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The Weak Type $(1, 1)$ Estimates of Maximal Functions on the Laguerre Hypergroup

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Abstract. In this paper, we discuss various maximal functions on the Laguerre hypergroup **K** including the heat maximal function, the Poisson maximal function, and the Hardy–Littlewood maximal function which is consistent with the structure of hypergroup of **K**. We shall establish the weak type $(1, 1)$ estimates for these maximal functions. The L^p estimates for $p > 1$ follow from the interpolation. Some applications are included.

1 Introduction

It is well known that various maximal functions play a very important role in harmonic analysis. The most common way is to establish first the weak type $(1, 1)$ estimates for the Hardy–Littlewood maximal function by using of some kind of covering lemmas (*cf.* [4, 8]) and then control other maximal functions by the Hardy–Littlewood maximal function. But this method can not be used for the Laguerre hypergroup **K**, because there is not a suitable quasi-distance on **K**. In this paper, we shall discuss various maximal functions on the Laguerre hypergroup **K** including the heat maximal function, the Poisson maximal function, and the Hardy–Littlewood maximal function which is consistent with the structure of hypergroup of **K**. Starting with the heat semigroup, we first establish the weak type $(1, 1)$ estimate for the Poisson maximal function by using of the Hopf–Dunford–Schwartz ergodic theorem as in [7]. Then we use the Poisson maximal function to control the Hardy–Littlewood maximal function, which, in turn, gives the control of the heat maximal function.

The paper is organized as follows. In Section 2, we collect some results about the Laguerre hypergroup, which we will use in the sequel. In Section 3, we give the explicit expressions of the heat kernel and the Poisson kernel. The pointwise estimates of both kernels are derived from their expressions. The weak type $(1, 1)$ estimates for various maximal functions are given in Section 4. Some applications are also included.

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2 Preliminaries

In this section, we set some notations and collect some basic results about the Laguerre hypergroup. We refer the reader to [12], [6] and [9] for detail.

Let $\mathbf{K} = [0, \infty) \times \mathbf{R}$ equipped with the measure

$$dm_\alpha(x,t) = \frac{1}{\pi\Gamma(\alpha+1)} x^{2\alpha+1}\, dx\, dt, \qquad \alpha \geq 0.$$

We denotes by $L^p_\alpha(\mathbf{K})$ the spaces of measurable functions on $\mathbf{K}$ such that $\|f\|_{\alpha,p} < +\infty$, where

$$\|f\|_{\alpha,p} = \Big(\int_{\mathbf{K}} |f(x,t)|^p\, dm_\alpha(x,t) \Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$
$$\|f\|_{\alpha,\infty} = \operatorname{esssup}_{(x,t)\in\mathbf{K}} |f(x,t)|.$$

For $(x,t) \in \mathbf{K}$, the generalized translation operators $T^{(\alpha)}_{(x,t)}$ are defined by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta}, s+t+xy\sin\theta)\, d\theta & \text{if } \alpha = 0, \\ \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f(\sqrt{x^2+y^2+2xyr\cos\theta}, s+t+xyr\sin\theta) r(1-r^2)^{\alpha-1}\, dr\, d\theta & \text{if } \alpha > 0. \end{cases}$$

It is known that $T^{(\alpha)}_{(x,t)}$ satisfies

$$\|T^{(\alpha)}_{(x,t)} f\|_{\alpha,p} \leq \|f\|_{\alpha,p}. \tag{2.1}$$

Let $M_b(\mathbf{K}\times\mathbf{K})$ denote the space of bounded Radon measures on $\mathbf{K}$. The convolution on $M_b(\mathbf{K})$ is defined by

$$(\mu * \nu)(f) = \int_{\mathbf{K}\times\mathbf{K}} T^{(\alpha)}_{(x,t)} f(y,s)\, d\mu(x,t)\, d\nu(y,s).$$

It is easy to see that $\mu * \nu = \nu * \mu$. If $f, g \in L^1_\alpha(\mathbf{K})$ and $\mu = fm_\alpha$, $\nu = gm_\alpha$, then $\mu * \nu = (f * g)m_\alpha$, where $f * g$ is the convolution of functions f and g defined by

$$(f * g)(x,t) = \int_{\mathbf{K}} T^{(\alpha)}_{(x,t)} f(y,s) g(y,-s)\, dm_\alpha(y,s).$$

The following lemma follows from (2.1).

