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# The Diffeomorphism Type of Canonical Integrations Of Poisson Tensors on Surfaces

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*Abstract.* A surface $\Sigma$ endowed with a Poisson tensor $\pi$ is known to admit a *canonical integration*, $\mathcal{G}(\pi)$, which is a 4-dimensional manifold with a (symplectic) Lie groupoid structure. In this short note we show that if $\pi$ is not an area form on the 2-sphere, then $\mathcal{G}(\pi)$ is diffeomorphic to the cotangent bundle $T^*\Sigma$. This extends results by the author and by Bonechi, Ciccoli, Staffolani, and Tarlini.

## 1 Introduction

A Poisson structure on a manifold $M$ is given by a bivector $\pi$ closed under the Schouten bracket. Equivalently, it is defined by a bracket $\{\cdot,\cdot\}$ on smooth functions such that $(C^\infty(M),\{\cdot,\cdot\})$ becomes a Lie algebra over the reals and $X_f := \{f,\cdot\}$ is a derivation for every function $f$; the Hamiltonian vector fields $X_f$, $f \in C^\infty(M)$, define a (possibly singular) integrable distribution, its maximal integral submanifolds carrying a symplectic form.

Poisson structures can have rather complicated symplectic foliations. Under some well-known conditions [3] a Poisson structure $(M,\pi)$ can be "symplectically desingularized". This means that there exist a symplectic manifold $(S,\omega)$ and surjective submersion $\phi: S \to M$, such that for any $f \in C^\infty(M)$ the Hamiltonian vector fields $X_f$ and $X_{\phi^* f}$ are $\phi$-related and $X_{\phi^* f}$ is complete whenever $X_f$ is complete.

Poisson manifolds that can be "symplectically desingularized" are referred to as *integrable*. The reason is that if there exist symplectic desingularizations for $(M,\pi)$, then there is a canonical one $\mathbf{s}: (\mathcal{G}(\pi),\omega_{\mathcal{G}}) \to (M,\pi)$ [3] called the *canonical integration*, which is the unique (symplectic) Lie groupoid [10] over $M$ with 1-connected $\mathbf{s}$-fibers which integrates the Lie algebroid structure defined by $\pi$ on $T^*M$. Suffice it to say here that the Lie groupoid structure includes the source and target maps $\mathbf{s},\mathbf{t}: \mathcal{G}(\pi) \to M$, a bisection of units $u: M \to \mathcal{G}(\pi)$, and a partial associative composition law in which "arrows" $g,h \in \mathcal{G}(\pi)$ can be composed whenever $\mathbf{s}(g) = \mathbf{t}(h)$; the symplectic form $\omega_{\mathcal{G}}$ is compatible with the Lie groupoid structure (*i.e.,* it is *multiplicative*).

There are several good reasons to study the symplectic geometry of canonical integrations of integrable Poisson structures. One of them is trying to understand the role of multiplicative symplectic structures among symplectic structures. Another one is that Poisson structures describe classical physical systems that one would like

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to quantize. If the Poisson structure is integrable, then it is natural consider the geometric quantization of the canonical symplectic integration (taking into account as well the groupoid structure) [2, 5].

For an integrable Poisson structure $(M, \pi)$, the standard construction of its canonical integration $(\mathcal{G}(\pi), \omega_{\mathcal{G}})$ is by symplectic reduction on an infinite-dimensional symplectic manifold [1,3]. Since $\mathcal{G}(\pi)$ is the leaf space of a foliation (of finite codimension), it is extremely difficult to describe $\mathcal{G}(\pi)$ as a manifold, let alone its symplectic geometry. Even more difficulties arise at the level of general topology, since, as it is often the case for leaf spaces, $\mathcal{G}(\pi)$ need not be Hausdorff. Still, it makes sense to try to describe $(\mathcal{G}(\pi), \omega_{\mathcal{G}})$ in the lowest possible dimension, *i.e.*, when $M$ is a surface.

Let us start by recalling that any bivector $\pi$ on a surface $\Sigma$ defines a Poisson structure and that all Poisson structures on surfaces are integrable [1].

Regarding general topology issues, the canonical integration of a Poisson structure on a surface is always Hausdorff (see [1] for the proof for Poisson structures on $\mathbb{R}^2$ with 1-connected symplectic leaves, and [6] for the case of arbitrary Poisson structures and surfaces).

