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# Indecomposable representations in characteristic two of the simple groups of order not divisible by eight

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The indecomposable representations in characteristic two of the groups PSL$(2, q)$ where $q$ is congruent to 3 or 5 modulo 8 are classified. For $q = 3$ or 5 the classification is obtained by explicit construction of modules, using the Green correspondence to prove completeness. For larger $q$, the classification is obtained using equivalences between appropriate categories of modules.

## 1. Introduction

Let $k$ be an algebraically closed field of characteristic 2. The simple groups with Sylow subgroup $V = C_2 \times C_2$ are the groups PSL$(2, q)$, $q \equiv 3$ or 5 (mod 8) and $q > 3$ (see Gorenstein, [5], p. 420). Their character tables (see Dornhoff, [2], Section 38) will be needed in Sections 2 and 3 below. Conlon's list of indecomposable modules for $kV$ and $kA_4$, given in [1], will be used. Frequent use will be made of the Green correspondence of [6]. The form of this correspondence convenient for our purposes is as follows.

Let $G$ be a finite group, $P$ a 2-subgroup, and $H$ a subgroup of $G$ containing $N_G(P)$. Let $M$ be an indecomposable $kG$ module with vertex $P$. Then there is a unique indecomposable $kH$ module $N$ with vertex $P$ such that $N$ is a component (indecomposable direct summand) of the

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restriction $M_H$ , and every other component of $M_H$ has vertex $P^x \cap H$ , for some $x \in G - H$ .

In Sections 5 and 6 the notation $M = f(M)$ will be used.

For $A_4$ and $A_5$ this classification method carries over to any field $k$ of characteristic 2 containing a primitive cube root of unity. The indecomposable representations that are referred to below as continuous are then parametrised by irreducible polynomials.

## 2. The 2-blocks of PSL(2, $q$) , $q \equiv 3 \pmod 8$

An examination of the character table of PSL(2, $q$) , $q \equiv 3 \pmod 8$ , shows that:

(a) its principal 2-block consists of 4 complex characters, 1 (degree 1) , $\eta_1$ (degree $\frac{1}{2}(q-1)$) , $\eta_2$ (degree $\frac{1}{2}(q-1)$) , and $\psi$ (degree $q$) ;

(b) the only other 2-blocks are $(q-3)/4$ blocks of defect 0 and $(q-3)/8$ blocks of defect 1 ;

(c) modulo characters in non-principal blocks, the product characters from the principal block are

$$\eta_1\eta_1 = \eta_2 , \quad \eta_1\eta_2 = 1 , \text{ and } \eta_2\eta_2 = \eta_1 .$$

As the representation theory of the blocks of defect 0 and 1 is known (see, for example, Dornhoff, [3], Section 68), we restrict attention to the principal 2-block. Since the number of irreducible modular representations is the number of 2-regular conjugacy classes, the principal 2-block has three irreducible modular representations. The group has a Sylow subgroup $Q$ , elementary abelian of order $q$ , whose normaliser $N(Q)$ is the image of the group of upper triangular matrices. The orbits of the action of $N(Q)$ on the set of linear characters of $Q$ consist of 1, $\frac{1}{2}(q-1)$ , and $\frac{1}{2}(q-1)$ elements respectively. Hence the restriction of any (characteristic 0 or 2 ) representation of $N(Q)$ to $Q$ is the sum of representations each of which is the restriction of 1, $\eta_1$ or $\eta_2$ . Hence $\eta_1$ and $\eta_2$ remain irreducible in characteristic 2 , and the Cartan matrix for the block must be

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} .$$

The trivial representation of $kN(Q)$ induced to $\mathrm{PSL}(2, q)$ is projective of dimension $q + 1$, and may be interpreted as the space of all functions on the projective line over $\mathrm{GF}(q)$. It has a unique minimal subrepresentation - the space of constant functions - and a unique maximal subrepresentation - the space of all functions with sum of values zero. The quotient of these two subrepresentations is readily calculated to be a direct sum $\eta_1 \oplus \eta_2$. Thus $\pi_0$, the above representation, is the projective cover of $1$, and so, by (c) above, $\pi_0 \otimes \eta_1 = \pi_1$ and $\pi_0 \otimes \eta_2 = \pi_2$ (modulo representations in other blocks) are the projective covers of $\eta_1$ and $\eta_2$ respectively. Thus the quotient of the maximal submodule of $\pi_i$ by its minimal submodule is $1 \oplus \eta_j$, $\{i, j\} = \{1, 2\}$.

