NON-LOCAL ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Let G be a bounded open set of R^n with a smooth boundary ∂G. We consider the following elliptic boundary-value problem:

$$Au = f \quad \text{on } G; \qquad B_j u = \sum_{k=1}^{m} L_{jk} C_k u \quad \text{on } \partial G, \qquad j = 1, \ldots, m,$$

where A and B_j are, respectively singular integro-differential operators on G and on ∂G, of orders $2m$ and r_j with $r_j < 2m$; C_k are boundary differential operators, and L_{jk} are linear operators, bounded in a sense to be specified.

Let A_2 be the realization of A as an operator on $L^2(G)$ with the above boundary conditions. When the symbols σ_A, σ_j of A and B_j satisfy a strengthened Shapiro-Lopatinskiĭ condition, we show, in § 2, that A_2 is a Fredholm operator, the generalized eigenfunctions of A_2 are complete in $L^2(G)$ and $(A_2 + \lambda I)^{-1}$ exists for large $|\lambda|$, $\arg \lambda = \theta$. We also prove the existence of a solution of $(A_2 + \lambda I)u = f(x, T_1 u, \ldots, T_{2m-1} u)$, where T_j are bounded, linear operators from $W^{j,2}(G)$ into $L^2(G)$, $f(x, \zeta_1, \ldots, \zeta_{2m-1})$ has a linear growth in $(\zeta_1, \ldots, \zeta_{2m-1})$.

The proofs depend on a result on elliptic boundary-value problems $\{A; B_j\}$ containing a large parameter λ, which is given in § 3. The notation, the definitions, and the results are given in § 1.

Non-local elliptic boundary-value problems have been studied by Agranovič (**2**), Beals (**4**), Browder (**6**), Schechter (**8**), and others.

1. Let G be a bounded open set of R^n, regular of class C^∞ with boundary ∂G. The generic point x of G is $x = (x_1, \ldots, x_n)$. Set $D_j = i^{-1}\partial/\partial x_j$, $j = 1, \ldots, n$. For each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, we write:

$$D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j} \quad \text{and} \quad |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

Let s be a non-negative integer; we denote by $W^{s,2}(G)$ the space

$$W^{s,2}(G) = \{u\colon u \text{ in } L^2(G), D^\alpha u \text{ in } L^2(G); |\alpha| \leqq s\}$$

(the derivatives are taken in the sense of the theory of distributions). $W^{s,2}(G)$ is a Hilbert space with the norm

$$||u||_{s,2} = \left\{ \sum_{|\alpha| \leqslant s} ||D^\alpha u||^2_{L^2(G)} \right\}^{\frac{1}{2}}$$

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and the obvious inner product. Set

$$|||u|||_{s,2} = \{||u||_{s,2}^2 + |\lambda|^{s/m}||u||_{0,2}^2\}^{\frac{1}{2}},$$

then

$$|||u|||_{s,2} \leqq \left(\sum_{k=0}^{s} |\lambda|^{k/m}||u||_{s-k,2}^2\right)^{\frac{1}{2}} \leqq C|||u|||_{s,2}$$

(cf. Agranovič and Višik (**3**, p. 64)).

Let ϕ_k, $k = 1, \ldots, N$, be those functions of the finite partition of unity whose supports intersect the boundary ∂G. For $s \geqq 0$, we define $W^{s,2}(\partial G)$ as the completion of $C^\infty(\partial G)$ with respect to the norm

$$||u||_{s,2}' = \left(\sum_{k=1}^{N} ||\phi_k u||_{W^{s,2}(R^{n-1})}^2\right)^{\frac{1}{2}},$$

where $||\phi_k u||_{W^{s,2}(R^{n-1})}$ is taken in local coordinates and is defined by means of the Fourier transforms:

$$||\phi_k u||_{W^{s,2}(R^{n-1})} = \left\{\int_{E^{n-1}} (1 + |\xi|^{2s})|F(\phi_k u)|^2 d\xi\right\}^{\frac{1}{2}}.$$

The space $W^{s,2}(\partial G)$ is a Hilbert space. It neither depends on the choice of local coordinates nor on the choice of the partition of unity. We set

$$|||u|||_{s,2}' = (||u||_{s,2}'^2 + |\lambda|^{s/m}||u||_{0,2}'^2)^{\frac{1}{2}}.$$

We have that

$$|||u|||_{s-\frac{1}{2},2}' \leqq C|||u|||_{s,2}.$$

Let $u(x)$ be in $C^k(R_+^n)$, $R_+^n = \{x\colon x_n > 0\}$. Then the Hestenes formula defines a smooth continuation L of u to Lu in $C^k(R^n)$. If u is in $W^{k,2}(R_+^n)$, then $||Lu||_{W^{s,2}(R^n)} \leqq C||u||_{W^{s,2}(R_+^n)}$, $s = 0, \ldots, k$.

DEFINITION 1.1. (i) *A is said to be an operator of order k in $W^{s,2}(G)$ if A is a bounded linear mapping from $W^{s,2}(G)$ into $W^{s-k,2}(G)$. s and k are two non-negative integers with $s \geqq k$.*

(ii) *A is said to be an operator almost of order $k - 1$ on $W^{s,2}(G)$ if A may be decomposed into $A = A_\epsilon' + A_\epsilon''$, where A_ϵ' is an operator of order k in $W^{s,2}(G)$ with norm less than ϵ and A_ϵ'' is an operator of order $k - 1$ in $W^{s,2}(G)$. ϵ is any given positive number.*

Consider the singular integral operators

$$A_{mk}u(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} Y_{mk}(x - y)|x - y|^{-n}u(y)\, dy, \qquad u \in L^2(R^n),$$

where $Y_{mk}(x)$ are the spherical functions on the unit sphere in R^n.

Let $\sigma(x, \xi)$ be a positive homogeneous function of ξ of degree 0. We expand $\sigma(x, \xi)$ as follows:

$$\sigma(x, \xi) = \sum_{m,k} \gamma_m a_{mk}(x) Y_{mk}(\xi), \qquad \gamma_0 = 1.$$

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The operator $A = \sum_{m,k} a_{mk}(x)A_{mk}$ associated with σ is a homogeneous singular integral operator on R^n with symbol σ. It is of *class* (p, q) *if*

$$\sigma(x, \xi) \in C^p(R^n; W^{q,2}(\Sigma)),$$

where $C^p(R^n; W^{q,2}(\Sigma))$ is the space of functions $f(x, \cdot)$ on R^n with values in $W^{q,2}(\Sigma)$ and having x-continuous derivatives of order $\leqq p$ in $W^{q,2}(\Sigma)$. Σ is the unit sphere in R^n.

DEFINITION 1.2. *A singular integro-differential operator of class* (p, q) *of order* s *in* $W^{k,2}(R^n)$, $s \geqq k$, *is an operator of the form*:

$$A = \sum_{|\alpha|=s} A_\alpha D^\alpha + T,$$

where A_α *are homogeneous singular integral operators in* R^n, *of class* (p, q), *and* T *is an arbitrary linear operator almost of order* $s - 1$ *in* $W^{k,2}(R^n)$. A *is homogeneous if* $T = 0$.

The symbol of A,

$$\sigma_A(x, \xi) = \sum_{|\alpha|=s} \sigma_\alpha(x, \xi)\xi^\alpha,$$

is a positive homogeneous function of order s *with respect to* ξ, $\sigma_\alpha(x, \xi)$ *is the symbol of* A_α.

