2
Second quantization

In the previous chapter we considered first quantization of particles, where the operators of coordinate and momentum, $\boldsymbol{x}$ and $\boldsymbol{p}$ respectively, are represented in the coordinate space by

$$\boldsymbol{x} = x\,, \qquad \boldsymbol{p} = -\mathrm{i}\frac{\partial}{\partial x} \qquad \boxed{= \text{ first quantization}}\,. \tag{2.1}$$

In the language of path integrals, first quantization is associated with integrals over trajectories in the coordinate space.

While propagators can be easily represented as path integrals, it is very difficult to describe, in this language, a (nongeometric) self-interaction of a particle, since this would correspond to extra weights for self-intersecting paths. For the free case (or a particle in an external gauge field) there are no such extra weights, and the transition amplitude is completely described by the classical action of the particle.

In the operator formalism, self-interactions of a particle are described using second quantization – this is where the transition from quantum mechanics to quantum field theory begins. Second quantization is a quantization of fields, and is associated with path integrals over fields, which is the subject of this chapter. We demonstrate how perturbation theory and the Schwinger–Dyson equations can be derived using path integrals.

2.1 Integration over fields

Let us define the following (Euclidean) *partition function*:

$$Z = \int \mathcal{D}\varphi(x)\,\mathrm{e}^{-S}, \tag{2.2}$$

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where the action in the exponent is given for the free case by

$$S_{\rm free}[\varphi] \;=\; \frac{1}{2}\int {\rm d}^d x \left((\partial_\mu \varphi)^2 + m^2\varphi^2\right). \tag{2.3}$$

The measure $\mathcal{D}\varphi(x)$ is defined analogously to Eq. (1.66):

$$\int \mathcal{D}\varphi(x)\cdots \;=\; \prod_x \int_{-\infty}^{+\infty} {\rm d}\varphi(x)\cdots, \tag{2.4}$$

where the product runs over all space points x and the integral over ${\rm d}\varphi$ is the Lebesgue one.

The propagator is given by the average

$$G(x,y) \;=\; \langle\,\varphi(x)\,\varphi(y)\,\rangle\,, \tag{2.5}$$

where a generic average is defined by the formula

$$\langle\, F[\varphi]\,\rangle \;=\; Z^{-1}\int \mathcal{D}\varphi(x)\,{\rm e}^{-S[\varphi]}F[\varphi]\,. \tag{2.6}$$

The notation is obvious since on the RHS of Eq. (2.6) we average over all field configurations with the same weight as in the partition function (2.2). The normalization factor of Z^{-1} provides the necessary property of an average

$$\langle\,1\,\rangle \;=\; 1\,. \tag{2.7}$$

Since the free action (2.3) is Gaussian, the average (2.5) equals

$$G(x-y) \;=\; \left\langle y\left|\frac{1}{-\partial^2+m^2}\right|x\right\rangle \tag{2.8}$$

which is identical to (1.61). Therefore, we have obtained the same propagator (1.89) as in the previous chapter.

Problem 2.1 By discretizing the (Euclidean) space, derive

$$Z^{-1}\int\prod_x {\rm d}\varphi_x \exp\left(-\tfrac{1}{2}\sum_{x,y}\varphi_x D_{xy}\varphi_y\right)\varphi_x\varphi_y \;=\; D^{-1}_{xy}\,. \tag{2.9}$$

Solution The Gaussian integral can be calculated using the change of variable

$$\varphi_x \;\to\; \varphi'_x \;=\; \sum_y \left(D^{-1/2}\right)_{xy}\varphi_y \tag{2.10}$$

which results in Eq. (2.9).

Note that the integrals over $\varphi(x)$ are convergent in Euclidean space. If a discretization of space is introduced, the path integrals in Eqs. (2.2) or (2.6) are defined rigorously.

Remark on Minkowski-space formulation

In Minkowski space, perturbation theory is well-defined since the Gaussian path integral, which determines the propagator

$$\int \mathcal{D}\varphi \; e^{iS} \;\; = \;\; \int \mathcal{D}\varphi \; e^{i\int d^dx\, \varphi \boldsymbol{D} \varphi}, \tag{2.11}$$

equals

$$\langle\, \varphi_x \varphi_y \,\rangle \;\; = \;\; \left\langle y \left| \frac{1}{i\boldsymbol{D}} \right| x \right\rangle, \tag{2.12}$$

where $\boldsymbol{D}$ is a proper operator in Minkowski space.

