A = \{0, 3a\}$ , so that $\zeta_2(N) = A$ and $(0 : \zeta_2(N)) = \{0, 2a, 4a, b, b+2a, b+4a\}$ . Moreover $N = \zeta_2(N) \oplus (0 : \zeta_2(N))$ ; hence $N$ is $\zeta_2$-decomposable.

However $J_2(N) = \{0, 2a, 4a\}$ .

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