The canonical integration $\mathcal{G}(\pi)$ is a 4-dimensional manifold whose diffeomorphism type is known in few cases. For the trivial Poisson structure and for area forms it is well known that $\mathcal{G}(\pi)$ is diffeomorphic to the cotangent bundle. When $\Sigma = S^2$ and $\pi$ is a Poisson homogeneous structure (the corresponding foliation having one symplectic leaf and quadratic singularity at the north pole say), the canonical integration is also diffeomorphic to the cotangent bundle. Finally, $\mathcal{G}(\pi)$ is diffeomorphic to $\mathbb{R}^4$ for any Poisson structure on $\mathbb{R}^2$ [1, 6].

Let us finish this brief survey by recalling that the symplectomorphism type of $(\mathcal{G}(\pi), \omega_{\mathcal{G}})$ is only known in the two extreme cases: for both the trivial Poisson structure on an arbitrary surface and the Poisson structure associated with an area form on $\Sigma$ compact with empty boundary, and different from the 2-sphere,[1] the canonical integration is symplectomorphic to the cotangent bundle with its standard symplectic form (see [7] for the proof in the case of area forms).

The purpose of this short note is to fully describe the diffeomorphism type of canonical integrations of Poisson structures on surfaces.

***Theorem 1.1*** *Let $\pi$ be a Poisson tensor on a surface $\Sigma$ that is not an area form on $S^2$. Then there exists a diffeomorphism of fibrations*

$$\begin{array}{ccc} \mathcal{G}(\pi) & \xrightarrow{\qquad} & T^*\Sigma \\ & {}_{\mathbf{s}}\searrow \quad \swarrow_{\pi} & \\ & \Sigma & \end{array}$$

*taking the units of the groupoid to the zero section of the cotangent bundle.*

[1] For $S^2$ endowed with an area form $\omega$, the canonical integration is symplectomorphic to $(S^2 \times S^2, \omega \oplus -\omega)$

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## 2 Proof of the Theorem

The starting point is a slight refinement of a result of Meigniez [9], in which we characterize those surjective submersions that are the total space of a vector bundle.

***Proposition 2.1*** *Let $p: Q \to B$ be a surjective submersion between (Hausdorff) manifolds satisfying the following properties.*

(i) *For all $b \in B$ the fiber $p^{-1}(b)$ is diffeomorphic to $\mathbb{R}^n$.*
(ii) *The submersion has a section $\sigma: B \to Q$.*

*Then there exists a rank n vector bundle $\pi: E \to B$ and a diffeomorphism of fibrations*

$$\begin{array}{ccc} Q & \xrightarrow{\quad\Psi\quad} & E \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

*taking $\sigma(B)$ to the zero section of E*

**Proof** Because of property (i), [9, Corollary 31] implies that $p: Q \to B$ is a locally trivial fibration. Very briefly, there are two fundamental steps to prove local triviality. First, it is enough to do it for the pullback of $Q$ to paths in $B$; this dimensional reduction uses a topological argument involving fibrations of path spaces. Secondly, the property that each fiber can be exhausted by closed balls (the fiber is diffeomorphic to Euclidean space) and any two (embedded) balls are isotopic, is used to trivialize the submersion over a path (though the proof is by no means straightforward).

Next, by property (ii) we can reduce the structural group of $p: Q \to B$ to the diffeomorphisms that fix the origin $\mathrm{Diff}(\mathbb{R}^n, 0)$.

The proof of the proposition amounts to showing that we can further reduce the structural group to the linear group $\mathrm{Gl}(n, \mathbb{R})$.

It is worth pointing out that if the structural group were a finite-dimensional Lie group $G$, then standard bundle theory says that the existence of reduction of the structural group to a subgroup $H$ is equivalent to the existence of a section of the associated bundle with fiber the homogeneous space $G/H$. If particular, if $H$ were a deformation retract of $G$, well-known arguments of obstruction theory would imply that such a section always exists.

In our case the existence of a retraction of $\mathrm{Diff}(\mathbb{R}^n, 0)$ into $\mathrm{Gl}(n, \mathbb{R})$ is not enough to reduce the structural group. The reason is that no matter which reasonable topology we use on $\mathrm{Diff}(\mathbb{R}^n, 0)$, the evaluation map is not smooth. Specifically, we need to find a "smooth retraction" $H: [0,1] \times \mathrm{Diff}(\mathbb{R}^n, 0) \to \mathrm{Diff}(\mathbb{R}^n, 0)$, in the sense that for any manifold $N$ and any smooth map $\Phi: N \times \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(n, \cdot) \in \mathrm{Diff}(\mathbb{R}^n, 0)$, the composition $H(t, \Phi): [0,1] \times N \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth (for the purposes of the application of obstruction theory the manifold $N$ will always be a sphere).