The $k$-algebra $\mathrm{end}_{\mathrm{PSL}(2,q)}(\pi_0 \oplus \pi_1 \oplus \pi_2)$ is thus of dimension $12$, and its isomorphism type is independent of $q$ $(\equiv 3 \pmod 8)$.

## 3. The 2-blocks of $\mathrm{PSL}(2, q)$, $q \equiv 5 \pmod 8$

An examination of the character table of $\mathrm{PSL}(2, q)$, $q \equiv 5 \pmod 8$ shows that

(a) its principal 2-block consists of 4 complex characters, $1$ (degree 1), $\xi_1$ (degree $\tfrac{1}{2}(q+1)$), $\xi_2$ (degree $\tfrac{1}{2}(q+1)$), and $\psi$ (degree $q$);

(b) the only other 2-blocks are $(q-1)/4$ 2-blocks of defect 0 and $(q-5)/8$ 2-blocks of defect 1;

(c) modulo characters in non-principal blocks, the product characters from the principal block are

$$\xi_1\xi_1 = 1 + \xi_1 + \psi , \quad \xi_1 \cdot \xi_2 = \psi , \quad \xi_2 \cdot \xi_2 = 1 + \xi_2 + \psi .$$

As before, we restrict attention to the principal block and calculate that it has three irreducible modular representations. As before, let $N(Q)$ be the image of the group of upper triangular matrices. The order of

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$N(Q)$ is $\frac{1}{2}q(q-1)$ , and it has two characters of degree $\frac{1}{2}(q-1)$ together with $\frac{1}{2}(q-1)$ linear characters.

If both $\xi_1$ and $\xi_2$ remained irreducible in characteristic 2 , no decomposition of $\psi$ could occur. Hence at least one of $\xi_1$ and $\xi_2$ does decompose. Since the outer automorphism of the group interchanges them, they must in fact both decompose. Hence each must decompose into a trivial irreducible plus one of degree $\frac{1}{2}(q-1)$ . So the Cartan matrix of the block is

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} .$$

Let $\eta_1, \eta_2$ denote the two non-trivial irreducible modular representations in the block. Then (c) above shows that the corresponding Brauer characters (also denoted by $\eta_1, \eta_2$ ) have products, modulo characters in other blocks, as follows:

$$\eta_1\eta_1 = 2.1 + \eta_2 \ , \quad \eta_1 \cdot \eta_2 = 0 \ , \quad \eta_2\eta_2 = 2.1 + \eta_1 \ .$$

Simple calculations show that if $\pi_0, \pi_1, \pi_2$ are the projective covers of $1, \eta_1, \eta_2$ respectively, $\eta_1 \otimes \pi_1 = \pi_0$ , modulo projective summands in other blocks. Hence the component of $\eta_1 \otimes \eta_1$ in the principal block has a unique trivial submodule and a unique trivial quotient. It is therefore uniserial, with composition factors $1, \eta_2, 1$ . Also, as $\eta_1 \otimes \pi_2 = \pi_1$ , modulo projective summands in other blocks, the only possible composition series for $\pi_2$ is $\eta_2, 1, \eta_1, 1, \eta_2$ . Likewise $\pi_1$ is uniserial, with composition series $\eta_1, 1, \eta_2, 1, \eta_1$ . A consideration of $\eta_1 \otimes \pi_1$ and $\eta_2 \otimes \pi_2$ shows that $\pi_0$ has uniserial submodules with composition series $1, \eta_2, 1, \eta_1$ , and $1, \eta_1, 1, \eta_2$ , and uniserial quotient modules with the same sequences of factors taken in reverse order. Thus the quotient of the maximal submodule of $\pi_0$ by its minimal submodule is the direct sum of two uniserial modules.

The $k$-algebra $\mathrm{end}_{\mathrm{PSL}(2,q)}\left(\pi_0 \oplus \bar{\pi}_1 \oplus \pi_2\right)$ is thus of dimension 18 and its isomorphism type is independent of $q$ $\left(\equiv 5 \pmod 8\right)$ .