DEFINITION 1.3. (i) $A = R\tilde{A}L$ (*where* $\tilde{A}$ *is a homogeneous singular integro-differential operator of class* (p, q) *of order* s *in* $W^{k,2}(R^n)$, $s \geqq k$, R *is the restriction operator of functions from* R^n *to* R_+^n, *and* L *is the extension operator of functions from* R_+^n *to* R^n) *is a singular integro-differential operator of class* (p, q) *and of order* s *in* $W^{k,2}(R_+^n)$.

(ii) A *is called an admissible singular integro-differential operator on* R_+^n *if for* $x_n = 0$ *we have that*

$$\sigma_A(x', 0, \xi', \xi_n) = \sum_{k=0}^{s} \sigma_k(x', \xi')\xi_n^k$$

and $\sigma_s(x', \xi')$ *does not depend on* ξ_n.

Hence, if σ is the symbol of an admissible singular integro-differential operator of class (p, q) of order s in $W^{k,2}(R_+^n)$, then $\sigma_k(x', \xi')$ are positively homogeneous of degree $s - k$ and are in $C^p(R^{n-1}; W^{q-\frac{1}{2},2}(\Sigma'))$, where Σ' is the unit sphere in R^{n-1}.

Let $\{N_k\}$ be a finite open covering of $\mathrm{cl}(G)$ and $\{\phi_k\}$ a finite partition of unity corresponding to N_k. Denote by ψ_k an infinitely differentiable function with compact support in N_k and $\psi_k = 1$ on the support of ϕ_k.

We shall consider singular integro-differential operators on G of the form

$$A = \sum_k \phi_k A_k \psi_k + T, \tag{1.1}$$

where T is an operator almost of order $2m - 1$ in $W^{s,2}(G)$ $(s \geqq 2m)$ and A_k is an admissible singular integro-differential operator of order $2m$ on R_+^n if N_k is a boundary neighbourhood, and on R^n, otherwise.

We consider also operators on ∂G of the form

$$B_j = \sum_k{}' \phi_k B_{jk} \psi_k + T_j, \qquad j = 1, \ldots, m, \tag{1.2}$$

where the summation is taken over all the k corresponding to boundary neighbourhoods N_k. B_{jk} are given by

$$B_{jk} = \sum_{l=0}^{r_j} B_{jk}{}^l D_n{}^l,$$

where $B_{jk}{}^l$ are singular integro-differential operators on R^{n-1}, homogeneous of orders $r_j - l$. T_j is an operator almost of order -1 from $W^{s,2}(G)$ into $W^{s-r_j-\frac{1}{2},2}(\partial G)$.The symbol σ_A of A is defined as follows: it is a function $\sigma_A(P, \xi)$ such that for points P in N_k, x in local coordinates, it coincides with the symbol $\sigma_{A_k}(x, \xi)$ of A_k. Similarly for σ_{B_j}.

DEFINITION 1.4. *An admissible singular integro-differential operator A on G of the form* (1.1) *is said to be elliptic at a point P of G if*

$$\sigma_{Ak}(x, \xi) \neq 0 \quad \textit{for } \xi \neq 0; \qquad P \in N_k \cap G,$$

and elliptic on G if it is elliptic at each point of G.

The definition is invariant with respect to the choice of coordinate neighbourhoods and local coordinates.

A_k is said to be *properly elliptic* at $x_0 = (x', 0)$ if $\sigma_{Ak}(x', \xi', \zeta) = 0$, considered as a polynomial in the complex variable ζ, has m roots in the upper half ζ-plane and m roots in the lower half-plane. Throughout the paper, we shall assume that the A_k are properly elliptic on R^n.

DEFINITION 1.5. *The elliptic boundary-value problem $\{A; B_j, j = 1, \ldots, m\}$ on G, where A and B_j are of the form* (1.1) *and* (1.2), *is said to be regular if for each k corresponding to boundary neighbourhoods N_k we have that*

$$\operatorname{Det}\left(\int_C \zeta^{r-1} \sigma_{B_{jk}}(x', \xi', \zeta)[\sigma_{Ak}(x', \xi', \zeta)]^{-1} d\zeta\right) \neq 0,$$

where $r, j = 1, \ldots, m$ and C is a closed Jordan rectifiable curve in the upper half ζ-plane containing all the m roots of $\sigma_{Ak}(x', \xi', \zeta) = 0$.

ASSUMPTION (1). *Let $\{A; B_j, j = 1, \ldots, m\}$ be a regular elliptic boundary problem on G. A and B_j are of the form* (1.1) *and* (1.2).

We assume that there exists a θ, $0 \leqq \theta < 2\pi$, such that for every k corresponding to boundary neighbourhoods N_k we have that

$$\operatorname{Det}\left(\int_C \zeta^{r-1} \sigma_{B_{jk}}(x', \xi', \zeta)[\sigma_{A_k}(x', \xi', \zeta) + \lambda]^{-1} d\zeta\right) \neq 0,$$

where $r, j = 1, \ldots, m$, $\arg \lambda = \theta$, $|\lambda| \geqq \lambda_0 > 0$, and C is as in Definition 1.5.

We now state the main results of the paper.

THEOREM 1.1. *Let* $\{A; B_j, j = 1, \ldots, m\}$ *be a regular elliptic boundary-value problem on* G. *The admissible singular integro-differential operator* A *is of the form* (1.1), *of class* $(s - 2m, q)$, *and of order* $2m$. $s \geqq 2m$ *and* $q > (n-1)/2$. *The* B_j *are of the form* (1.2), *of class* $(s - r_j, q - \frac{1}{2})$, *and of orders* r_j *with* $r_j < 2m - 1$.

Suppose that there exists a θ *such that Assumption* (1) *is satisfied. Then*

(1) *For all* u *in* $W^{s,2}(G)$, *we have that*

$$|||u|||_{s,2} \leqq C\left\{|||(A + \lambda I)u|||_{s-2m,2} + \sum_{j=1}^{m} |||B_j u|||'_{s-r_j-\frac{1}{2},2}\right\},$$

where $\arg \lambda = \theta$, $|\lambda| \geqq \lambda_0 > 0$, *and* C *is independent of* λ *and* u.

(2) *For any* $(f, g_1, \ldots, g_m)$ *in*

$$W^{s-2m,2}(G) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(\partial G), \qquad s \geqq 2m,$$

there exists a unique solution u *in* $W^{s,2}(G)$ *of*

$$(A + \lambda I)u = f \text{ on } G, \quad B_j u = g_j \text{ on } \partial G, \qquad j = 1, \ldots, m.$$

The proof of the theorem is long and will be given in § 3. The theorem has been proved by Agranovič and Višik (**3**) for the case when the operators A and B_j are differential operators (cf., also, Agmon (**1**)).