It cannot be said *a priori* whether a nonperturbative formulation of a given (interacting) theory via the path integral in Minkowski space exists since the weight factor is complex and the integral may be divergent.

2.2 Grassmann variables

Path integrals over anticommuting Grassmann variables are used to describe fermionic systems.

The Grassmann variables ψ_x and $\bar{\psi}_y$ obey the anticommutation relations

$$\{\psi_y, \psi_x\} \;=\; 0\,, \qquad \{\bar{\psi}_y, \bar{\psi}_x\} \;=\; 0\,, \qquad \{\bar{\psi}_y, \psi_x\} \;=\; 0\,. \tag{2.13}$$

Consequently, the square of a Grassmann variable vanishes

$$\psi_x^2 \;=\; 0 \;=\; \bar{\psi}_x^2\,. \tag{2.14}$$

The path integral over the Fermi fields equals

$$\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \; e^{-\int d^dx\, \bar{\psi} \boldsymbol{D} \psi} \;\; = \;\; \det \boldsymbol{D} \tag{2.15}$$

while an analogous integral over the Bose fields is

$$\int \mathcal{D}\varphi^\dagger\, \mathcal{D}\varphi \, e^{-\int d^dx\, \varphi^\dagger \boldsymbol{D} \varphi} \;\; = \;\; (\det \boldsymbol{D})^{-1}\,. \tag{2.16}$$

Problem 2.2 Define integrals over Grassmann variables.

Solution Assuming that ψ and $\bar{\psi}$ belong to the same Grassmann algebra, the Berezin integrals are defined by

$$\left.\begin{aligned} \int d\psi_x \;&=\; 0 \;=\; \int d\bar{\psi}_x\,, \\ \int d\psi_x\, \psi_y \;&=\; \delta_{xy} \;=\; \int d\bar{\psi}_x\, \bar{\psi}_y\,, \\ \int d\psi_x\, \bar{\psi}_y \;&=\; 0 \;=\; \int d\bar{\psi}_x\, \psi_y\,. \end{aligned}\right\} \tag{2.17}$$

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The simplest interesting integral is

$$\int \mathrm{d}\bar{\psi}_x \, \mathrm{d}\psi_x \, \mathrm{e}^{-\bar{\psi}_x \psi_x} \quad = \quad 1 \,. \tag{2.18}$$

Equation (2.15) can now be easily derived by representing $\langle y \,|\boldsymbol{D}|\, x\rangle$ in the diagonal form, expanding the exponential up to a term which is linear in all the Grassmann variables and calculating the integrals of this term according to Eq. (2.17). See more details in the book by Berezin [Ber86] (§3 of Part I).

The average over the Fermi fields, defined with the same weight as in Eq. (2.15), equals

$$\left\langle \, \psi(x) \, \bar{\psi}(y) \, \right\rangle \quad = \quad \left\langle y \left| \boldsymbol{D}^{-1} \right| x \right\rangle \tag{2.19}$$

which is the same as for bosons.

Note that the fermion partition function (2.15) can be rewritten according to Eq. (1.166) as

$$\det \boldsymbol{D} \quad = \quad \mathrm{e}^{\mathrm{Tr} \ln \boldsymbol{D}}. \tag{2.20}$$

The analogous formula for bosons (2.16) is rewritten as

$$(\det \boldsymbol{D})^{-1} \quad = \quad \mathrm{e}^{-\mathrm{Tr} \ln \boldsymbol{D}}. \tag{2.21}$$

The relative difference of sign between the exponents on the RHS of Eqs. (2.20) and (2.21) is a famous minus sign that emerges for closed fermionic loops which contribute to the (logarithm of the) partition function.

2.3 Perturbation theory

The cubic self-interaction of the scalar field is described by the action

$$S[\varphi] \quad = \quad \int \mathrm{d}^d x \left(\frac{1}{2} \left(\partial_\mu \varphi \right)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{3!} \varphi^3 \right), \tag{2.22}$$

where λ is the coupling constant.

To construct perturbation theory, we expand the exponential in λ and calculate the Gaussian averages with the free action (2.3).