## 4. Construction of certain modules

Let $G$ be a finite group. Let $E_1, \ldots, E_{n+1}, F_1, \ldots, F_n$ be irreducible $kG$ modules (not necessarily distinct), and $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ uniserial $kG$ modules such that:

the $U_i$ and $V_i$ are reducible, except possibly $U_1$ and $V_n$ ;

$U_i$ has socle $E_i$ and irreducible quotient $F_i$ ;

$V_i$ has socle $E_{i+1}$ and irreducible quotient $F_i$ ;

$\ker\left(U_i \oplus V_i \to F_i\right)$ is a quotient of the projective cover of $F_i$ ;

$\mathrm{coker}\left(E_{i+1} \to U_{i+1} \oplus V_i\right)$ is a submodule of the injective hull of $E_{i+1}$ .

Let

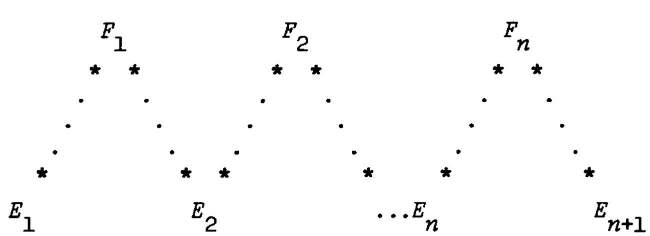
$$P = \ker\left(U_1 \oplus V_1 \to F_1\right) \oplus \ldots \oplus \ker\left(U_n \oplus V_n \to F_n\right)$$

and

$$Q = \mathrm{im}\left(E_2 \to U_2 \oplus V_1\right) + \ldots + \mathrm{im}\left(E_n \to U_n \oplus V_{n-1}\right) ,$$

and let $M = P/Q$ .

In future we shall denote the modules $M$ so constructed by diagrams of the form




where the asterisks stand for the remaining composition factors of the $U_i$

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and $V_i$ .

## 5. The indecomposable $kA_4$ modules

The indecomposable $kV$ modules are, adapting the notation of Conlon [1]:

| | Dimension | Dimension of Socle | |
|---|---|---|---|
| $A_n$ | $2n + 1$ | $n$ | $n > 0$ |
| $B_n$ | $2n + 1$ | $n + 1$ | $n \geq 0$ |
| $C_n(\gamma)$ | $2n$ | $n$ | $n > 0$ , $\gamma \in k \cup \{\infty\}$ . |

For typographical convenience we write $G$ for the group $A_4$ .

The modules $A_n$ and $B_n$ are characterised by their dimension and the dimension of their socle. The module $B_0$ (= '$A_0$' ) is the irreducible $kV$ module. All the above modules have vertex $V$ except for $C_1(0)$, $C_1(1)$ , and $C_1(\infty)$ .

The modules $A_n$, $B_n$, $C_n(\omega)$ , and $C_n(\omega^2)$ (where $\omega$ is a fixed root of $t^2 + t + 1 = 0$ ) each have three non-isomorphic extensions to $kG$ , which we shall call the discrete indecomposable $kG$ modules. If $\lambda \neq \omega, \omega^2$ , $C_n(\lambda)^G$ is indecomposable, and its restriction to $V$ is $C_n(\lambda) \oplus C_n(1+\lambda^{-1}) \oplus C_n((1+\lambda)^{-1})$ (see Conlon, [1]). We denote $C_n(\lambda)^G$ by $K_n(\sigma)$ , where $\sigma = (\lambda+\omega)^3/(\lambda+\omega^2)^3$ , and we shall call these modules continuous. (If $\lambda = \infty$ , take $\sigma = 1$ .)

We now describe a construction of the above indecomposable $kQ$ modules using the method of Section 4. The results of Section 2 show that $kG$ has exactly six reducible uniserial modules, all of composition length 2 , the unique uniserial extensions of pairs of irreducible modules. If we denote the trivial irreducible representation of $kG$ by '0' and those that take value $\omega, \omega^2$ on (123) by '1', '2' respectively, it is clear that the modules constructible by the method of Section 4 can be described

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by diagrams of the form

$$\cdots {}_0{}^1{}_2{}^0{}_1{}^2{}_0{}^1{}_2 \cdots .$$

We shall denote the module $M$ described by the above diagram by $[i, m, j]$ if there are $m + 1$ symbols $0$ in the upper line, $i$ symbols to the left of the extreme left upper $0$, and $j$ symbols to the right of the extreme right upper $0$. Thus $0 \leq i, j \leq 5$. If no $0$'s appear in the upper line, we take $m$ to be $-1$, and $i, j$ are measured from where the compatibility requirements of the construction would have to place $0$'s in the upper line. For example, $[5, -1, 3]$ denotes ${}_1{}^2{}_0$, and we write $[3, -1, 3]$, $[5, -1, 1]$, and $[1, -1, 5]$ for the three irreducible representations. In this and subsequent sections the triples $[0, 0, 0]$, $[2, -1, 4]$, $[4, -1, 2]$ are excluded. In all cases the dimension of $[i, m, j]$ is $i + 6m + j + 1$.