Lemma 2.1 *Let $f \in L^1_\alpha(\mathbf{K})$ and $g \in L^p_\alpha(\mathbf{K})$, $1 \leq p \leq \infty$. Then*

$$\|f * g\|_{\alpha,p} \leq \|f\|_{\alpha,1}\, \|g\|_{\alpha,p}.$$

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$(\mathbf{K},*,i)$ is a hypergroup in the sense of Jewett (*cf.* [5], [1]), where i denotes the involution defined by $i(x,t)=(x,-t)$. If $\alpha=n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbf{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathbf{H}^n$.

The dilations on $\mathbf{K}$ are defined by

$$\delta_r(x,t)=(rx,r^2t),\quad r>0.$$

It is clear that the dilations are consistent with the structure of hypergroup. Let

$$f_r(x,t)=r^{-(2\alpha+4)}f\Big(\frac{x}{r},\frac{t}{r^2}\Big). \tag{2.2}$$

Then we have

$$\|f_r\|_{\alpha,1}=\|f\|_{\alpha,1}.$$

We also introduce a homogeneous norm defined by $|(x,t)|=(x^4+4t^2)^{\frac14}$ (*cf.* [10]). Then we can define the ball centered at $(0,0)$ of radius r, *i.e.*, the set $B_r=\{(x,t)\in\mathbf{K}:|(x,t)|<r\}$.

Remark 2.2 It seems difficult to define balls or cubes on $\mathbf{K}$ as on the Euclidean space. This is the reason that we don't have suitable covering lemmas.

Let $f\in L^1_\alpha(\mathbf{K})$. Set $x=\rho(\cos\theta)^{\frac12}$, $t=\frac12\rho^2\sin\theta$. We get

$$\int_{\mathbf{K}}f(x,t)\,dm_\alpha(x,t)$$
$$=\frac{1}{2\pi\Gamma(\alpha+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\infty f\Big(\rho(\cos\theta)^{\frac12},\frac12\rho^2\sin\theta\Big)\rho^{2\alpha+3}(\cos\theta)^\alpha\,d\rho\,d\theta.$$

If f is radial, *i.e.*, there is a function ψ on $[0,\infty)$ such that $f(x,t)=\psi\big(|(x,t)|\big)$, then

$$\begin{aligned}\int_{\mathbf{K}}f(x,t)\,dm_\alpha(x,t)&=\frac{1}{2\pi\Gamma(\alpha+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta)^\alpha\,d\theta\int_0^\infty\psi(\rho)\rho^{2\alpha+3}\,d\rho\\&=\frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}\int_0^\infty\psi(\rho)\rho^{2\alpha+3}\,d\rho.\end{aligned}$$

Specifically,

$$m_\alpha(B_r)=\frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}r^{2\alpha+4}. \tag{2.3}$$

We consider the partial differential operator

$$L=-\Big(\frac{\partial^2}{\partial x^2}+\frac{2\alpha+1}{x}\frac{\partial}{\partial x}+x^2\frac{\partial^2}{\partial t^2}\Big).$$

L is positive and symmetric in $L^2_\alpha(\mathbf{K})$, and is homogeneous of degree 2 with respect to the dilations defined above. When $\alpha = n-1$, L is the radial part of the sublaplacian on the Heisenberg group $\mathbf{H}^n$. We call L the generalized sublaplacian.

