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ON INC-EXTENSIONS AND POLYNOMIALS WITH UNIT CONTENT

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ABSTRACT. It is proved that if u is an element of a faithful algebra over a commutative ring R, then u satisfies a polynomial over R which has unit content if and only if the extension $R \subset R[u]$ has the imcomparability property. Applications include new proofs of results of Gilmer–Hoffmann and Papick, as well as a characterization of the P-extensions introduced by Gilmer and Hoffmann.

1. **Introduction and summary.** Each ring considered in this note will be assumed commutative, with unit. We also understand by an inclusion (extension) of rings that the smaller ring is a subring of the larger and possesses the same multiplicative identity. Our use of the inclusion symbol will not preclude the possibility of equality.

The principal motivation for our work arises from the following unpublished characterization of integrality in terms of the lying-over (LO) and incomparability (INC) properties. One proof of this result follows readily from Zariski's main theorem (as formulated, for example, in [3]); a simpler proof is available by pursuing the reasoning in [6, Exercise 19, p. 42]. We first learned of this result in May, 1970, and believe that it should be attributed to Graham Evans.

FOLKLORE THEOREM. *For rings $R \subset T$, the following are equivalent*:

(1) *T is an integral extension of R*;

(2) *For any inclusions of rings $R \subset A \subset B \subset T$, the extension $A \subset B$ satisfies both* LO *and* INC;

(3) *For any inclusions of rings $R \subset A \subset T$ and any element $u \in T$, the extension $A \subset A[u]$ satisfies both* LO *and* INC.

It seems natural to ask which weakening of integrality (condition (1) above) is characterized by the condition(s) obtained by deleting references to LO in (2) and (3) above. As detailed in Corollary 4 in Section 3, the answer is: the notion of P-extension recently introduced by Gilmer and Hoffmann [4]. To review and expand upon the terminology in [4], given rings $R \subset T$ and an element $u \in T$, we say that u is *primitive over* R in case u is a root of a

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polynomial $f \in R[X]$ with unit content, i.e. such that the coefficients of f generate the unit ideal of R; if each element of T is primitive over R, then $R \subset T$ is said to be a *P-extension*. It is useful to record the following consequence of [4, Theorem 1]: u is primitive over R (if and) only if u is a root of a polynomial $g \in R[X]$ such that at least one coefficient of g is a unit of R.

Corollary 4, the above-mentioned analogue of Evans' result, is an easy consequence of our main result, which may now be stated.

THEOREM. *For any rings $R \subset T$ and any element $u \in T$, u is primitive over R if and only if the extension $R \subset R[u]$ satisfies* INC.

Section 2 is devoted to proving the result just stated. Included among its applications, which are collected in Section 3, are a characterization of integrality in terms of primitivity (see Remark 8(c)) and new proofs of some results of Gilmer–Hoffmann [4] and Papick [9]. Of these, we mention here only Corollary 5, which uses INC to recapture the connection, established in [4, Theorem 5], between P-extensions and (integral closures being) Prüfer domains.

2. **Proof of theorem.** The "only if" half may be proved by modifying one of the standard proofs that integral extensions satisfy INC. Indeed, if u is primitive over R but $R \subset R[u]$ does not satisfy INC, then the failure of INC supplies distinct comparable prime ideals $Q_1 \subset Q_2$ of $R[u]$ such that $Q_1 \cap R = Q_2 \cap R$ ($=$, say, P). Note that $Q_1R[u]_{R \setminus P}$ and $Q_2R[u]_{R \setminus P}$ are then distinct comparable prime ideals of $R[u]_{R \setminus P}$ which each meet R_P in PR_P; that is, $R_P \subset R[u]_{R \setminus P}$ also fails to satisfy INC. If v is the image of u in $R[u]_{R \setminus P}$, then $R[u]_{R \setminus P} = R_P[v]$; moreover, v is primitive over R_P since u is primitive over R. Hence, R may be assumed quasilocal, with maximal ideal P. Now, the integral domain $B = R[u]/Q_1$ is generated as an algebra over $A = R/P$ by $w = u + Q_1$. As w is primitive over the field A, it follows that w is integral over A and so B is integral over A, whence B is a field (cf. [6, Theroem 44]). This contradicts the presence of the nonzero prime ideal Q_2/Q_1 in B, and completes the proof of the "only if" half.