Direct verification shows that the modules $[i, m, j]$ are indecomposable, with restriction to $V$ as in the table on page 414. It follows that the $[i, m, j]$ afford a complete list of the discrete indecomposable $kA_4$-modules.

The module $K_m(\sigma)$ may be obtained from the module $M = [3, m-1, 3]$. Let $v_0 \ldots v_n$ be basis vectors for the socle factors of type $0$ in the diagram for $M$, ordering from left to right, and let $N$ be the submodule generated by

$$v_m - c_{m-1}v_{m-1} - \cdots - c_0v_0 ,$$

where

$$(t-\sigma)^m = t^m - c_{m-1}t^{m-1} - \cdots - c_0 .$$

It is easy to verify that $M/N$ is indecomposable and has a submodule isomorphic to $K_1(\sigma)$, which implies that $M/N$ is isomorphic to $K_m(\sigma)$.

| $kA_4$ modules $m \geq 0$ | | | Dimension | Dimension of socle | Restriction to $V$ |
|---|---|---|---|---|---|
| $[0, m, 0]$ | $[2, m-1, 4]$ | $[4, m-1, 2]$ | $6m + 1$ | $3m$ | $A_{3m}$ |
| $[1, m, 5]$ | $[3, m-1, 3]$ | $[5, m-1, 1]$ | $6m + 1$ | $3m + 1$ | $B_{3m}$ |
| $[1, m, 0]$ | $[3, m, 4]$ | $[5, m, 2]$ | $6m + 2$ | $3m + 1$ | $C_{3m+1}(\omega)$ |
| $[0, m, 1]$ | $[2, m, 5]$ | $[4, m, 3]$ | $6m + 2$ | $3m + 1$ | $C_{3m+1}(\omega^2)$ |
| $[0, m, 2]$ | $[2, m, 0]$ | $[4, m-1, 4]$ | $6m + 3$ | $3m + 1$ | $A_{3m+1}$ |
| $[1, m, 1]$ | $[3, m-1, 5]$ | $[5, m-1, 3]$ | $6m + 3$ | $3m + 2$ | $B_{3m+1}$ |
| $[1, m, 2]$ | $[3, m, 0]$ | $[5, m-1, 4]$ | $6m + 4$ | $3m + 2$ | $C_{3m+2}(\omega)$ |
| $[0, m, 3]$ | $[2, m, 1]$ | $[4, m-1, 5]$ | $6m + 4$ | $3m + 2$ | $C_{3m+2}(\omega^2)$ |
| $[0, m, 4]$ | $[2, m, 2]$ | $[4, m, 0]$ | $6m + 5$ | $3m + 2$ | $A_{3m+2}$ |
| $[1, m, 3]$ | $[3, m, 1]$ | $[5, m-1, 5]$ | $6m + 5$ | $3m + 3$ | $B_{3m+2}$ |
| $[1, m, 4]$ | $[3, m, 2]$ | $[5, m, 0]$ | $6m + 6$ | $3m + 3$ | $C_{3m+3}(\omega)$ |
| $[0, m, 5]$ | $[2, m, 3]$ | $[4, m, 1]$ | $6m + 6$ | $3m + 3$ | $C_{3m+3}(\omega^2)$ |

## 6. The indecomposable $kA_5$ modules

NOTE. If $P$ is any non-trivial 2-subgroup of $A_5$, the normaliser of $P$ in $A_5$ is contained in a subgroup $H$ isomorphic to $A_4$, and, for $x \notin H$, $P^x \cap H = 1$. Thus if $M$ is an indecomposable $kA_5$ module with vertex $P$, then $M_H = f(M) \oplus N$, where $N$ is projective. Let $V$ be the Sylow 2-subgroup of $A_5$ contained in $H$, and let $a, b$ be generators for $V$. Then $N_V \simeq (kV)^n$, and $n = \dim(a-1)(b-1)M$.

The indecomposable $kA_5$ modules $M$ such that $f(M) \simeq [i, m, j]$ will be called discrete, and those with $f(M) \simeq K_m(\sigma)$ will be called continuous.