Let $L^{(\alpha)}_m$ be the Laguerre polynomial of degree m and order α defined in terms of the generating function by

$$\sum_{m=0}^{\infty} s^m L^{(\alpha)}_m(x) = \frac{1}{(1-s)^{\alpha+1}} \exp\Big(-\frac{xs}{1-s}\Big). \tag{2.4}$$

For $(\lambda, m) \in \mathbf{R} \times \mathbf{N}$, we put

$$\varphi_{(\lambda,m)}(x,t) = \frac{m!\,\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} e^{i\lambda t} e^{-\frac{1}{2}|\lambda| x^2} L^{(\alpha)}_m(|\lambda| x^2).$$

The following proposition summarizes some basic properties of functions $\varphi_{(\lambda,m)}$.

Proposition 2.3 *The function $\varphi_{(\lambda,m)}$ satisfies that*

(i) $\|\varphi_{(\lambda,m)}\|_{\alpha,\infty} = \varphi_{(\lambda,m)}(0,0) = 1$,
(ii) $\varphi_{(\lambda,m)}(x,t)\varphi_{(\lambda,m)}(y,s) = T^{(\alpha)}_{(x,t)}\varphi_{(\lambda,m)}(y,s)$,
(iii) $L\varphi_{(\lambda,m)} = 4|\lambda|(m + \frac{\alpha+1}{2})\varphi_{(\lambda,m)}$.

Let $f \in L^1_\alpha(\mathbf{K})$, the generalized Fourier transform of f is defined by

$$\widehat{f}(\lambda, m) = \int_{\mathbf{K}} f(x,t)\varphi_{(-\lambda,m)}(x,t)\, dm_\alpha(x,t).$$

It is easy to known that

$$(f * g)\widehat{\ }(\lambda, m) = \widehat{f}(\lambda, m)\widehat{g}(\lambda, m)$$

and

$$\widehat{f_r}(\lambda, m) = \widehat{f}(r^2\lambda, m).$$

Let $d\gamma_\alpha$ be the positive measure defined on $\mathbf{R} \times \mathbf{N}$ by

$$\int_{\mathbf{R}\times\mathbf{N}} g(\lambda, m)\, d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)} \int_{\mathbf{R}} g(\lambda, m)|\lambda|^{\alpha+1}\, d\lambda.$$

Write $L^p_\alpha(\widehat{\mathbf{K}})$ instead of $L^p(\mathbf{R}\times\mathbf{N}, d\gamma_\alpha)$. We have the following Plancherel formula.

$$\|f\|_{\alpha,2} = \|\widehat{f}\|_{L^2_\alpha(\widehat{\mathbf{K}})}, \quad f \in L^1_\alpha(\mathbf{K}) \cap L^2_\alpha(\mathbf{K}).$$

Then the generalized Fourier transform can be extended to the tempered distributions. We also have the inverse formula of the generalized Fourier transform.

$$f(x,t) = \int_{\mathbf{R}\times\mathbf{N}} \widehat{f}(\lambda, m)\varphi_{\lambda,m}(x,t)\, d\gamma_\alpha(\lambda, m)$$

provided $\widehat{f} \in L^1_\alpha(\widehat{\mathbf{K}})$.

3 Heat Kernel and Poisson Kernel

Let $\{H^s\} = \{e^{-sL}\}$ be the heat semigroup generated by L. There is a unique smooth function $h\big((x,t),s\big) = h_s(x,t)$ on $\mathbf{K} \times (0,+\infty)$ such that

$$H^s f(x,t) = f * h_s(x,t)$$

where h_s is the heat kernel associated to L. The following proposition due to Stempak [9].

Proposition 3.1 *The heat kernel h_s satisfies that*

(i) $(\partial_s + L)h_s = 0$, *on* $\mathbf{K} \times (0,+\infty)$,
(ii) $h_s(x,t) \geq 0$, $h_s(x,-t) = h_s(x,t)$, $\int_{\mathbf{K}} h_s(x,t)\, dm_\alpha(x,t) = 1$,
(iii) $h_{s_1} * h_{s_2} = h_{s_1+s_2}$,
(iv) $h_{r^2 s}(rx, r^2 t) = r^{-(2\alpha+4)} h_s(x,t)$.