THEOREM 1.2. *Suppose that the hypotheses of Theorem* 1.1 *are satisfied. Let* C_k, $k = 1, \ldots, m$, *be a set of boundary differential operators of orders* ν_k *with* $\nu_k < 2m$. *Let* L_{jk}, $j, k = 1, \ldots, m$, *be a set of compact (or bounded) linear operators from* $W^{s-\nu_k-\frac{1}{2},2}(\partial G)$ into $W^{s-r_j-\frac{1}{2},2}(\partial G)$ *(or into* $W^{s-r_j-\frac{1}{2}+\epsilon,2}(\partial G)$ *for some* $\epsilon > 0$*). Then*

(i) *there exists a positive constant* M, *independent of* λ $(\arg \lambda = \theta)$ *and* u, *such that, for all* u *in* $W^{s,2}(G)$,

$$|||u|||_{s,2} \leqq M\left\{|||(A + \lambda I)u|||_{s-2m,2} + \sum_{j=1}^{m} \left|\left|\left|\left(B_j - \sum_{k=1}^{m} L_{jk}C_k\right)u\right|\right|\right|'_{s-r_j-\frac{1}{2},2}\right\},$$
$$s \geqq 2m, \ |\lambda| \geqq \lambda_0 > 0;$$

(ii) *let* A_2 *be the realization of* A *as an operator on* $L^2(G)$ *with null boundary conditions*

$$B_j u - \sum_{k=1}^{m} L_{jk}C_k u = 0 \text{ on } \partial G, \qquad j = 1, \ldots, m.$$

Then $(A_2 + \lambda I)^{-1}$ *exists and is defined on all of* $L^2(G)$. *It is a compact operator on* $L^2(G)$ *with* $||A_2 + \lambda I)^{-1}|| \leqq M/|\lambda|$ *for* $|\lambda| \geqq \lambda_0 > 0$.

THEOREM 1.3. *Suppose that the hypotheses of Theorem* 1.2 *are satisfied. Then*

(i) *there exists a positive constant* M *such that, for all* u *in* $W^{s,2}(G)$,

$$||u||_{s,2} \leqq M\left\{||Au||_{s-2m,2} + ||u||_{0,2} + \sum_{j=1}^{m} \left|\left|\left(B_j - \sum_{k=1}^{m} L_{jk}C_k\right)u\right|\right|'_{s-r_j-\frac{1}{2},2}\right\},$$
$$s \geqq 2m;$$

(ii) *A_2 is a Fredholm operator and* $\operatorname{ind}(A_2) = 0$ (cf. Schechter (**8**).

THEOREM 1.4. *Suppose that the hypotheses of Theorem* 1.2 *are satisfied for* $s = 2m$. *Suppose, further, that there exist* θ_k, $k = 1, \ldots, N$, $0 \leqq \theta_k < 2\pi$, *for which assumption* (1) *is satisfied and such that the plane is divided by these rays* $\arg \lambda = \theta_k$ *into angles which are less than* $2m\pi/n$. *Then the generalized eigenfunctions of* A_2 *are complete in* $L^2(G)$.

The theorem extends for the case $p = 2$, a result of Agmon (**1**).

THEOREM 1.5. *Suppose that the hypotheses of Theorem* 1.2 *are satisfied for* $s = 2m$. *Let* $f(x, \zeta_1, \ldots, \zeta_{2m})$ *be a function measurable in x on G, continuous in* $(\zeta_1, \ldots, \zeta_{2m})$ *with* $f(x, 0, \ldots, 0) \neq 0$. *Suppose, further, that there exists a positive constant M such that*

$$|f(x, \zeta_1, \ldots, \zeta_{2m})| \leqq M\left\{1 + \sum_{j=1}^{2m-1} |\zeta_j|\right\}.$$

Let $T_1, \ldots, T_{2m-1}$ *be bounded linear operators from* $W^{j,2}(G)$ *into* $L^2(G)$ *and let* T_{2m} *be a bouuded linear operator from* $W^{2m-\epsilon,2}(G)$ *into* $L^2(G)$, $0 < \epsilon$. *Then*

(i) *for* $|\lambda| \geqq \lambda_0 > 0$, *there exists a non-trivial solution u in* $W^{2m,2}(G)$ *of the elliptic boundary-value problem*

$$(A + \lambda I)u = f(x, T_1 u, \ldots, T_{2m}u) \text{ on } G,$$

$$B_j u = \sum_{k=1}^{m} L_{jk} C_k u \text{ on } \partial G, \qquad j = 1, \ldots, m;$$

(ii) *let* $(g_1, \ldots, g_m)$ *be in*

$$\prod_{j=1}^{m} W^{2m-r_j-\frac{1}{2},2}(\partial G).$$

There exists a solution u in $W^{2m,2}(G)$ *of* $(A + \lambda I)u = f(x, T_1 u, \ldots, T_{2m}u)$ *on* G; $B_j u = g_j$ *on* ∂G.

2. In this section we shall give the proofs of Theorems 1.2–1.5, assuming Theorem 1.1.

Proof of Theorem 1.2. (1) We establish the a-priori estimate. Suppose that part (i) of the theorem is not true. Then for any λ with $\arg \lambda = \theta$, $|\lambda| \geqq \lambda_0 > 0$, there would exist $\{u_n\}$ with

$$|||u_n|||_{s,2} = 1$$

and

$$|||(A + \lambda)u_n|||_{s-2m,2} + ||u_n||_{0,2} + \sum_{j=1}^{m} \left|\left|\left|\left(B_j - \sum_{k=1}^{m} L_{jk}C_k\right)u_n\right|\right|\right|'_{s-r_j-\frac{1}{2},2} \to 0.$$

From the weak compactness of the unit ball in a Hilbert space, we obtain a subsequence, which we may assume to be the original one, such that $u_n \to u$ weakly in $W^{s,2}(G)$ as $n \to \infty$. Since $u_n \to 0$ in $L^2(G)$, we have that $u = 0$. Since G is a bounded open set of R^n, regular of class C^∞, it follows from the

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Sobolev imbedding theorem that $u_n \to 0$ in $W^{s-1,2}(G)$ and $u_n \to 0$ weakly in $W^{s-\frac{1}{2},2}(\partial G)$ as $n \to \infty$. The operator $\sum_{k=1}^{m} L_{jk}C_k$ is a compact linear mapping from $W^{s-\frac{1}{2},2}(\partial G)$ into $W^{s-r_j-\frac{1}{2},2}(\partial G)$, being the composition of a linear mapping from $W^{s-\frac{1}{2},2}(\partial G)$ into $W^{s-\nu_k-\frac{1}{2},2}(\partial G)$ and a compact mapping from $W^{s-\nu_k-\frac{1}{2},2}$ into $W^{s-r_j-\frac{1}{2},2}(\partial G)$.

Therefore $\sum_{k=1}^{m} L_{jk}C_k u_n \to 0$ in $W^{s-r_j-\frac{1}{2},2}(\partial G)$ as $n \to \infty$.

Hence, $B_j u_n \to 0$ in $W^{s-r_j-\frac{1}{2},2}(\partial G)$ as $n \to \infty$, $j = 1, \ldots, m$. In a similar fashion, we show that

$$\lambda^{(s-r_j-\frac{1}{2})/2m} B_j u_n \to 0 \text{ in } L^2(\partial G), \qquad j = 1, \ldots, m.$$

On the other hand, from Theorem 1.1, we obtain the following:

$$|||u_n|||_{s,2} \leqq M\left\{|||(A+\lambda)u_n|||_{s-2m,2} + \sum_{j=1}^{m} |||B_j u_n|||'_{s-r_j-\frac{1}{2},2}\right\}.$$

Thus $|||u_n|||_{s,2} \to 0$ as $n \to \infty$, which is a contradiction. Now take $|\lambda|$ sufficiently large and we obtain the a-priori estimate.