To order λ^2, the expansion is

$$\begin{aligned}
\left\langle \, \varphi(x) \, \varphi(y) \, \right\rangle \quad = \quad & \left\langle \, \varphi(x) \, \varphi(y) \, \right\rangle_{\text{free}} \\
& \times \left(1 - \left\langle \frac{\lambda}{3!} \int \mathrm{d}^d x_1 \, \varphi^3(x_1) \, \frac{\lambda}{3!} \int \mathrm{d}^d x_2 \, \varphi^3(x_2) \right\rangle_{\text{free}} \right) \\
& + \left\langle \varphi(x) \, \frac{\lambda}{3!} \int \mathrm{d}^d x_1 \, \varphi^3(x_1) \, \frac{\lambda}{3!} \int \mathrm{d}^d x_2 \, \varphi^3(x_2) \, \varphi(y) \right\rangle_{\text{free}} \\
& + \cdots .
\end{aligned} \tag{2.23}$$

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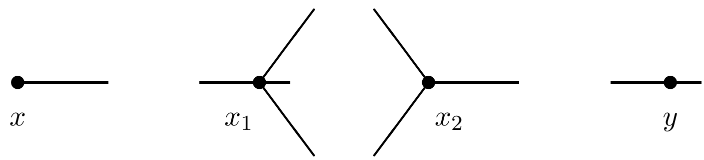

Fig. 2.1. Diagrammatic representation of the term in the third line of Eq. (2.23).

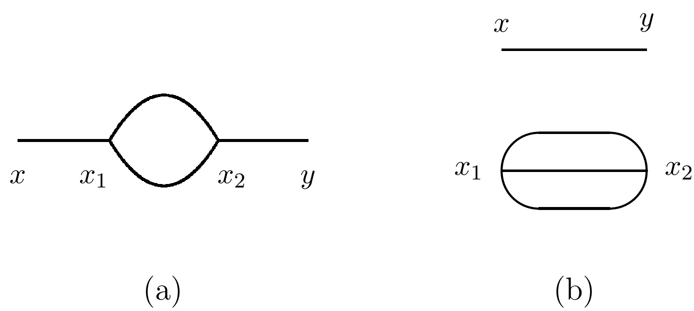

Fig. 2.2. Some of the Feynman diagrams which appear from (2.23) after the Wick pairing.

The term which is linear in λ (as well as the Gaussian average of any odd power of φ) vanishes owing to the reflection symmetry $\varphi \to -\varphi$ of the Gaussian action. The term displayed in the third line of Eq. (2.23) is depicted graphically in Fig. 2.1.

Further calculation of the RHS of Eq. (2.23) is based on the free average

$$\langle\, \varphi(x_i)\, \varphi(x_j)\, \rangle_{\text{free}} \quad = \quad G(x_i - x_j) \tag{2.24}$$

and the rules of Wick pairing of Gaussian averages, which allow us to represent the average of a product as the sum of all possible products of pair averages. Some of the diagrams which emerge after the Wick contraction are depicted in Fig. 2.2. These diagrams are called *Feynman diagrams*.

The diagram shown in Fig. 2.2b is disconnected. Its disconnected part with two loops cancels with the same contribution from Z^{-1} (which yields the factor in the first term on the RHS of Eq. (2.23)). It is a general property that only connected diagrams are left in $\langle\, \varphi(x_i)\, \varphi(x_j) \rangle$.

Let us note finally that the combinatoric factor of $1/2$ is reproduced correctly for the diagram of Fig. 2.2a. For an arbitrary diagram, this procedure of pairing reproduces the usual combinatoric factor, which is equal to the number of automorphisms of the diagram (i.e. the symmetries of a given graph).

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2.4 Schwinger–Dyson equations

Feynman diagrams can be derived alternatively by iterating the coupling constant of the set of Schwinger–Dyson equations which is a quantum analog of the classical equation of motion.

To derive the Schwinger–Dyson equations, let us utilize the fact that the measure (2.4) is invariant under an arbitrary shift of the field

$$\varphi(x) \quad \rightarrow \quad \varphi(x) + \delta\varphi(x)\,. \tag{2.25}$$

This invariance is obvious since the functional integration goes over all the fields, while the shift (2.25) is just a transformation from one field configuration to another.