Remark 3.2 We recall that there are a confusion to some extent between the notation h_s and the dilation (2.2). In fact, let $f = h_1$, then $h_s = f_r$ with $s = r^2$.

As a direct consequence of Proposition 3.1, we have:

Lemma 3.3 *The heat semigroup $\{H^s\}$ satisfies that*

(i) $\lim_{s\to 0} \|H^s f - f\|_{\alpha,p} = 0$ *for* $f \in L^p_\alpha(\mathbf{K})$, $1 \leq p < \infty$,
(ii) $\|H^s f\|_{\alpha,p} \leq \|f\|_{\alpha,p}$, $1 \leq p \leq \infty$,
(iii) H^s *is self-adjoint on* $L^2_\alpha(\mathbf{K})$,
(iv) $H^s f \geq 0$ *if* $f \geq 0$,
(v) $H^s 1 = 1$.

We shall give an explicit expression of the heat kernel h_s in terms of Euclidean Fourier transform with respect to the variable t.

Lemma 3.4

$$h_s(x,t) = \int_{\mathbf{R}} \Big(\frac{\lambda}{2\sinh(2\lambda s)}\Big)^{\alpha+1} e^{-\frac{1}{2}\lambda\coth(2\lambda s)x^2} e^{i\lambda t}\, d\lambda.$$

Proof It is known that

$$\widehat{h_s}(\lambda, m) = e^{-4(m+\frac{\alpha+1}{2})|\lambda| s}$$

(*cf.* [9]). By the inverse formula of the generalized Fourier transform,

$$\begin{aligned} h_s(x,t) &= \int_{\mathbf{R}\times\mathbf{N}} \widehat{h_s}(\lambda,m)\varphi_{\lambda,m}(x,t)\, d\gamma_\alpha(\lambda,m) \\ &= \int_{\mathbf{R}} \sum_{m=0}^{\infty} e^{-4(m+\frac{\alpha+1}{2})|\lambda| s} e^{-\frac{1}{2}|\lambda| x^2} L_m^{(\alpha)}(|\lambda| x^2)|\lambda|^{\alpha+1} e^{i\lambda t}\, d\lambda \\ &= \int_{\mathbf{R}} h_s^\lambda(x) e^{i\lambda t}\, d\lambda. \end{aligned}$$

By the generating function identity (2.4) for the Laguerre polynomials, we have

$$
\begin{aligned}
h_s^\lambda(x) &= \Big(\sum_{m=0}^{\infty} e^{-4m|\lambda|s} L_m^{(\alpha)}(|\lambda|x^2)\Big)\, e^{-2(\alpha+1)|\lambda|s} e^{-\frac{1}{2}|\lambda|x^2} |\lambda|^{\alpha+1} \\
&= \frac{1}{(1-e^{-4|\lambda|s})^{\alpha+1}} \exp\Big(-\frac{|\lambda|x^2 e^{-4|\lambda|s}}{1-e^{-4|\lambda|s}}\Big)\, e^{-2(\alpha+1)|\lambda|s} e^{-\frac{1}{2}|\lambda|x^2} |\lambda|^{\alpha+1} \\
&= \Big(\frac{\lambda}{2\sinh(2\lambda s)}\Big)^{\alpha+1} e^{-\frac{1}{2}\lambda\coth(2\lambda s)x^2}.
\end{aligned}
$$

This completes the proof of Lemma 3.4. ∎

The pointwise estimate of the heat kernel $h_s(x,t)$ can be derived from its expression as same as in [11].