(2) Let A_2 be a linear operator on $L^2(G)$ defined as follows:

$$D(A_2) = \left\{u: u \text{ in } W^{2m,2}(G),\ Au \text{ in } L^2(G);\right.$$
$$\left. B_j u = \sum_{k=1}^{m} L_{jk}C_k u \text{ on } \partial G,\ j = 1, \ldots, m\right\},$$
$$A_2 u = Au \quad \text{if } u \text{ is in } D(A_2).$$

A_2 is densely defined. Indeed, we have that $C_c^{\infty}(G) \subset D(A_2)$. From the a-priori estimate and Proposition 16.1 of Agranovič (**2**, p. 99), we deduce that $(A_2 + \lambda I)$ is a closed operator on $L^2(G)$ with $N(A_2 + \lambda I) = \{0\}$. We show that $R(A_2 + \lambda I) = L^2(G)$. Let f be any element of $L^2(G)$, v an element of $W^{2m,2}(G)$, and suppose that $0 \leqq t \leqq 1$. Consider the following elliptic boundary-value problem

$$(A + \lambda I)u = f \quad \text{on } G, \qquad B_j u = t\sum_{k=1}^{m} L_{jk}C_k v \quad \text{on } \partial G, \qquad j = 1, \ldots, m.$$

From Theorem 1.1, we know that there exists a unique solution u in $W^{2m,2}(G)$ of the above problem. Define the following non-linear mapping $T(t)$ from $[0, 1] \times W^{2m,2}(G)$ into $W^{2m,2}(G)$:

$$T(t)v = u,$$

where u is the unique solution of the above boundary-value problem. If we can show that $T(1)u = u$, i.e. $T(1)$ has a fixed point, then u is in $D(A_2)$ and is in $R(A_2 + \lambda I)$. Since f is an arbitrary element of $L^2(G)$, we have that $R(A_2 + \lambda I) = L^2(G)$. We verify that $T(t)$ satisfies the hypotheses of the Leray-Schauder fixed-point theorem.

PROPOSITION 2.1. *$T(t)$ is a completely continuous operator from $[0, 1] \times W^{2m,2}(G)$ into $W^{2m,2}(G)$.*

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Proof. $T(t)$ is continuous. Let $t_n \to t$, $v_n \to v$ in $W^{2m,2}(G)$. From Theorem **1.1** we obtain the following:

$$||u_n||_{2m,2} \leqq M\left\{||f||_{0,2} + \sum_{j,k=1}^{m} |||L_{jk}C_k(t_n v_n)|||'_{2m-r_j-\frac{1}{2},2}\right\}.$$

Thus

$$||u_n - u||_{2m,2} \leqq M \sum_{j,k=1}^{m} |||L_{jk}C_k(t_n v_n - tv)|||'_{2m-r_j-\frac{1}{2},2}.$$

We immediately have that $u_n \to u$ in $W^{2m,2}(G)$. $T(t)$ is compact. Indeed, suppose that $||v_n||_{2m,2} \leqq M$. Then from the weak compactness of the unit ball in a Hilbert space, we have that $v_n \to v$ weakly in $W^{2m,2}(G)$, hence also weakly in $W^{2m-\frac{1}{2},2}(\partial G)$. But $\sum_{k=1}^{m} L_{jk}C_k$ is a compact operator from $W^{2m-\frac{1}{2},2}(\partial G)$ into $W^{2m-r_j-\frac{1}{2},2}(\partial G)$, thus

$$\sum_{k=1}^{m} L_{jk}C_k v_n \to \sum_{k=1}^{m} L_{jk}C_k v \quad \text{in } W^{2m-r_j-\frac{1}{2},2}(\partial G)$$

as well as in $L^2(\partial G)$. Therefore $u_n \to u$ in $W^{2m,2}(G)$.

PROPOSITION 2.2. *$I - T(0)$ is a homeomorphism of $W^{2m,2}(G)$ into itself. If $[I - T(t)]v = 0$, $0 < t \leqq 1$, then $||v||_{2m,2} \leqq M$, where M is independent of t.*

Proof. The first assertion follows directly from Theorem 1.1. Suppose that $T(t)v = v$; then v is the solution of the boundary-value problem

$$(A + \lambda I)v = f \quad \text{on } G, \qquad B_j v = \sum_{k=1}^{m} L_{jk}C_k(tv) \quad \text{on } \partial G, \qquad j = 1, \ldots, m.$$

In the first part of the proof of the theorem, we may, instead of considering the operator L_{jk}, take the operator tL_{jk}; then we have that

$$||v||_{2m,2} \leqq M\, ||f||_{0,2},$$

where M is independent of λ, v, and t.

Proof of Theorem 1.2 (*continued*). The operator $T(t)$ satisfies all the conditions of the Leray-Schauder fixed-point theorem (the uniform continuity condition of the theorem is not necessary as observed by Browder in (**7**)). Therefore, $T(1)u = u$. Thus $R(A_2 + \lambda I) = L^2(G)$ and hence, $(A_2 + \lambda I)^{-1}$ exists and is defined on all of $L^2(G)$. Since the injection mapping from $W^{2m,2}(G)$ into $L^2(G)$ is compact, $(A_2 + \lambda I)^{-1}$ is a compact linear mapping of $L^2(G)$ into itself and, moreover, from the a-priori estimate, it follows that

$$||(A_2 + \lambda I)^{-1}|| \leqq M/|\lambda| \quad \text{for } |\lambda| \geqq \lambda_0 > 0.$$

The theorem is proved.

Proof of Theorem 1.3. (1) We establish the a-priori estimate by contradiction. It is similar to the first part of the proof of Theorem 1.2. We obtain a contradiction by using the following estimate of Proposition 16.3 of Agranovič (**2**, p. 101):

$$||u||_{s,2} \leqq M\left\{||Au||_{s-2m,2} + ||u||_{0,2} + \sum_{j=1}^{m} ||B_j u||'_{s-r_j-\frac{1}{2},2}\right\}.$$

(2) By standard arguments, we deduce from the a-priori estimate that A_2 is closed, $N(A_2)$ is of finite dimension, and that $R(A_2)$ is closed in $L^2(G)$. Hence A_2 is a semi-Fredholm operator.

We now show that if Assumption (1) is satisfied, then A_2 is a Fredholm operator and $\operatorname{ind}(A_2) = \dim N(A_2) - \operatorname{codim} R(A_2) = 0$. From Theorem 1.2, we have that

$$(A_2 + \lambda I)(A_2 + \lambda I)^{-1} = I,$$

where I is the identity operator on $L^2(G)$. Thus

$$A_2(A_2 + \lambda I)^{-1} = I - \lambda(A_2 + \lambda I)^{-1}.$$

Since $(A_2 + \lambda I)^{-1}$, considered as a mapping from $L^2(G)$ into itself, is compact, it follows from a well-known argument that $I - \lambda(A_2 + \lambda I)^{-1}$ is a Fredholm operator and $\operatorname{ind}(I - \lambda(A_2 + \lambda I)^{-1}) = 0$. Hence $A_2(A_2 + \lambda I)^{-1}$ is a Fredholm operator and $\operatorname{ind}(A_2(A_2 + \lambda I)^{-1}) = 0$. We can easily show that $R(A_2) = R(A_2(A_2 + \lambda I)^{-1})$ and $N(A_2) = N(A_2(A_2 + \lambda I)^{-1})$. Therefore, $\operatorname{ind}(A_2) = \operatorname{ind}(A_2(A_2 + \lambda I)^{-1}) = 0$.