Since the measure is invariant, the path integral in the average (2.6) does not change under the shift (2.25):

$$\int \mathrm{d}^d x\, \delta\varphi(x) \int \mathcal{D}\varphi\ \mathrm{e}^{-S[\varphi]} \left[-\frac{\delta S[\varphi]}{\delta\varphi(x)} F[\varphi] + \frac{\delta F[\varphi]}{\delta\varphi(x)} \right] \;=\; 0. \tag{2.26}$$

Since $\delta\varphi(x)$ is arbitrary, Eq. (2.26) results in the following *quantum equation of motion*

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} \quad \stackrel{\text{w.s.}}{=} \quad \hbar \frac{\delta}{\delta\varphi(x)}, \tag{2.27}$$

where we have written explicitly the dependence on Planck's constant $\hbar$. It appears this way since the action S is divided by $\hbar$ in Eq. (2.2) when $\hbar$ is restored.

We have put the symbol "w.s." on the top of the equality sign in Eq. (2.27) to emphasize that it is to be understood in the weak sense, i.e. it is valid under averaging when applied to a functional $F[\varphi]$. In other words, the variation of the action on the LHS of Eq. (2.27) can always be substituted by the variational derivative on the RHS when integrated over fields with the same weight as in Eq. (2.6). Therefore, one arrives at the following functional equation:

$$\left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} F[\varphi] \right\rangle \;=\; \hbar \left\langle \frac{\delta F[\varphi]}{\delta\varphi(x)} \right\rangle. \tag{2.28}$$

This equation is quite similar to that which Schwinger considered within the framework of his variational technique.

2.5 Commutator terms

In order to show how Eq. (2.28) reproduces Eq. (1.34) for the free propagator, let us choose

$$F[\varphi] \;=\; \varphi(y)\,. \tag{2.29}$$

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Substituting into Eq. (2.28) and calculating the variational derivative, one obtains

$$\left(-\partial^2 + m^2\right) \langle \varphi(x)\, \varphi(y) \rangle \;=\; \hbar \left\langle \frac{\delta \varphi(y)}{\delta \varphi(x)} \right\rangle \;=\; \hbar\, \delta^{(d)}(x-y)\,, \tag{2.30}$$

which coincides with Eq. (1.34).

The LHS of Eq. (2.30) emerges from the variation of the free classical action (2.3)

$$\frac{\delta S_{\text{free}}}{\delta \varphi(x)} \;=\; \left(-\partial^2 + m^2\right) \varphi(x) \tag{2.31}$$

while the RHS, which results from the variational derivative, emerges in the operator formalism from the canonical commutation relations

$$\delta(x_0 - y_0)\,[\boldsymbol{\varphi}(x_0, \vec{x})\,, \dot{\boldsymbol{\varphi}}(y_0, \vec{y})] \;=\; \mathrm{i}\, \delta^{(d)}(x-y) \tag{2.32}$$

as is explained in Sect. 1.1.

For this reason, the RHS of Eq. (2.30) and, more generally, the RHS of Eq. (2.28) are called *commutator terms*. The variational derivative on the RHS of Eq. (2.27) plays the role of the conjugate momentum in the operator formalism. The calculation of this variational derivative in Euclidean space is equivalent to differentiating the T-product and using canonical commutation relations in Minkowski space.

When Planck's constant vanishes, $\hbar \to 0$, the RHS of Eq. (2.27) (or Eq. (2.28)) vanishes. Therefore it reduces to the classical equation of motion for the field φ:

$$\frac{\delta S[\varphi]}{\delta \varphi(x)} \;=\; 0\,. \tag{2.33}$$

This implies that the path integral over fields has a saddle point as $\hbar \to 0$ which is given by Eq. (2.33).

Another lesson we have learned is that the average (2.5), which is defined via the Euclidean path integral, is associated with the Wick-rotated T-product. We have already seen this property in the previous chapter in the language of first quantization. More generally, the Euclidean average (2.6) is associated with the vacuum expectation value of $\langle 0 \,|\, \boldsymbol{T} F[\boldsymbol{\varphi}] \,|\, 0 \rangle$ in Minkowski space.

2.6 Schwinger–Dyson equations (continued)

The set of the Schwinger–Dyson equations for an interacting theory can be derived analogously to the free case.