Lemma 3.5 *There are positive constants C and A such that*

$$h_s(x,t) \le C s^{-\alpha-2} e^{-\frac{A}{s}|(x,t)|^2}.$$

Proof In view of Proposition 3.1, we may assume that $s = 1$ and $t \ge 0$. By Lemma 3.4,

$$h_1(x,t) = \int_{-\infty}^{\infty} \Big(\frac{\lambda}{2\sinh(2\lambda)}\Big)^{\alpha+1} e^{-\frac{1}{2}\lambda\coth(2\lambda)x^2} e^{i\lambda t}\, d\lambda.$$

Let

$$f(x,\lambda) = \Big(\frac{\lambda}{2\sinh(2\lambda)}\Big)^{\alpha+1} e^{-\frac{1}{2}\lambda\coth(2\lambda)x^2}.$$

Then $f(x,\lambda)$ can be extended to a holomorphic function of λ in the strip $|\Im\lambda| < \frac{\pi}{2}$. Hence by Cauchy theorem,

$$h_1(x,t) = \lim_{R\to\infty} (I_R^1 + I_R^2 + I_R^3)$$

where

$$
\begin{aligned}
I_R^1 &= \int_0^{\frac{\pi}{4}} i e^{i(-R+i\sigma)t} f(x,-R+i\sigma)\, d\sigma, \\
I_R^2 &= \int_{-R}^{R} e^{i(\sigma+i\frac{\pi}{4})t} f\Big(x,\sigma+i\frac{\pi}{4}\Big)\, d\sigma, \\
I_R^3 &= -\int_0^{\frac{\pi}{4}} i e^{i(R+i\sigma)t} f(x,R+i\sigma)\, d\sigma.
\end{aligned}
$$

It is easy to see that I_R^1 and I_R^3 go to zero as $R \to \infty$. Thus we have

$$h_1(x,t) = e^{-\frac{\pi}{4}t} \int_{-\infty}^{\infty} e^{i\sigma t} f\Big(x,\sigma+i\frac{\pi}{4}\Big)\, d\sigma.$$

Therefore,

$$h_1(x,t) \le Ce^{-\frac{\pi}{4}t}. \tag{3.1}$$

Because $\lim_{\lambda\to 0} \lambda \coth(2\lambda) = \frac{1}{2}$, we also have

$$h_1(x,t) \le Ce^{-\frac{1}{4}x^2}. \tag{3.2}$$

The lemma follows from the estimate (3.1) and (3.2). ■

Now we turn to the Poisson kernel. Let $\{P^s\} = \{e^{-s\sqrt{L}}\}$ be the Poisson semigroup. There is an unique smooth function $p\big((x,t),s\big) = p_s(x,t)$ on $\mathbf{K} \times (0,+\infty)$, which is called the Poisson kernel, such that

$$P^s f(x,t) = f * p_s(x,t).$$

The Poisson kernel can be calculated by the subordination. Using the identity

$$e^{-\beta\sqrt{L}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\mu}}{\sqrt{\mu}} e^{-\frac{\beta^2}{4\mu}L}\, d\mu, \quad \beta > 0,$$

we get

$$p_s(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\mu}}{\sqrt{\mu}} h_{\frac{s^2}{4\mu}}(x,t)\, d\mu. \tag{3.3}$$

From (3.3) and Proposition 3.1 we obtain:

Proposition 3.6 *The Poisson kernel p_s satisfies that*

(i) $(\partial_s^2 - L)p_s = 0$, *on* $\mathbf{K} \times (0,+\infty)$,
(ii) $p_s(x,t) \ge 0$, $p_s(x,-t) = p_s(x,t)$, $\int_{\mathbf{K}} p_s(x,t)\, dm_\alpha(x,t) = 1$,
(iii) $p_{s_1} * p_{s_2} = p_{s_1+s_2}$,
(iv) $p_s(x,t) = s^{-(2\alpha+4)} p_1(\frac{x}{s}, \frac{t}{s^2})$.

The following lemma gives the explicit expression of the Poisson kernel p_s.