Let $\lambda_t: \mathbb{R}^n \to \mathbb{R}^n$ be the dilation by factor $t \in \mathbb{R} > 0$. Then the standard retraction taking a diffeomorphism fixing the origin to its linearization at the origin,

$$\begin{array}{rcl} H: [0,1] \times \mathrm{Diff}(\mathbb{R}^n, 0) & \longrightarrow & \mathrm{Diff}(\mathbb{R}^n, 0) \\ (t, \phi) & \longmapsto & \lambda_{1/t} \circ \phi \circ \lambda_t, \end{array}$$

is "smooth" in the above sense. We recall how the argument goes (in fact, that embedded balls in Euclidean space are isotopic is proved is a similar fashion). We can assume without loss of generality that we have a smooth map

$$\Phi: \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n, \quad (u, x) \mapsto \Phi(u, x),$$

which for each $u \in \mathbb{R}^n$ is a diffeomorphism fixing the origin $\Phi_u(x)$. To show the smoothness of

$$\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, u, x) \mapsto \frac{\Phi(u, tx)}{t},$$

we regard $\Phi(u, tx)$ as a family of functions on the variable $t$, and consider the auxiliary family of functions $G_{(t,u,x)}(s) = \Phi(u, stx)$, $s \in [0, 1]$. By the fundamental theorem of calculus

$$\Phi(u, tx) = G_{(t,u,x)}(1) = \int_0^1 \frac{d}{ds} G_{(t,u,x)}(s) ds = \bigg( \int_0^1 \Big( \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(u, stx) x_j \Big) ds \bigg) t.$$

Therefore,

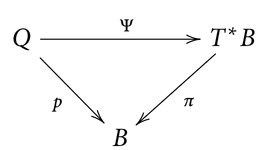
$$\frac{\Phi(u, tx)}{t} = \int_0^1 \Big( \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(u, stx) x_j \Big) ds$$

is smooth, and its value at zero is indeed the differential of $\Phi_u(x)$ at the origin

$$\lim_{t \to 0} \frac{\Phi(u, tx)}{t} = \int_0^1 \Big( \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(u, 0) x_j \Big) ds = \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(u, 0) x_j. \qquad \blacksquare$$

When $P$ is a symplectic manifold carrying a Lagrangian section, then much more can be said.

***Corollary 2.2*** *Let $p: Q \to B$ be as in Proposition 2.1. Assume further that $Q$ carries a symplectic structure so that the graph of the section $\sigma$ is a Lagrangian submanifold. Then there exists a diffeomorphism of fibrations*




*taking $\sigma(B)$ to the zero section of the cotangent bundle.*

**Proof** By Proposition 2.1 we can assume without loss of generality that $p: Q \to B$ is a vector bundle. A vector bundle is isomorphic to the normal bundle of the graph of either of its sections. If the graph is Lagrangian, basic symplectic linear algebra [8] implies the normal bundle of the graph is isomorphic to $T^*B$, and this proves the corollary. $\blacksquare$

**Proof of Theorem 1.1** By [6, Corollary 3], $\mathbf{s}: \mathcal{G}(\pi) \to \Sigma$ is a locally trivial fibration with fiber diffeomorphic to $\mathbb{R}^2$. Because it is a symplectic Lie groupoid, the units are a Lagrangian section [10], and therefore the theorem follows from Corollary 2.2. $\blacksquare$

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The proof of Theorem 1.1 is purely topological. It would be desirable to have a proof in the spirit of Lie theory, namely, to find a connection on the Lie algebroid $(T^*M, [\cdot,\cdot]_\pi)$ whose (contravariant) exponential map exp: $T^*M \to \mathcal{G}(\pi)$ is a diffeomorphism, and hence deduce that $\mathcal{G}(\pi)$ is of "exponential type". Such a proof might also shed some light on whether $(\mathcal{G}(\pi), \omega_{\mathcal{G}})$ is the standard or an exotic symplectic structure on the cotangent bundle. The reason is that it is tempting to try to prove that canonical integrations are standard cotangent bundles by using a global version of the symplectic realization construction in [4]. The kind of problems one encounters for the latter approach are analogous to those appearing when trying to show that an exponential provides a diffeomorphism from the Lie algebroid to the canonical integration.

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