Proof of Theorem 1.4. Since $(A_2 + \lambda I)^{-1}$ is a compact linear mapping of $L^2(G)$ into itself, the spectrum of A_2 is discrete and the eigenspaces are of finite dimension. With the hypotheses of the theorem, it follows from Theorem 3.2 of Agmon (**1**, pp. 128–129) that the generalized eigenfunctions of A_2 are complete in $L^2(G)$. Indeed, the proof in (**1**) depends only on the compactness of $(A_2 + \lambda I)^{-1}$ and on an estimate on the growth of the resolvent operator as in Theorem 1.2.

Proof of Theorem 1.5. Let v be an element of $W^{2m,2}(G)$ and suppose that $0 \leqq t \leqq 1$. Consider the following elliptic boundary-value problem:

$$(A + \lambda I)u = f(x, tT_1 v, \ldots, tT_{2m} v) \quad \text{on } G,$$

$$B_j u = \sum_{k=1}^{m} L_{jk} C_k u \quad \text{on } \partial G, \qquad j = 1, \ldots, m.$$

Since

$$|f(x, \zeta_1, \ldots, \zeta_{2m})| \leqq M\left\{1 + \sum_{j=1}^{2m-1} |\zeta_j|\right\},$$

$f(x, tT_1 v, \ldots, tT_{2m} v)$ is in $L^2(G)$. Define the non-linear mapping $\mathfrak{T}(t)$ from $[0, 1] \times W^{2m,2}(G)$ into $W^{2m,2}(G)$ as follows:

$$\mathfrak{T}(t)v = u,$$

where u is the unique solution of the above boundary-value problem. It follows from Theorem 1.2 that $\mathfrak{T}(t)$ is well-defined.

To prove the theorem, we show that $\mathfrak{T}(t)$ satisfies the hypotheses of the Leray-Schauder fixed-point theorem. The proof is essentially the same as that given in (**10**). It suffices to note that since T_{2m} is a bounded linear mapping from $W^{2m-\epsilon,2}(G)$ into $L^2(G)$, it is a compact linear mapping from $W^{2m,2}(G)$ into $L^2(G)$.

A similar argument (taking into account Theorem 1.1) gives the existence of a solution in $W^{2m,2}(G)$ of

$$(A + \lambda I)u = f(x, T_1u, \ldots, T_{2m}u) \quad \text{on } G, \quad B_ju = g_j \text{ on } \partial G, \quad j = 1, \ldots, m.$$

Finally, we note that with the estimate on $\|(A_2 + \lambda I)^{-1}\|$ of Theorem 1.2 for all λ with $|\arg \lambda| \leqq \pi/2$, we may show the existence of a solution of a non-local parabolic boundary-value problem of the form

$$\begin{aligned} \frac{\partial u}{\partial t} + Au &= f(x, t) && \text{on } G \times [0, T]; \\ B_ju &= \sum_{k=1}^{m} L_{jk}C_ku && \text{on } \partial G \times [0, T], \qquad j = 1, \ldots, m; \\ u(x, 0) &= u_0(x) && \text{on } G, \end{aligned}$$

by using a result of Sobolevskiĭ (**9**) (cf. **10**).

3. We proceed to prove Theorem 1.1. As usual, we consider first the case of a half-space with A and B_j having constant symbols, then the case when A and B_j have symbols depending on x, but close (in a sense to be specified) to constant symbols, and finally, the case of a bounded open set G of R^n.

THEOREM 3.1. *Let $\{A; B_j, j = 1, \ldots, m\}$ be a regular elliptic boundary-value problem on $R_+^n = \{x\colon x_n > 0\}$. The homogeneous singular integro-differential operators A and B_j are of orders $2m$, r_j ($r_j < 2m - 1$) with constant symbols $\sigma_A(\xi)$ in $W^{q,2}(\Sigma)$; $\sigma_j(\xi')$ in $W^{q-\frac{1}{2},2}(\Sigma')$, $q > (n-1)/2$. Suppose that there exists a θ, $0 \leqq \theta < 2\pi$, for which Assumption* (1) *is verified. Then*

(i) $$|||u|||_{s,2} \leqq M\left\{|||(A + \lambda I)u|||_{s-2m,2} + \sum_{j=1}^{m} |||B_ju|||'_{s-r_j-\frac{1}{2},2}\right\}$$

for all u in $W^{s,2}(R_+^n)$ and for all $|\lambda| \geqq \lambda_0 > 0$, $\arg \lambda = \theta$. M is independent of λ, u and $s \geqq 2m$.

(ii) *The mapping $\mathscr{A}u = \{(A + \lambda I)u, B_1u, \ldots, B_mu\}$ of $W^{s,2}(R_+^n)$ into*

$$W^{s-2m,2}(R_+^n) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(R^{n-1})$$

is 1-1 *and onto for large $|\lambda|$.*

Proof. We follow (**3**) closely (cf. also **2** and **5**).

(i) To prove the a-priori estimate, it suffices to show it for u in $C_c^\infty(R_+^n \cup R^{n-1})$. Since A is an admissible singular integro-differential operator on R_+^n, we have that

$$A = \sum_{k=0}^{2m} A_kD_n^k;$$

similarly,

$$B_j = \sum_{k=0}^{r_j} B_{jk} D_n^{\ k},$$

where A_k and B_{jk} are singular integro-differential operators on R^{n-1}, homogeneous of orders $2m - k$ and $r_j - k$, respectively, with constant symbols.

(a) Consider $(A + \lambda I)u = Lf = f_0(x)$ on R^n, where L is the extension of f to R^n. By taking the Fourier transform, we obtain

$$(\sigma_A(\xi) + \lambda)\hat{u} = \hat{f}_0(\xi) = \left(\sum_{k=0}^{2m} \sigma_k(\xi')\xi_n^{\ k} + \lambda \right)\hat{u}.$$

A computation as in (3) yields $|||u|||_{s,2} \leqq C|||f|||_{s-2m,2}$.

(b) Consider the boundary-value problem:

$$(A + \lambda I)w = 0 \text{ on } R_+^{\ n}, \qquad B_j w = g_j - B_j u \text{ on } R^{n-1}, \quad j = 1, \ldots, m.$$

By taking the Fourier transform with respect to the tangential variables $\hat{x} = (x_1, \ldots, x_{n-1})$, we obtain

$$\sum_{k=0}^{2m} \sigma_k(\xi') D_n^{\ k} \hat{w}(\xi', x_n) + \lambda \hat{w}(\xi', x_n) = 0, \qquad x_n > 0,$$

$$\sum_{k=0}^{r_j} \sigma_{jk}(\xi') D_n^{\ k} \hat{w}(\xi', 0) = \hat{h}_j(\xi') = \hat{g}_j - \sum_{k=0}^{r_j} \sigma_{jk}(\xi') D_n^{\ k} \hat{u}(\xi', 0), \qquad j = 1, \ldots, m,$$

where $\hat{w}$ and $\hat{g}_j$ denote the Fourier transforms of w and g_j with respect to $\hat{x}$. We seek a solution of the form

$$\hat{w}(\xi', x_n) = \sum_{\tau=1}^{m} p_\tau(\xi') \int_{C_{\lambda,\xi'}} \zeta^{\tau-1} \exp(i\zeta x_n)[\sigma_A(\xi', \zeta) + \lambda]^{-1} d\zeta,$$

where $C_{\lambda,\xi'}$ is a closed Jordan rectifiable curve in the upper half ζ-plane, containing in its interior all the m roots of

$$\lambda + \sum_{k=0}^{2m} \sigma_k(\xi')\zeta^k = 0,$$

considered as a polynomial in ζ. We are reduced to showing the solvability of a system of m equations with m unknowns, $p_\tau(\xi')$. Since Assumption (1) is verified, the system may be solved in a unique fashion. If we set

$$c_{\tau j}(\xi', \lambda) = \int_{C_{\lambda,\xi'}} \zeta^{\tau-1} \sigma_j(\xi', \zeta)[\sigma_A(\xi', \zeta) + \lambda]^{-1} d\zeta$$

and if $Q_{\tau j}(\xi', \lambda)$ are the elements of the inverse of the transpose of the matrix $(c_{\tau j})$, then

$$\hat{w}(\xi', x_n) = \sum_{\tau, j=1}^{m} Q_{\tau j}(\xi', \lambda)\hat{h}_j(\xi', \lambda) \int_{C_{\lambda,\xi'}} \zeta^{\tau-1} \exp(i\zeta x_n)[\sigma_A(\xi', \zeta) + \lambda]^{-1} d\zeta.$$

To take the inverse Fourier transform of $\hat{w}(\xi', x_n)$, we need the following lemma.