Lemma 3.7

$$p_s(x,t) = \frac{4s}{\sqrt{\pi}} \Gamma\Big(\alpha + \frac{5}{2}\Big) \int_0^\infty \Big(\frac{\lambda}{\sinh\lambda}\Big)^{\alpha+1} \big((s^2 + x^2\lambda\coth\lambda)^2 + (2\lambda t)^2\big)^{-\frac{2\alpha+5}{4}}$$
$$\cos\Big(\Big(\alpha + \frac{5}{2}\Big) \arctan\Big(\frac{2\lambda t}{s^2 + x^2\lambda\coth\lambda}\Big)\Big)\, d\lambda.$$

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Proof By Proposition 3.6, we only need to prove this lemma for the case $s = 1$. From (3.3) and Lemma 3.4, we have

$$\begin{aligned}
p_1(x,t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\mu}}{\sqrt{\mu}} h_{\frac{1}{4\mu}}(x,t)\, d\mu \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\mu}}{\sqrt{\mu}} \int_{\mathbf{R}} \Big(\frac{\lambda}{2\sinh(\frac{\lambda}{2\mu})} \Big)^{\alpha+1} e^{-\frac{1}{2}\lambda \coth(\frac{\lambda}{2\mu}) x^2} e^{i\lambda t}\, d\lambda\, d\mu \\
&= \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-\mu} \mu^{\alpha+\frac{3}{2}} \int_0^\infty \Big(\frac{\lambda}{\sinh \lambda} \Big)^{\alpha+1} e^{-\mu x^2 \lambda \coth \lambda} \cos(2\mu\lambda t)\, d\lambda\, d\mu \\
&= \frac{4}{\sqrt{\pi}} \int_0^\infty \Big(\frac{\lambda}{\sinh \lambda} \Big)^{\alpha+1} \Big(\int_0^\infty \mu^{\alpha+\frac{3}{2}} e^{-\mu(1+x^2\lambda\coth\lambda)} \cos(2\mu\lambda t)\, d\mu \Big)\, d\lambda.
\end{aligned}$$

By the integral transformation formula

$$\int_0^\infty x^{\nu-1} e^{-ax} \cos(xy)\, dx = \Gamma(\nu)(a^2+y^2)^{-\frac{\nu}{2}} \cos\Big(\nu \arctan\Big(\frac{y}{a}\Big)\Big), \quad \nu > 0, a > 0$$

(*cf.* [3]), we get

$$\begin{aligned}
p_1(x,t) = \frac{4}{\sqrt{\pi}} \Gamma\Big(\alpha + \frac{5}{2}\Big) \int_0^\infty \Big(\frac{\lambda}{\sinh\lambda} \Big)^{\alpha+1} \big((1 + x^2 \lambda \coth \lambda)^2 + (2\lambda t)^2 \big)^{-\frac{2\alpha+5}{4}} \\
\cos\Big(\Big(\alpha + \frac{5}{2}\Big) \arctan\Big(\frac{2\lambda t}{1 + x^2\lambda\coth\lambda} \Big) \Big)\, d\lambda.
\end{aligned}$$

The proof is completed. ■

As an easy consequence of Lemma 3.7, we have the following pointwise estimate of the Poisson kernel $p_s(x,t)$.

Lemma 3.8 *There is a positive constant C such that*

$$p_s(x,t) \le C\, s\big(s^2 + |(x,t)|^2\big)^{-(\alpha+\frac{5}{2})}.$$

4 Weak Type $(1,1)$ Estimates

We will consider following maximal functions.

The heat maximal function M_H is defined by

$$M_H f(x,t) = \sup_{s>0} |H^s f(x,t)| = \sup_{s>0} |(f * h_s)(x,t)|.$$

The Poisson maximal function M_P is defined by

$$M_P f(x,t) = \sup_{s>0} |P^s f(x,t)| = \sup_{s>0} |(f * p_s)(x,t)|.$$

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The Hardy–Littlewood maximal function is defined by

$$M_B f(x,t) = \sup_{r>0} \frac{1}{m_\alpha(B_r)} \int_{B_r} T^{(\alpha)}_{(x,t)}(|f|)(y,s)\, dm_\alpha(y,s) = \sup_{r>0}(|f| * b_r)(x,t),$$

where $b(x,t) = \frac{1}{m_\alpha(B_1)} \chi_{B_1}(x,t)$.