Let us consider the cubic interaction which is described by the action (2.22). Choosing again $F[\varphi]$ to be given by Eq. (2.29) and calculating the variation of the action (2.22), one obtains

$$\left(-\partial^2 + m^2\right) \langle\, \varphi(x)\, \varphi(y)\, \rangle + \frac{\lambda}{2} \left\langle\, \varphi^2(x)\, \varphi(y) \right\rangle \;=\; \delta^{(d)}(x-y)\,. \qquad (2.34)$$

Problem 2.3 Rederive Eq. (2.34) by analyzing Feynman diagrams.

Solution Let us introduce the Fourier-transformed two- and three-point Green functions

$$G(p) \;=\; \int \mathrm{d}^d x\, \mathrm{e}^{-\mathrm{i}px} \langle\, \varphi(x)\, \varphi(0)\, \rangle\,, \qquad (2.35)$$

and

$$G_3(p,q,-p-q) \;=\; \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{e}^{-\mathrm{i}px - \mathrm{i}qy} \langle\, \varphi(x)\, \varphi(y)\, \varphi(0)\, \rangle\,. \qquad (2.36)$$

Let us also denote the free momentum-space propagator as

$$G_0(p) \;=\; \frac{1}{p^2+m^2}\,. \qquad (2.37)$$

The perturbative expansion of G_3 starts from

$$G_3(p,q,-p-q) \;=\; -\lambda\, G_0(p)\, G_0(q)\, G_0(p+q) + \cdots\,. \qquad (2.38)$$

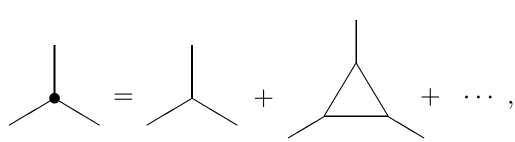
It is standard to truncate three external legs, introducing the vertex function

$$\Gamma(p,q,-p-q) \;=\; G_3(p,q,-p-q)\, G^{-1}(p)\, G^{-1}(q)\, G^{-1}(p+q) \qquad (2.39)$$

with a perturbative expansion which starts from λ:

$$\Gamma(p,q,-p-q) \;=\; -\lambda - \lambda^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} G_0(k-p)\, G_0(k)\, G_0(k+q) + \cdots\,. \qquad (2.40)$$

This expansion can be represented diagrammatically as



(2.41)

where the filled circle on the LHS represents the exact vertex and the thin lines are associated with the bare propagator (2.37).

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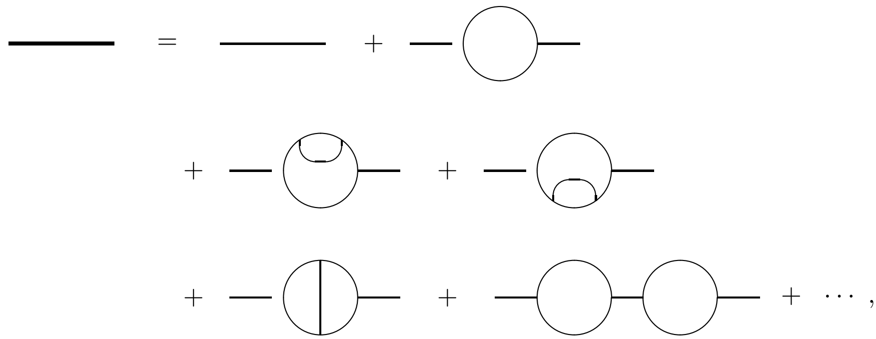
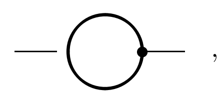
An analogous expansion of the propagator is



(2.42)

where the bold line represents the exact propagator. It is commonly rewritten as an equation for the self-energy $G_0^{-1}(p) - G^{-1}(p)$, which involves only the one-particle irreducible (1PI) diagrams. The last diagram shown on the RHS of Eq. (2.42) is not 1PI.