(a) The discrete indecomposable modules.

For convenience we denote the three irreducible $kA_5$ modules by $\alpha$, $\beta$, $\gamma$ instead of $1$, $\eta_1$, $\eta_2$, and their projective covers by $\pi_\alpha$, $\pi_\beta$, $\pi_\gamma$, respectively. Section 3 shows that $kA_5$ has 15 nonprojective uniserial modules, each of which is a subquotient of one of $\pi_\beta$ and $\pi_\gamma$. Indeed $\pi_\beta$ may be described as the matrix representation which takes the generators $(12)(34)$, $(123)$, and $(12345)$ of $A_5$ respectively to the matrices

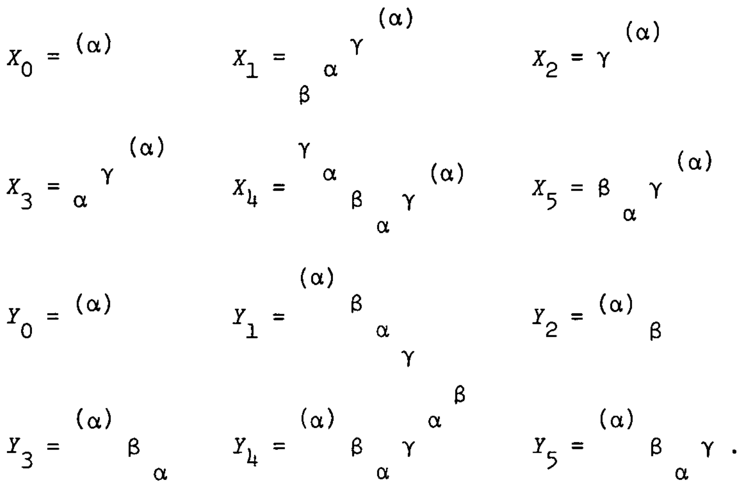
$$\begin{pmatrix} 1&1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \quad \begin{pmatrix} \omega&0&0&\upsilon&0&0&1&0\\ 0&\upsilon&\omega&\omega&\omega&\upsilon&\upsilon&1\\ 0&0&1&\omega&0&0&0&0\\ 0&0&0&\upsilon&0&0&\omega&0\\ 0&0&0&0&\omega&\upsilon&\upsilon&\upsilon\\ 0&0&0&0&0&1&\upsilon&0\\ 0&0&0&0&0&0&\omega&0\\ 0&0&0&0&0&0&0&\upsilon \end{pmatrix} \quad \begin{pmatrix} \omega&\omega&0&1&1&0&0&0\\ \upsilon&0&1&\omega&0&\omega&0&0\\ 0&0&1&\upsilon&\upsilon&1&0&0\\ 0&0&0&\upsilon&\upsilon&0&1&1\\ 0&0&0&\omega&0&1&\upsilon&0\\ 0&0&0&0&0&1&\omega&\omega\\ 0&0&0&0&0&0&\omega&\omega\\ 0&0&0&0&0&0&\upsilon&0 \end{pmatrix}$$

where $\omega$ is a fixed root of $t^2 + t + 1$ and $\upsilon = 1 + \omega$ is the other. A similar description of $\pi_\gamma$ is obtained by interchanging $\omega$ and $\upsilon$ in the above matrices.

The modules constructed by the method of Section 4 from the 15 nonprojective uniserial $kA_5$ modules may be described by diagrams of type

$$\text{(1)} \qquad \ldots {}^{\alpha}\,\beta\,{}_{\alpha}\,\gamma\,{}^{\alpha}\,\beta\,{}_{\alpha}\,\gamma\,{}^{\alpha} \ldots .$$

We denote by $[i, m, j]^*$ the module whose diagram is of the form (1) with $m + 1$ factors $\alpha$ in the top line and subdiagrams $X_i$, $Y_j$ to the left and right (respectively) of the extreme top line $\alpha$'s as follows:




The bracketed (α) indicates the position of the extreme factors α in the top line relative to the subdiagrams. Provided that $i + j \geq 6$ , we can assign a meaning to $[i, -1, j]^*$ as before.

THEOREM. *The modules* $[i, m, j]^*$ *are indecomposable, and afford a complete set of discrete indecomposable* $kA_5$ *modules, since* $f[i, m, j]^* = [i, m, j]$ .