LEMMA 3.1. (i) *Let*

$$\phi_{\alpha\beta}(\xi, x_n) = \int_C \zeta^\alpha \xi^\beta \exp(i\zeta x_n)[\sigma_A(\xi, \zeta) + \lambda]^{-1} d\zeta.$$

Then

$$\phi_{\alpha\beta}(\xi, x_n) = O(|\xi| + |\lambda|^{1/2m})^{\alpha+\beta+1-2m} \exp(-dx_n(|\xi|^2 + |\lambda|^{1/m})^{\frac{1}{2}}),$$

where $d = \min\{\operatorname{Im}\zeta : \zeta \in C\} > 0$.

(ii) $Q_{rj}(\xi, \lambda) = O(|\xi| + |\lambda|^{1/2m})^{2m-r-r_j}$, $r, j = 1, \ldots, m$.

Proof. Set $\lambda = \mu^{2m}$ and make the following change of variables:

$$\xi' = \xi(|\xi|^2 + |\mu|^2)^{-\frac{1}{2}}, \qquad \mu' = \mu(|\xi|^2 + |\mu|^2)^{-\frac{1}{2}}, \qquad \zeta' = \zeta(|\xi|^2 + |\mu|^2)^{-\frac{1}{2}}.$$

(1) We have that

$$\phi_{\alpha\beta}(\xi, x_n) = (|\xi|^2 + |\mu|^2)^{(\alpha+\beta+1-2m)/2} \phi_{\alpha\beta}(\xi', x_n(|\xi|^2 + |\mu|^2)^{\frac{1}{2}}),$$

where

$$\phi_{\alpha\beta}(\xi, x_n) = \int_C \zeta^\alpha \xi^\beta \exp(i\zeta x_n)[\sigma_A(\xi, \zeta) + \mu^{2m}]^{-1} d\zeta.$$

(i) As $|\xi| \to \infty$, $|\xi'| \to 1$ and $|\mu'| \to 0$. Thus, the roots with positive imaginary parts of

$$(\mu')^{2m} + \sum_{k=0}^{2m} \sigma_k(\xi')\zeta^k = 0$$

tend continuously to those of

$$\sum_{k=0}^{2m} \sigma_k(I)\zeta^k = 0.$$

Hence, there exists a closed curve C_1 independent of μ and ξ containing all the m roots with positive imaginary parts of

$$(\mu)^{2m} + \sum_{k=0}^{2m} \sigma_k(\xi)\zeta^k = 0 \quad \text{for large } |\xi|.$$

Therefore, for large $|\xi|$, we have that

$$|\phi_{\alpha\beta}(\xi, x_n)| \leqq M \exp(-dx_n(|\xi|^2 + |\mu|^2)^{\frac{1}{2}})(|\xi|^2 + |\mu|^2)^{(\alpha+\beta+1-2m)/2}.$$

(ii) For small $|\xi|$, as $|\xi| \to 0$, $|\xi'| \to 0$ and $|\mu'| \to 1$. Thus, all the roots with positive imaginary parts of $(\mu')^{2m} + \sigma_A(\xi', \zeta) = 0$ tend continuously to those with positive imaginary parts of $1 + \sigma_A(0, \zeta) = 0$. Again, we have a curve C_2, in the upper half ζ-plane, independent of both μ and ξ containing all the m roots of $(\mu')^{2m} + \sigma_A(\xi', \zeta) = 0$ for small $|\xi|$. Thus,

$$\phi_{\alpha\beta}(\xi, x_n) \leqq M \exp(-dx_n(|\xi|^2 + |\mu|^2)^{\frac{1}{2}})(|\xi|^2 + |\mu|^2)^{(\alpha+\beta+1-2m)/2}.$$

Combining (i) and (ii) we obtain the first part of the lemma.

(2) Arguing as above, we have that

$$Q_{rj}(\xi, \lambda) = O(|\xi| + |\lambda|^{1/m})^{2m-r-r_j}.$$

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Proof of Theorem 3.1. (i) (continued). As in (**3**), using Lemma 3.1 and the Parseval formula, we obtain:

$$|||w|||_{s,2} \leqq C \sum_{j=1}^{m} |||h_j|||'_{s-r_j-\frac{1}{2},2}.$$

Thus

$$|||w|||_{s,2} \leqq C\left\{|||u|||_{s,2} + \sum_{i=1}^{m} |||g_j|||'_{s-r_j-\frac{1}{2},2}\right\} \leqq$$

$$C\left\{|||f|||_{s-2m,2} + \sum_{j=1}^{m} |||g_j|||'_{s-r_j-\frac{1}{2}}\right\}.$$

Therefore, if v is such that $(A + \lambda I)v = f$ on R_+^n, $B_j v = g_j$ on R^{n-1}, we obtain

$$|||v|||_{s,2} \leqq C\left\{|||f|||_{s-2m,2} + \sum_{j=1}^{m} |||g_j|||'_{s-r_j-\frac{1}{2},2}\right\}.$$

(ii) Let $(f, g_1, \ldots, g_m)$ be an element of

$$W^{s-2m,2}(R_+^n) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(R^{n-1}).$$

Then the unique solution u in $W^{s,2}(R_+^n)$ of

$$(A + \lambda I)u = f \quad \text{on } R_+^n, \qquad B_j u = g_j \quad \text{on } R^{n-1}, \quad j = 1, \ldots, m,$$

is given by

$$u(x) = F^{-1}\{[\sigma_A(\xi) + \lambda]^{-1}F(Lf)\}\big|_{R_+^n}$$
$$+ \sum_{j=1}^{m} (F')^{-1}\left\{\sum_{\tau=1}^{m} Q_{\tau j} \int_C \zeta^{\tau-1}\exp(i\zeta x_n)[\sigma_A(\xi', \zeta) + \lambda]^{-1}d\zeta\right\}F'g_j\Big|_{R_+^n},$$

where F' denotes the Fourier transform with respect to $\hat{x}$.

Because of Lemma 3.1, the expression is well-defined.