Conversely, suppose that $R \subset R[u]$ satisfies INC; our task is to show that u is primitive over R. We first treat the case in which R is quasilocal, with maximal ideal M. Let h be the R-algebra homomorphism $R[X] \to R[u]$ sending X to u. If the assertion is false, then $\ker(h) \subset MR[X]$, since any polynomial in $R[X] \setminus MR[X]$ has unit content. By the second isomorphism theorem, h induces an R-algebra isomorphism between $R[X]/MR[X]$ and $R[u]/MR[u]$. As the former is isomorphic to the ring of polynomials in one variable over the field R/M, it follows that $MR[u]$ is a non-maximal prime ideal of $R[u]$. The desired contradiction (to the hypothesis that $R \subset R[u]$ satisfies INC) arises since $MR[u] \cap R = M = N \cap R$ for each maximal ideal N of $R[u]$ which contains $MR[u]$.

If R is not necessarily quasilocal, then for each maximal ideal M of R, let u_M denote the image of u in $R[u]_{R\backslash M}$. As in the proof of the "only if" half, the extension $R_M \subset R[u]_{R\backslash M} = R_M[u_M]$ inherits INC from $R \subset R[u]$. Consequently by the quasilocal case treated in the preceding paragraph, there exists $f_M \in R_M[X]$, with unit content, such that $f_M(u_M) = 0 \in R_M[u_M]$. Hence, for each M, there exists $g_M \in R[X]$ such that $g_M(u) = 0 \in R[u]$ and at least one coefficient of g_M lies in $R\backslash M$. If I is the content of u, that is the ideal of R generated by the coefficients of all the polynomials over R which u satisfies, the preceding argument gives $I \not\subset M$ for each M. Thus, $I = R$. The proof may now be completed by appealing to [4, Corollary 1]; for completeness, we provide the remaining detail. As $1 \in I$, we have $1 = \Sigma r_i c_i$, where $r_i \in R$ and c_i is the coefficient of X^{k_i} in some $f_i \in R[X]$ such that $f_i(u) = 0$. Set $k = \max k_i$ and $f = \Sigma r_i f_i X^{k-k_i}$. Since $f(u) = 0$ and the coefficient of X^k in f is $\Sigma r_i c_i = 1$, the proof is complete.

3. **Applications.** If $R \subset T$ are rings and X is an indeterminate over T, it is easy to see that $R \subset T$ will inherit any of the properties going-up (GU), going-down (GD), LO and INC from $R[X] \subset T[X]$. The converses in the first three cases have each received attention. Indeed, if $R \subset T$ satisfies GU (resp., GD), then $R[X] \subset T[X]$ need not satisfy GU (resp., GD). The assertion regarding GU was established by Kaplansky [5] who actually proved that T is integral over R whenever $R[X] \subset T[X]$ satisfies GU; building on [5], Dawson and Dobbs [1, Example 3.9] established the assertion about GD. However, $R[X] \subset T[X]$ will inherit LO from $R \subset T$: this was shown by McAdam [7, Proposition 1] for integral domains, and was noted in [5] for the general case. The next result and remark treat the case of INC.

Corollary 1. *Let $R \subset T$ be rings, such that $T = R[u]$ for some $u \in T$. Let X be an indeterminate over T. Then $R[X] \subset T[X]$ satisfies* INC *if (and only if) $R \subset T$ satisfies* INC.

Proof. If $R \subset T$ satisfies INC, our theorem implies that u is primitive over R; *a fortiori*, u is then primitive over $R[X]$. As $T[X] = (R[X])[u]$, another application of the theorem shows that $R[X] \subset T[X]$ also satisfies INC, as desired. The parenthetical assertion was observed above, but may also be proved using the theorem.