Resumming the diagrams according to definition (2.41) of the exact vertex Γ and the exact propagator G, the propagator equation can be represented graphically as

$$G_0(p) - G_0(p)\, G^{-1}(p)\, G_0(p) \;=$$  , (2.43)

where the bold lines represent the exact propagator G, while the external (thin) ones are associated with the bare propagator G_0. One vertex on the RHS of Eq. (2.43) is exact and the other is bare. It does not matter which one is exact and which one is bare since we can collect the diagrams of Eq. (2.42) into the exact vertex either on the LHS or on the RHS. Equation (2.43) can be written analytically as

$$G_0^{-1}(p) - G^{-1}(p) \;=\; -\frac{\lambda}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d} G(q)\,\Gamma(-q,p,q-p)\,G(p-q)\,. \tag{2.44}$$

Multiplying Eq. (2.44) by $G(p)$ and using the definition (2.36), we obtain the Fourier transform of Eq. (2.34).

Note that Eq. (2.34) is not closed. It relates the two-point average (propagator) to the three-point average (which is associated with a vertex). The closed set of the Schwinger–Dyson equations can be obtained for the n-point averages

$$G_n(x_1, x_2, \cdots, x_n) \;=\; \langle\, \varphi(x_1)\,\varphi(x_2)\cdots\varphi(x_n)\rangle\,. \tag{2.45}$$

They are also called the correlators, in analogy with statistical mechanics, or the n-point Green functions, in analogy with the Green functions in Minkowski space.

Choosing

$$F[\varphi] \;=\; \varphi(x_2)\cdots\varphi(x_n) \tag{2.46}$$

and calculating the variational derivative, one finds from Eq. (2.28) the following chain of equations:

$$\begin{aligned}\left(-\partial^2+m^2\right)\langle\,\varphi(x)\,\varphi(x_2)\cdots\varphi(x_n)\,\rangle+\frac{\lambda}{2}\langle\,\varphi^2(x)\,\varphi(x_2)\cdots\varphi(x_n)\,\rangle \\ = \sum_{j=2}^{n}\delta^{(d)}(x-x_j)\left\langle\varphi(x_2)\cdots\underline{\varphi(x_j)}\cdots\varphi(x_n)\right\rangle,\end{aligned} \tag{2.47}$$

where $\underline{\varphi(x_j)}$ denotes that the corresponding term $\varphi\,(x_j)$ is missing in the product. Using the notation (2.45), Eq. (2.47) can be rewritten as

$$\begin{aligned}\left(-\partial^2+m^2\right)G_n(x,x_2,\ldots,x_n)+\frac{\lambda}{2}G_{n+1}(x,x,x_2,\ldots,x_n) \\ = \sum_{j=2}^{n}\delta^{(d)}(x-x_j)\,G_{n-2}(x_2,\ldots,\underline{x_j},\ldots,x_n)\end{aligned} \tag{2.48}$$

with $\underline{x_j}$ again denoting the missing argument.

Remark on connected correlators

The n-point correlators (2.45) include both connected and disconnected parts. The presence of disconnected parts is most easily seen in the free case when all connected parts disappear, while G_n for even n is given by the Wick pairing as is discussed in Sect. 2.3.

The correlators can also be defined by introducing the generating functional, which is a functional of an external source $J(x)$:

$$Z[J] \;=\; \int \mathcal{D}\varphi\,(x)\ \mathrm{e}^{-S+\int \mathrm{d}^dx\,J(x)\,\varphi(x)}, \tag{2.49}$$

and varying with respect to the source. The n-point correlators (2.45) are then given by

$$G_n(x_1,\ldots,x_n) \;=\; \frac{1}{Z[J]}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\,, \tag{2.50}$$

while the connected parts are given by

$$\langle\,\varphi(x_1)\cdots\varphi(x_n)\,\rangle_{\rm conn} \;=\; \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\ln Z[J]. \tag{2.51}$$

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This is because

$$W[J] \quad = \quad \ln Z[J] \tag{2.52}$$

involves only a set of connected diagrams, while disconnected ones emerge in $Z[J]$ after the exponentiation. We have already touched on this property in Sect. 2.3 to order λ^2. The functional $W[J]$ is called, for this reason, the generating functional for connected diagrams.