THEOREM 3.2. *Let* $\{A; B_j, j = 1, \ldots, m\}$ *be a regular elliptic boundary-value problem on* R_+^n. *The singular integro-differential operators* A *and* B_j *are of orders* $2m$ *and* r_j $(r_j < 2m - 1)$, *respectively. Suppose that there exists a* θ, $0 \leqq \theta < 2\pi$, *for which Assumption* (1) *is satisfied. Suppose further that*

$$\max_x ||\sigma_A(\xi, x) - \sigma_A(\xi, 0)||_{q,2} + \sum_{j,k} \max_x ||\sigma_{jk}(x', \xi') - \sigma_{jk}(0, \xi')||_{q-\frac{1}{2},2} \leqq \delta$$

for x *near* 0. *Then*

(1) *There exists a constant* M *independent of* λ, $\arg \lambda = \theta$, *and of* u *such that*

$$|||u|||_{s,2} \leqq M\left\{|||(A + \lambda)u|||_{s-2m,2} + \sum_{j=1}^{m} |||B_j u|||'_{s-r_j-\frac{1}{2},2}\right\} s \geqq 2m;$$

(2) *For every* $(f, g_1, \ldots, g_m)$ *in*

$$W^{s-2m,2}(R_+^n) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(R^{n-1})$$

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there exists a unique solution u in $W^{s,2}(R_+^n)$ *of* $(A+\lambda)u = f$ *on* R_+^n; $B_j u = g_j$ *on* R^{n-1}, $j = 1, \ldots, m$.

Proof. We prove the a-priori estimate. We denote by A_0 and B_{j0} the principal parts of A and B_j, and by $A_0(0)$ and $B_{j0}(0)$ the homogeneous singular integro-differential operators with symbols $\sigma_A(0, \xi)$ and $\sigma_j(0, \xi')$. From Theorem 3.1, we obtain

$$|||u|||_{s,2} \leqq M\left\{|||(A_0(0)+\lambda)u|||_{s-2m,2} + \sum_{j=1}^{m} |||B_{j0}(0)u|||'_{s-r_j-\frac{1}{2},2}\right\}$$
$$\leqq M\Big\{|||(A+\lambda)u|||_{s-2m,2} + |||(A_0(0)-A_0)u|||_{s-2m,2}$$
$$+ |||(A-A_0)u|||_{s-2m,2} + \sum_{j=1}^{m} |||B_j u|||'_{s-r_j-\frac{1}{2},2} + |||(B_{j0}-B_j)u|||'_{s-r_j-\frac{1}{2},2}$$
$$+ |||(B_{j0}-B_{j0}(0))u|||'_{s-r_j-\frac{1}{2},2}\Big\}.$$

(i) Since A is an admissible singular integro-differential operator on R_+^n, it may be written as: $A = R\tilde{A}L + T$, where T is an operator almost of order $2m-1$ on $W^{s,2}(R_+^n)$.

Therefore, $||(A-A_0)u||_{s-2m,2} \leqq \epsilon||u||_{s,2} + C(\epsilon)||u||_{s-1,2}$ and

$$|\lambda|^{(s-2m)/2m}||(A-A_0)u||_{0,2} \leqq \epsilon|\lambda|^{(s-2m)/2m}||u||_{2m,2} + C(\epsilon)|\lambda|^{(s-2m)/2m}||u||_{2m-1,2}.$$

But

$$||u||_{2m-1,2} \leqq \epsilon/C(\epsilon)||u||_{2m,2} + K(\epsilon)||u||_{0,2},$$
$$|||(A-A_0)u|||_{s-2m,2} \leqq 2\epsilon|||u|||_{s,2} + C_2(\epsilon)|\lambda|^{-1/2m}|||u|||_{s,2}.$$

(ii) Similarly,

$$B_j = \sum_{k=0}^{r_j} B_{jk}D_n^k + \sum_{k=0}^{r_j} T_{jk}D_n^k,$$

where T_{jk} are linear operators almost of orders $r_j - k - 1$ on $W^{s-\frac{1}{2},2}(R^{n-1})$. Thus

$$|||(B_j - B_{j0})u|||'_{s-r_j-\frac{1}{2},2} \leqq \epsilon|||u|||'_{s-\frac{1}{2},2} + C_3(\epsilon)|\lambda|^{-1/2m}|||u|||_{s,2}.$$

(iii) We consider $|||(A_0 - A_0(0))u|||_{s-2m,2}$. If $\sigma_A(x,\xi)$ is the symbol of A, then the symbol $\sigma_{\tilde{A}}(x,\xi)$ of $\tilde{A}$ may be obtained from $\sigma_A(x,\xi)$ by the Hestenes formula and, moreover,

$$\max_x||\sigma_{\tilde{A}}(x,\xi) - \sigma_{\tilde{A}}(0,\xi)||_{q,2} \leqq C\max_x||\sigma_A(x,\xi) - \sigma_A(0,\xi)||_{q,2},$$

where C does not depend on σ_A. Thus

$$|||(A_0 - A_0(0))u|||_{s-2m,2} \leqq C_2|||(\tilde{A}_0 - \tilde{A}_0(0))Lu|||_{s-2m,2}.$$

Using Proposition 8.3 of Agranovič (**2**, p. 47), we have that

$$|||(\tilde{A}_0 - \tilde{A}_0(0))Lu|||_{s-2m,2} \leqq C_3\delta|||u|||_{s,2} + C_4(\sigma_A)|\lambda|^{-1/2m}|||u|||_{s,2}.$$

(iv) A similar argument yields:

$$|||(B_{j0} - B_{j0}(0))u|||'_{s-r_j-\frac{1}{2},2} \leqq C\delta|||u|||_{s,2} + C_5(\sigma_j)|\lambda|^{-1/2m}|||u|||_{s,2}.$$

Therefore, by taking δ small and $|\lambda|$ sufficiently large, we obtain the a-priori estimate of the theorem.

(2) We now show that $\mathscr{A}$ has a right inverse. It follows from Theorem 3.1 that $\mathscr{A}_0(0)$ has a right inverse $\mathfrak{T}_0$; thus

$$\begin{aligned}\mathscr{A}\mathfrak{T}_0 &= \mathscr{A}_0(0)\mathfrak{T}_0 + (\mathscr{A} - \mathscr{A}_0)\mathfrak{T}_0 + (\mathscr{A}_0 - \mathscr{A}_0(0))\mathfrak{T}_0 \\ &= I + (\mathscr{A} - \mathscr{A}_0)\mathfrak{T}_0 + (\mathscr{A}_0 - \mathscr{A}_0(0))\mathfrak{T}_0.\end{aligned}$$

Set

$$g = (g_1, \ldots, g_m) \quad \text{and} \quad |||(f, g)|||_{s,2} = |||f|||_{s-2m,2} + \sum_{j=1}^{m} |||g_j|||'_{s-r_j-\frac{1}{2},2}.$$

Let $u = \mathfrak{T}_0(f, g)$ with $\mathscr{A}_0(0)\mathfrak{T}_0(f, g) = (f, g)$ (Theorem 3.1). Then a computation, as in the first part, yields

$$|||(\mathscr{A}_0 - \mathscr{A}_0(0))\mathfrak{T}_0(f, g)|||_{s,2} \leqq \tfrac{1}{4}|||(f, g)|||_{s,2}$$

for δ small and $|\lambda|$ sufficiently large. Also,

$$|||(\mathscr{A} - \mathscr{A}_0)\mathfrak{T}_0(f, g)||| \leqq \tfrac{1}{4}|||(f, g)|||_{s,2}$$

since $(\mathscr{A} - \mathscr{A}_0)$ is an operator almost of order -1 from $W^{s,2}(R_+^n)$ into $W^{s-2m,2}(R_+^n) \times \Pi_{j=1}^m W^{s-r_j-\frac{1}{2},2}(R^{n-1})$. Let

$$Q = (\mathscr{A} - \mathscr{A}_0)\mathfrak{T}_0 + (\mathscr{A}_0 - \mathscr{A}_0(0))\mathfrak{T}_0;$$

then $|||Q(f, g)|||_{s,2} \leqq \frac{1}{2}|||(f, g)|||_{s,2}$. Hence $(I + Q)^{-1}$ exists. Take $\mathfrak{T} = \mathfrak{T}_0(I + Q)^{-1}$, then $\mathscr{A}\mathfrak{T} = I$.