Remark 2. One cannot remove the hypothesis in Corollary 1 that T may be generated as an R-algebra by one element. For example, let $R \subset T$ be distinct fields, such that R is algebraically closed in T. Then $R \subset T$ satisfies INC, by default (and no $u \in T$ satisfies $T = R[u]$). Showing that $R[X] \subset T[X]$ does not satisfy INC amounts to proving that $P \cap R[X] = 0$ for some non-zero prime ideal P of $T[X]$. Take P to be the ideal of $T[X]$ generated by $X - t$, where t is a chosen element of $T\backslash R$. If there exists non-zero $g \in P \cap R[X]$, then factoring

g in the ring of polynomials over an algebraic closure of T reveals that t is a root of g; that is, t is algebraic over R, a contradiction. Thus, $P \cap R[X] = 0$, as claimed.

The next result is the analogue, for INC, of a result for GD established by Dobbs [2, Theorem 2.6].

COROLLARY 3. *Let $R \subset T$ be rings, and let u be a unit (i.e. invertible element) of T. Then $R \subset R[u]$ satisfies* INC *if and only if $R \subset R[u^{-1}]$ satisfies* INC.

Proof. By the theorem, out task is to show: u is primitive over R if and only if u^{-1} is primitive over R. This, however, is apparent, for if u satisfies an nth degree polynomial $\Sigma r_i X^i \in R[X]$, then u^{-1} satisfies $\Sigma r_{n-i} X^i$.

We next give the promised analogue of Evans' result.

COROLLARY 4. *For rings $R \subset T$, the following are equivalent:*

(1) *T is a P-extension of R;*

(2) *For any inclusions of rings $R \subset A \subset B \subset T$, the extension $A \subset B$ satisfies* INC;

(3) *For any inclusions of rings $R \subset A \subset T$ and any element $u \in T$, the extension $A \subset A[u]$ satisfies* INC;

(4) *For any element $u \in T$, the extension $R \subset R[u]$ satisfies* INC.

Proof. Our theorem yields (1)$\Leftrightarrow$(4). Moreover, (2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) trivially.

Finally, if (2) fails, there exist distinct comparable prime ideals $Q_1 \subset Q_2$ of B such that $Q_1 \cap A = Q_2 \cap A$ (=, say, P). If $u \in Q_2 \backslash Q_1$, then $Q_1 \cap R[u]$ and $Q_2 \cap R[u]$ are distinct comparable prime ideals of $R[u]$ which each meet R in $P \cap R$; thus, $R \subset R[u]$ does not satisfy INC, and (4) fails. Hence, (4)$\Rightarrow$(2), and the proof is complete.

We next turn to the connection between P-extensions and Prüfer domains.

COROLLARY 5. (Gilmer–Hoffmann [4, Theorem 5]). *Let R be an integral domain, with quotient field K. Then the integral closure of R is a Prüfer domain if and only if $R \subset K$ is a P-extension.*

Proof. As noted by Papick [8, Proposition 2.26], the integral closure of R is a Prüfer domain if and only if $R \subset A$ satisfies INC for each overring A of R (that is, for each ring A between R and K). By Corollary 4, the latter condition is equivalent to the requirement that $R \subset K$ be a P-extension, which completes the proof.

The next result, which generalizes a comment in [9], shows that any "coherent pair" (to use the terminology recently introduced by Papick) must be a P-extension.

Corollary 6. *Let $R \subset T$ be integral domains, such that R is not a field. If each ring contained between R and T is coherent, then $R \subset T$ is a P-extension.*

Proof. By [9, Proposition 1], $R \subset A$ satisfies INC for each ring A contained between R and T. An application of Corollary 4 completes the proof.

Corollary 7 (Papick [9, Corollary 10]). *Let R be an integral domain. If each overring of R is coherent, then the integral closure of R is a Prüfer domain.*

Proof. We offer two proofs. The first merely amounts to combining Corollaries 5 and 6 (with $T = K$, the quotient field of R). Another proof, which does not rely on Corollary 5, proceeds as follows. Without loss of generality, R may be taken integrally closed and quasilocal. As Corollary 6 shows that $R \subset K$ is a P-extension, the celebrated lemma of Seidenberg (cf. [10, Theorem 6], [6, Theorem 67]) guarantees that R is a valuation domain, to complete the proof.