Remark on the LSZ reduction formula

The correlators $G_n(x_1, \ldots, x_n)$ (analytically continued to Minkowski space) determine the amplitude of the process when n, generally speaking, virtual particles produce k on-mass-shell particles. Let us denote this amplitude as $A_{n\to k}(q_1, \ldots, q_n; p_1, \ldots, p_k)$, where $q_1, \ldots, q_n$ and $p_1, \ldots, p_k$ are the four-momenta of the incoming and outgoing particles, respectively. Four-momentum conservation requires

$$q_1 + \cdots + q_n \quad = \quad p_1 + \cdots + p_k \,. \tag{2.53}$$

The Lehman–Symanzik–Zimmerman (LSZ) reduction formula reads as

$$\begin{aligned} & A_{n\to k}(q_1, \ldots, q_n; p_1, \ldots, p_k) \\ & \quad = \prod_{j=1}^{n} (q_j^2 + m^2) \prod_{i=1}^{k} \lim_{p_i^2 \to -m^2} (p_i^2 + m^2) \int \prod_{i=1}^{k} \frac{\mathrm{d}^d p_i}{(2\pi)^d} \int \prod_{j=1}^{n-1} \frac{\mathrm{d}^d q_j}{(2\pi)^d} \\ & \qquad \times \exp\left(-\mathrm{i} \sum_{i=1}^{k} p_i x_i + \mathrm{i} \sum_{j=1}^{n-1} q_j x_{k+j}\right) G_n(x_1, \ldots, x_{n+k-1}, 0) \,. \end{aligned} \tag{2.54}$$

The unusual sign of the square of the particle mass m arises from the Euclidean metric.

Equation (2.54) makes sense for timelike p_i, when $p_i^2 < 0$, while q_j^2 is arbitrary. The amplitude for the case of on-mass-shell incoming particles is given by Eq. (2.54) with $q_j^2 \to -m^2$.

2.7 Regularization

The ultraviolet divergences (i.e. those at small distances or large momenta) are an intrinsic property of quantum field theory which makes it different from the quantum mechanics of a finite number of degrees of freedom. The divergences emerge, roughly speaking, because of the delta-function in the canonical commutation relations.

The idea of regularization is to somehow smooth the effect of the delta-function. The usual procedures of regularization are to:

(1) smear the delta-function by
   - point splitting,
   - Schwinger proper-time regularization,
   - latticizing;
(2) add a negative-norm regulating term to the action
   - Pauli–Villars regularization;
(3) introduce higher derivatives in the kinetic term;*
(4) change the dimension, $4 \to 4-\epsilon$;
(5) regularize the measure in the path integral.

As an example of point splitting, let us consider the regularization when the delta-function in the commutator term is replaced by

$$\delta^{(d)}(x-y) \ \stackrel{\text{reg.}}{\Longrightarrow} \ \boldsymbol{R}\,\delta^{(d)}(x-y) \ = \ R(x,y)\,. \tag{2.55}$$

The regularizing operator $\boldsymbol{R}$ is, for instance,

$$\boldsymbol{R} \ = \ \mathrm{e}^{a^2(\partial^2-m^2)}\,, \tag{2.56}$$

where the parameter a with the dimension of length plays the role of an ultraviolet cutoff. The cutoff disappears as $a \to 0$ when

$$\left.\begin{aligned} \boldsymbol{R} \ &\to \ \mathbf{1}\,, \\ R(x,y) \ &\to \ \delta^{(d)}(x-y)\,. \end{aligned}\right\} \tag{2.57}$$

It is easy to calculate how the regularization (2.55) modifies the propagator. The result is

$$\begin{aligned} G_R(x-y) \ &= \ \frac{1}{-\partial^2+m^2}\boldsymbol{R}\,\delta^{(d)}(x-y) \\ &= \ \frac{1}{2}\int\limits_{a^2}^{\infty} \mathrm{d}\tau \ \mathrm{e}^{-\frac{1}{2}\tau m^2}\,\mathrm{e}^{-(x-y)^2/2\tau}\frac{1}{(2\pi\tau)^{d/2}}\,. \end{aligned} \tag{2.58}$$

The lower limit in the integral over the proper time τ is now a^2 rather than 0 as in the nonregularized expression (1.89). This particular method of point splitting coincides with the Schwinger proper-time regularization.

A regularization via point splitting can be performed nonperturbatively, while the dimensional regularization (which is listed in item (4)) is defined only within the framework of perturbation theory. The regularization of the measure listed in item (5) will be considered in the next chapter.

When a regularization is introduced, some of the first principles (called the axioms), on which quantum field theory is constructed, are violated. For instance, the regularization via the point splitting (2.55) violates locality.

* That is in the quadratic-in-fields part of the action.