Proof of Theorem 1.1. (1) We establish the a-priori estimate. Since G is a bounded open set of R^n, regular of class C^∞ (cf. **5**), there exist a finite open covering of cl(G) and a finite partition of unity ϕ_k corresponding to N_k. Let ψ_k be an infinitely differentiable function with compact support in N_k such that $\psi_k = 1$ on the support of ϕ_k. We have that

$$\mathscr{A} = \sum_{k=1}^{N} \phi_k \mathscr{A}_k \psi_k + T,$$

where T is an operator almost of order -1 from $W^{s,2}(G)$ into

$$W^{s-2m,2}(G) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(\partial G) \quad \text{and} \quad \mathscr{A}_k = (A_k + \lambda I, B_{1k}, \ldots, B_{mk}),$$

where A_k and B_{jk} are singular integro-differential operators on R_+^n and on R^{n-1}, respectively. We also have that

$$\mathscr{A}_k(\phi_k u) = \mathscr{A}_k(\phi_k\psi_k u) = \phi_k\mathscr{A}_k(\psi_k u) + T_k(\psi_k u),$$

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where T_k is an operator almost of order -1 from $W^{s,2}(R_+^n)$ into

$$W^{s-2m,2}(R_+^n) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(R^{n-1})$$

if N_k is a boundary neighbourhood and T_k is an operator almost of order -1 from $W^{s,2}(R_+^n)$ into $W^{s-2m,2}(R_+^n)$ if N_k is an interior neighbourhood (cf. **2**).

From Theorem 3.2 and an easy computation we obtain

$$|||\phi_k u|||_{s,2} \leqq M\Big\{|||\phi_k(A_k + \lambda)\psi_k u|||_{s-2m,2} + \epsilon|||\psi_k u|||_{s,2} + C(\epsilon)|\lambda|^{-1/2m}|||\psi_k u|||_{s,2} + \sum_{j=1}^{m} |||\phi_k B_{jk}(\psi_k u)|||'_{s-r_j-\frac{1}{2},2}\Big\}.$$

The norms are taken in local coordinates. On the other hand, we have that

$$\phi_k \mathscr{A}_k(\psi_k u) = \phi_k \mathscr{A}(\psi_k u) + \phi_k \tilde{T}_k(\psi_k u),$$

where $\tilde{T}_k$ is an operator of the same type as T_k. Therefore

$$|||\phi_k u|||_{s,2} \leqq M\Big\{|||\phi_k(A + \lambda)(\psi_k u)|||_{s-2m,2} + \epsilon|||\psi_k u|||_{s,2} + C(\epsilon)|\lambda|^{-1/2m}|||\psi_k u|||_{s,2} + \sum_{j=1}^{m} |||\phi_k B_j(\psi_k u)|||'_{s-r_j-\frac{1}{2},2}\Big\}.$$

We may write $\phi_k(A + \lambda)(\psi_k u) = \phi_k(A + \lambda)u + \phi_k(A + \lambda)(\psi_k - 1)u$ and, similarly, for $\phi_k B_{jk}(\psi_k u)$. The operator $\phi_k \mathscr{A}(\psi_k - 1)$ is again an operator almost of order -1 from $W^{s,2}(G)$ into

$$W^{s-2m,2}(G) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(\partial G).$$

Hence we finally obtain

$$|||u|||_{s,2} \leqq M\Big\{|||(A + \lambda)u|||_{s-2m,2} + \epsilon|||u|||_{s,2} + C(\epsilon)|\lambda|^{-1/2m}|||u|||_{s,2} + \sum_{j=1}^{m} |||B_j u|||'_{s-r_j-\frac{1}{2},2}\Big\}.$$

Taking ϵ small and $|\lambda|$ sufficiently large, we obtain the a-priori estimate.

(2) We now construct the inverse of $\mathscr{A}$. We have that

$$\mathscr{A}u = \sum_{k=1}^{N} \phi_k \mathscr{A}_k(\psi_k u) + Tu.$$

For each k, $\mathscr{A}_k$ has a right inverse R_k (Theorem 3.2). To simplify the notation, we write $g = (g_1, \ldots, g_m)$. Consider

$$R(f, g) = \sum_{r=1}^{N} \psi_r R_r(\phi_r f, \phi_r g).$$

R is a bounded linear operator from $W^{s-2m,2}(G) \times \Pi_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(\partial G)$ into $W^{s,2}(G)$. We have that

$$\mathscr{A}R(f,g) = \sum_{r,k=1}^{N} \phi_k \mathscr{A}_k[\psi_r R_r(\phi_r f, \phi_r g)\psi_k] + TR(f,g).$$

Set $u_r = \psi_r R_r(\phi_r f, \phi_r g)$. We also have that

$$\phi_k \mathscr{A}_k[\psi_k \psi_r u_r] = \phi_k \mathscr{A}_r[\psi_k \psi_r u_r] + \phi_k T_{rk} u_r$$

(cf. **2**, pp. 102, 75) T_{rk} is an operator almost of order -1 from $W^{s,2}(G)$ into

$$W^{s-2m,2}(G) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(\partial G).$$

Hence

$$\begin{aligned} \mathscr{A}R(f,g) = \sum_{r,k} & \phi_k \psi_k \psi_r \mathscr{A}_r R_r(\phi_r f, \phi_r g) + TR(f,g) \\ & + \sum_{r,k} \phi_k T_{rk}[\psi_r R_r(\phi_r f, \phi_r g)] \\ & + \sum_{r,k} \phi_k \{\mathscr{A}_r[\psi_k \psi_r R_r(\phi_r f, \phi_r g)] - \psi_k \psi_r \mathscr{A}_r R_r(\phi_r f, \phi_r g)\}. \end{aligned}$$

Consider the first sum. It is equal to (f, g). Set

$$|||(f,g)|||_s = |||f|||_{s-2m,2} + \sum_{j=1}^{m} |||g_j|||'_{s-r_j-\frac{1}{2},2}.$$

Then

$$|||TR(f,g)|||_s \leqq \epsilon|||(f,g)|||_s + C(\epsilon)|\lambda|^{-1/2m}|||(f,g)|||_s.$$

In a similar fashion, we obtain the same bound for the third sum. Since $\mathscr{A}_r[\psi_k \psi_r \cdot] - \psi_r \psi_k \mathscr{A}_r[\cdot]$ is an operator almost of order -1 from $W^{s,2}(R_+^n)$ into

$$W^{s-2m,2}(R_+^n) \times \prod_{j=1}^{m} W^{s-r_j-\frac{1}{2},2}(R^{n-1}),$$

we obtain the following upper bound for the last sum, namely,

$$\epsilon|||(f,g)|||_s + C(\epsilon)|\lambda|^{-1/2m}|||(f,g)|||_s.$$

Thus $\mathscr{A}R(f, g) = (f, g) + \mathfrak{T}(f, g)$ with $||\mathfrak{T}|| \leqq \frac{1}{2}$ for large $|\lambda|$. Hence $(I + \mathfrak{T})^{-1}$ exists and $\mathscr{A}^{-1} = R(I + \mathfrak{T})^{-1}$.

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