Remark 4.1 It is clear that M_B is consistent with the structure of hypergroup of $\mathbf{K}$. The Hardy–Littlewood maximal function on Euclidean space restricted on $[0,\infty)\times\mathbf{R}$ is worthless to harmonic analysis on $\mathbf{K}$.

We are going to prove the following results.

Theorem 4.2 *M_H, M_P and M_B are operators on* $\mathbf{K}$ *of weak type* $(1,1)$ *and strong type* (p,p) *for* $1 < p \leq \infty$.

Proof It is obvious that M_H, M_P and M_B are bounded on $L^\infty_\alpha(\mathbf{K})$. Then the L^p estimates for $p > 1$ follow from the weak type $(1,1)$ estimates and the interpolation.

We first establish the weak type $(1,1)$ estimate for the Poisson maximal function M_P. By the subordination identity (3.3),

$$P^s f(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\mu}}{\sqrt{\mu}} H^{\frac{s^2}{4\mu}} f(x,t)\, d\mu.$$

The same as in [7], we have

$$|P^s f(x,t)| \leq A \sup_{y>0} \left| \frac{1}{y} \int_0^y H^s f(x,t)\, ds \right|$$

where $A = \|y\phi'(y)\|_1$ and

$$\phi(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4y}} y^{-\frac{3}{2}}.$$

Therefore,

$$M_P f(x,t) \leq A \sup_{y>0} \left| \frac{1}{y} \int_0^y H^s f(x,t)\, ds \right|. \tag{4.1}$$

Lemma 3.3 shows that the heat semigroup H^s satisfies the conditions of the following Hopf–Dunford–Schwartz ergodic theorem (*cf.* [2, p. 690]).

Theorem (HDS) Let $(\mathcal{M}, dm)$ be a positive measure space. $\{T^t\}_{0<t<+\infty}$ is a semigroup of operators on $\mathcal{M}$ such that

(i) $\lim_{t\to 0} \|T^t f - f\|_2 = 0$ for $f \in L^2(\mathcal{M}, dm)$,
(ii) $\|T^t f\|_p \leq \|f\|_p$, $1 \leq p \leq \infty$,
(iii) Each T^t is self-adjoint on $L^2(\mathcal{M}, dm)$,
(iv) $T^t f \geq 0$ if $f \geq 0$,

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(v) $T^t 1 = 1$,

then the maximal function Mf defined by

$$Mf(x) = \sup_{s>0}\Big(\frac{1}{s}\int_0^s |T^t f(x)|\,dt\Big)$$

is of weak type $(1,1)$.

It follows from (4.1) and the Hopf–Dunford–Schwartz ergodic theorem that the Poisson maximal function M_P is of weak type $(1,1)$.

From Lemma 3.7, it is easy to see that $p_1(0,0) > 0$. There exist constants $c > 0$ and $\delta > 0$ such that $p_1(x,t) \geq c$ when $|(x,t)| \leq \delta$. Take $s_0 = \frac{1}{\delta}$, by Proposition 3.6, there exists a constant $C > 0$ such that

$$b(x,t) \leq C p_{s_0}(x,t).$$

This means

$$M_B f(x,t) \leq C M_P(|f|)(x,t).$$

Therefore the Hardy–Littlewood maximal function M_B is of weak type $(1,1)$.

It follows from Lemma 3.5 that there exists a constant $C > 0$ such that

$$M_H f(x,t) \leq C M_B f(x,t)$$

(see Proposition 4.4 below). Then the heat maximal function M_H is of weak type $(1,1)$. ■

As a consequence of the weak type $(1,1)$ estimate for the Hardy–Littlewood maximal function, we have the following Lebesgue differential theorem.

