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AN ANSWER TO VAN MILL'S QUESTION

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Abstract

van Mill *et al*. posed in 'Classes defined by stars and neighborhood assignments', *Topology Appl.* **154** (2007), 2127–2134 the following question: Is a star-compact space metrizable if it has a G_δ-diagonal? In this paper, we give a negative answer to this question.

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1. Introduction

Star-compact spaces are natural generalizations of countably compact spaces (strongly 1-star-compact spaces). van Mill *et al.* gave an example in [4] showing that star-compact spaces are strictly weaker than countably compact spaces. It is known that [2] a countably compact space with a G_δ-diagonal is compact metrizable. van Mill *et al.* posed the following question in [4].

QUESTION 1.1 [4]. Is a star-compact space metrizable if it has a G_δ-diagonal?

In this paper, we give a negative answer to the above question.

Throughout, a space will mean a topological space. Let A be a subset of a space X and $\mathcal{U}$ a family of subsets of X. The star, $St(A, \mathcal{U})$, of the set A with respect to $\mathcal{U}$ is the set $\bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$.

DEFINITION 1.2 [3]. Let $\mathcal{P}$ be a class (or a property) of a space X. X is said to be star-$\mathcal{P}$ (or star-determined by $\mathcal{P}$) if for any open cover $\mathcal{U}$ of the space X, there is a subspace $Y \subset X$ such that $Y \in \mathcal{P}$ and $St(Y, \mathcal{U}) = X$.

By Definition 1.2, a space X is said to be star-compact (respectively, star-determined by convergent sequences) if, for any open cover $\mathcal{U}$ of the space X, there is a compact subspace K (respectively, a convergent sequence S) of X such that $St(K, \mathcal{U}) = X$ (respectively, $St(S, \mathcal{U}) = X$).

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DEFINITION 1.3 [1]. A space X is said to be strongly 1-star-compact (respectively, 1-star-compact) if, for any open cover $\mathcal{U}$ of X, there exists a finite subset $F \subset X$ (respectively, a finite subfamily $\mathcal{V}$ of $\mathcal{U}$) such that $St(F, \mathcal{U}) = X$ (respectively, $St(\cup\mathcal{V}, \mathcal{U}) = X$).

It was shown that [1] the notions of strongly 1-star-compact and countably compact are equivalent in a T_2 space.

2. An example

In this section, we give an example which presents a negative answer to Question 1.1. The space used here was constructed by van Douwen *et al.* [1]. It was proved that the space is second countable and 1-star-compact. We show that it is also star-compact and has a G_δ-diagonal but it is not metrizable, which answers Question 1.1 negatively.

EXAMPLE 2.1. There exists a second countable star-compact space X which has a G_δ-diagonal but X is not metrizable.

PROOF. Let $Y = \bigcup\{[0, 1] \times \{n\} \mid n < \omega\}$ and $X = Y \cup \{a\}$, where $a \notin Y$. Topologize X as follows: basic neighborhoods of the point a take the form $\{a\} \cup \bigcup\{[0, 1) \times \{n\} \mid n > m\}$, where $m \in \omega$; basic neighborhoods of the other points of X are the usual induced metric open neighborhoods. One readily sees that the space X with this topology is T_2 second countable and the subspace $[0, 1] \times \{n\}$ is compact for all $n \in \omega$ [1].

Since the point a and the closed set $\{(1, n) \mid n \in \omega\}$ cannot be separated by open sets, X is not regular, and hence not metrizable.