We close by examining some possible analogies between primitivity and integrality.

Remark 8. (a) By Corollary 4, any P-extension satisfies INC. However, the converse is false. For example, if R is any integral domain whose integral closure is not a Prüfer domain and T is the quotient field of R, then $R \subset T$ satisfies INC by default, although (by Corollary 5) $R \subset T$ is not a P-extension.

(b) It is well known that an extension $R \subset T$ is integral if (and only if) T is generated as an R-algebra by integral elements. The analogue of this result for P-extensions and primitive elements is, however, false. For an example, let R be a G(oldman)-domain whose integral closure is not a Prüfer domain. (For an explicit example, consider distinct fields $F \subset L$, with F algebraically closed in L, and set $R = F + YL[[Y]]$.) As R is a G-domain, its quotient field T may be expressed as $T = R[u]$, where $u \in T$ satisfies $u^{-1} \in R$. (In the example, Y^{-1} is a suitable value for u.) Now, u is primitive over R since u satisfies the linear polynomial $u^{-1}X - 1$, but as in (a), $R \subset T$ is not a P-extension.

(c) In view of the preceding remarks, it is of some interest to note that primitivity figures in a characterization of integrality, as follows. An extension $R \subset T$ is integral if and only if the following two conditions hold: (1) T may be generated as an R-algebra by a set of elements each of which is primitive over R; (2) $A \subset A[u]$ satisfies GU whenever $R \subset A \subset T$ and $u \in T$.

Of course, the "only if" half is immediate. Conversely, to prove that (1) and (2) imply integrality: since GU$\Rightarrow$LO [6, Theorem 42], Evans' result reduces our task to proving that $A \subset A[u]$ satisfies INC whenever $R \subset A \subset T$ and $u \in T$. By Corollary 4, it is enough to show $R \subset R[u]$ satisfies INC for each $u \in T$. By (1), for any such u, there exist $u_1, \ldots, u_n$ such that $u \in R[u_1, \ldots, u_n]$ and each u_i is primitive over R. As u_{i+1} is primitive over $R[u_1, \ldots, u_i]$, our theorem shows that $R[u_1, \ldots, u_i] \subset R[u_1, \ldots, u_i, u_{i+1}]$ satisfies INC; then the extension $R \subset R[u_1, \ldots, u_n]$ satisfies INC, as it is obtained by n successive extensions which each satisfy INC. It therefore is enough to prove that the extension $R[u] \subset R[u_1, \ldots, u_n]$ satisfies GU (and hence, LO), for then $R \subset R[u]$ will indeed satisfy INC, as required. However, this is immediate from (2), by considering the tower

$$R[u] \subset R[u, u_1] \subset R[u, u_1, u_2] \subset \cdots \subset R[u, u_1, \ldots, u_n] = R[u_1, \ldots, u_n]$$

of n going-up extensions, to complete the proof.

COROLLARY 9 (Gilmer and Hoffmann [4, Theorem 4]). *Let $R \subset S \subset T$ be rings. If $R \subset S$ is integral and $S \subset T$ is a P-extension, then $R \subset T$ is a P-extension.*

Proof. For each $u \in T$, the extension $R[u] \subset S[u]$ inherits integrality from $R \subset S$ and, hence, $R[u] \subset S[u]$ satisfies GU and INC. Accordingly, it suffices to prove the next result.

COROLLARY 10. *Let $R \subset S \subset T$ be rings. If $R \subset S$ satisfies* INC, *if $S \subset T$ is a P-extension, and if $R[u] \subset S[u]$ satisfies* GU *for each $u \in T$, then $R \subset T$ is a P-extension.*

Proof. By Corollary 4, it is enough to prove that $R \subset R[u]$ satisfies INC for each $u \in T$. Since $S \subset T$ is a P-extension, our theorem yields that $S \subset S[u]$ satisfies INC. Then the extension $R \subset S[u]$ satisfies INC, as it results from the tower $R \subset S \subset S[u]$. As $R[u] \subset S[u]$ satisfies GU (and LO), consideration of the tower $R \subset R[u] \subset S[u]$ leads to $R \subset R[u]$ satisfying INC, as desired.

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