Corollary 4.3 *Suppose $f \in L^1_{\alpha,\mathrm{loc}}(\mathbf{K})$. Then*

$$\lim_{r\to 0}\frac{1}{m_\alpha(B_r)}\int_{B_r} |T^{(\alpha)}_{(x,t)} f(y,s) - f(x,t)|\,dm_\alpha(y,s) = 0, \quad a.e.\ (x,t) \in \mathbf{K}.$$

The proof of Corollary 4.3 is standard and we omit it.

As application, we give a result about approximations of the identity. Let $\varphi \in L^1_\alpha(\mathbf{K})$. We say that φ is well controlled if there exists a nonnegative and decreasing function ψ on $[0,\infty)$ such that $|\varphi(x,t)| \leq \psi\big(|(x,t)|\big)$ and $\psi\big(|(x,t)|\big) \in L^1_\alpha(\mathbf{K})$. Lemmas 3.5 and 3.8 show that the heat kernel h_1 and the Poisson kernel p_1 are well controlled.

Proposition 4.4 *Suppose φ is well controlled. Then there exists a constant $C > 0$ such that*

$$M_\varphi f(x,t) = \sup_{r>0} |(f * \varphi_r)(x,t)| \leq C M_B f(x,t).$$

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Proof Let ψ be a control function of φ.

$$
\begin{aligned}
&|(f * \varphi_r)(x,t)| \\
&\quad = \Big| \int_{\mathbf{K}} T^{(\alpha)}_{(x,t)} f(y,s) \varphi_r(y,-s)\, dm_\alpha(y,s) \Big| \\
&\quad \le \sum_{k=-\infty}^{\infty} r^{-(2\alpha+4)} \int_{2^k r \le |(y,s)| < 2^{k+1} r} \psi\Big(\frac{|(y,s)|}{r} \Big) |T^{(\alpha)}_{(x,t)} f(y,s)|\, dm_\alpha(y,s) \\
&\quad \le \sum_{k=-\infty}^{\infty} r^{-(2\alpha+4)} \psi(2^k) \int_{B_{2^{k+1} r}} |T^{(\alpha)}_{(x,t)} f(y,s)|\, dm_\alpha(y,s) \\
&\quad \le \Big(m_\alpha(B_1) \sum_{k=-\infty}^{\infty} 2^{(k+1)(2\alpha+4)} \psi(2^k) \Big) M_B f(x,t) \\
&\quad = 2^{2\alpha+4} \Big(m_\alpha(B_1) \sum_{k=-\infty}^{\infty} 2^{k(2\alpha+4)} \psi(2^k) \Big) M_B f(x,t)
\end{aligned}
$$

where we have used (2.3).

On the other hand, we have

$$
\begin{aligned}
\int_{\mathbf{K}} \psi\big(|(x,t)| \big)\, dm_\alpha(x,t) &= \sum_{k=-\infty}^{\infty} \int_{2^{k-1} \le |(y,s)| < 2^k} \psi\big(|(x,t)| \big)\, dm_\alpha(x,t) \\
&\ge m_\alpha(B_1) \sum_{k=-\infty}^{\infty} (2^{k(2\alpha+4)} - 2^{(k-1)(2\alpha+4)}) \psi(2^k) \\
&= (1 - 2^{-(2\alpha+4)}) \Big(m_\alpha(B_1) \sum_{k=-\infty}^{\infty} 2^{k(2\alpha+4)} \psi(2^k) \Big).
\end{aligned}
$$

Because $\psi\big(|(x,t)| \big) \in L^1_\alpha(\mathbf{K})$, the proposition is proved. ■

Corollary 4.5 *Suppose φ is well controlled. Then for $f \in L^p_\alpha(\mathbf{K})$ or $f \in C_c(\mathbf{K})$,*

$$
\lim_{r \to 0} (f * \varphi_r)(x,t) = A f(x,t), \quad \textit{a.e. } (x,t) \in \mathbf{K},
$$

where

$$
A = \int_{\mathbf{K}} \varphi(x,t)\, dm_\alpha(x,t).
$$

The proof of Corollary 4.5 is standard and we omit it.

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