Now, we show that X is star-compact. Let $\mathcal{U}$ be an open cover of X consisting of basic open sets of X. For each $n \in \omega$, there exists $U_n \in \mathcal{U}$ such that $(1, n) \in U_n$. For each $n \in \omega$, U_n can be represented as $U_n = V_n \times \{n\}$, where V_n is an open neighborhood of 1 in $[0, 1]$. Thus $V_n \setminus \{1\} \neq \emptyset$. Pick $x_n \in V_n \setminus \{1\}$ then $(x_n, n) \in U_n$ with $(x_n, n) \neq (1, n)$. Put $K_1 = \{(x_n, n) \mid n \in \omega\} \cup \{a\}$ then K_1 is a compact subset of X and $\{(1, n) \mid n \in \omega\} \subset St(K_1, \mathcal{U})$. For the point a, choose $U_a \in \mathcal{U}$ such that $a \in U_a$. Then there exists $m \in \omega$ such that $U_a = \{a\} \cup \bigcup\{[0, 1) \times \{n\} \mid n > m\}$. It is obvious that $U_a \subset St(K_1, \mathcal{U})$ since $U_a \cap K_1 \neq \emptyset$. Now, set $K_2 = \bigcup\{[0, 1] \times \{n\} \mid n \leq m\}$; then, being a finite union of compact sets, K_2 is compact. Obviously, $K_2 \subset St(K_2, \mathcal{U})$. Put $K = K_1 \cup K_2$; then K is a compact subset of X and $St(K, \mathcal{U}) = X$, which shows that X is star-compact.

It remains to show that X has a G_δ-diagonal. From the construction of the topology of X, we see that X has a countable base $\mathcal{B}$ consisting of basic open sets of X. Then $\mathcal{B} \times \mathcal{B}$ is a countable base of $X \times X$. It is easy to verify that every basic open set of X can be represented as the countable union of closed subsets of X. Thus every member of $\mathcal{B} \times \mathcal{B}$ can be represented as the countable union of closed subsets of $X \times X$. This, together with the fact that $\mathcal{B} \times \mathcal{B}$ is a countable base of $X \times X$, shows that $X \times X$ is perfect. X being T_2, we conclude that X has a G_δ-diagonal.

This completes the proof. □

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REMARK 2.2. Spaces star-determined by convergent sequences which are stronger than star-compact spaces are also generalizations of countably compact spaces (strongly 1-star-compact spaces). van Mill *et al.* showed in [4] that there is a space X star-determined by convergent sequences while X is not countably compact (strongly 1-star-compact spaces). The space X in Example 2.1 is actually star-determined by convergent sequences. From the proof of Example 2.1, we see that K_1 is in fact a convergent sequence with the limit point a. Since K_2 is compact, it is of course strongly 1-star-compact and thus there exists a finite subset $F \subset X$ such that $K_2 \subset St(F, \mathcal{U})$. Put $S = K_1 \cup F$; then S is a convergent sequence and $St(S, \mathcal{U}) = X$, which shows that X is star-determined by convergent sequences. Thus we have the following stronger result: *there exists a second countable space X, star-determined by convergent sequences, which has a G_δ-diagonal while X is not metrizable.*

REMARK 2.3. The subset $\{(1, n) \mid n \in \omega\}$ of the space X in Example 2.1 is a closed subset of X, but it is not star-compact (hence not star-determined by convergent sequences). So Example 2.1 also shows that a closed subspace of a star-compact (respectively, star-determined by convergent sequences) space need not be star-compact (respectively star-determined by convergent sequences).

REMARK 2.4. van Douwen *et al.* [1] showed that countably compact meta-compact spaces are compact and that the space in Example 2.1 is also meta-compact. From Example 2.1, we see that the condition *countably compact* cannot be replaced by *star-compact (star-determined by convergent sequences)*. That is: *there exists a star-compact (star-determined by convergent sequences) meta-compact space which is not compact (not even countably compact).*

References

[1] E. K. van Douwen, G. M. Reed and I. J. Tree, 'Star covering properties', *Topology Appl.* **39** (1991), 71–103.

[2] G. Gruenhage, 'Generalized metric spaces', in: *Handbook of Set-Theoretic Topology* (North-Holland, Amsterdam, 1984), pp. 423–501.

[3] M. Matveev, 'A survey on star covering properties', *Topology Atlas*, Preprint No. 330, 1998.

[4] J. van Mill, V. V. Tkachuk and R. G. Wilson, 'Classes defined by stars and neighborhood assignments', *Topology Appl.* **154** (2007), 2127–2134.

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