Proof. 1. Indecomposability. Throughout this argument we interpret $m$ so as to exclude uniserial modules.

(i) Direct calculation shows that if $h$ is any endomorphism of $[0, m, 0]^*$, $[3, m, 3]^*$, $[3, m, 0]^*$ , or $[0, m, 4]^*$ , there exists $c \in k$ such that $h - c1$ is nilpotent. Hence these modules are indecomposable.

(ii) Now let $M$ be $[4, m, 4]^*$, $[0, m, 4]^*$ , or $[4, m, 0]^*$ . Each of these has a submodule $N$ containing the radical of $M$ and isomorphic to $[0, m+1, 0]^*$ . The quotient $M/N$ is isomorphic to a non-trivial submodule of $\beta \oplus \gamma$ , and is the only possible such quotient of $M$ . It follows that $N$ is invariant under any endomorphism $h$ of $M$ , so that $(h-c1)^n$ annihilates $N$ for some $c \in k$ and some $n$ . Hence $(h-c1)^{n+1}$ annihilates $M$ , which is therefore indecomposable.

(iii) Now let $M$ be $[0, m, 2]^*$, $[2, m, 0]^*$ , or $[2, m, 2]^*$ . Each of these has a quotient isomorphic to $[0, m, 0]^*$ , with kernel $K$

isomorphic to a non-trivial submodule of $\beta \oplus \gamma$ , and containing the only such components of the socle of $M$ . The submodule $K$ is therefore invariant under any endomorphism $h$ of $M$ , which then induces an endomorphism $h'$ of $M/K$ . Hence $(h'-c1)^n$ annihilates $M/K$ (for some $c \in k$ , some $n$ ). Since $M$ has no quotient isomorphic to a non-trivial submodule of $\beta \oplus \gamma$ , $(h-c1)^n$ must annihilate $M$ , showing that $M$ is indecomposable.

(iv) Arguments similar to those of (ii) and (iii) above, using the submodules and quotients proved indecomposable at each step allow us to establish indecomposability for the remaining $[i, m, j]^*$ .

2. The Green correspondence. If $M = [i, m, j]^*$ , $\dim(a-1)(b-1)M$ is equal to the number of uniserial modules of composition length 4 used in the construction of $M$ . Hence $M_{A_4}$ has 1 projective summand if $i \equiv 1 \pmod 3$ or $j \equiv 2 \pmod 3$ , 2 if both congruences hold, and none otherwise.

If $M_{A_4}$ remains indecomposable, the isomorphism type of $f(M)$ is determined by the isomorphism type of the restriction of the initial obvious uniserial submodule or quotient in the diagram for $M$ . It is therefore easily verified that in such a case $f(M)$ is as stated. For example, the diagram

$$\begin{matrix} \alpha & & \\ & \beta & & \gamma \\ & & \alpha & \end{matrix} \quad \text{corresponds to the diagram} \quad \begin{matrix} 0 & & 2 & & 1 & \\ & 1 & & 0 & & 2 \end{matrix} .$$

If $M_{A_4}$ has one projective summand, then (unless $M$ is uniserial, in which case the correspondence is easily verified directly) one can establish the required correspondence either by factoring out in $M$ by the obvious appropriate uniserial submodule of composition length 4 and considering the restriction of the quotient to $A_4$ , or else by considering the restriction of the kernel of the natural homomorphism of $M$ onto its obvious appropriate uniserial quotient of composition length 4 .

The isomorphism type of $f(M)$ when $M_{A_4}$ has two projective summands may then be established by a similar procedure, given that the type of

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$f(M')$ is known when $M'_{A_4}$ has one projective summand.

The list of discrete indecomposable $kA_5$ modules in the principal 2-block is therefore complete.

(b) The continuous indecomposable modules.

We construct indecomposable $kA_5$ modules $K^*_m(\sigma)$ $(\sigma \in k - \{0\})$ such that $f(K^*_m(\sigma)) = \dot{K}_m(\sigma)$, using the modules $M = [3, m, 4]$. We choose basis vectors $v_0, \ldots, v_m$ for the socle factors of type $\alpha$ in the diagram for $M$, ordering from left to right. Let $N$ be the submodule of $M$ generated by

$$v_m - c_{m-1}v_{m-1} - \ldots - c_0 v_0 ,$$

where

$$(t-\sigma)^m = t^{m-1} - c_{m-1}t^{m-1} - \ldots - c_0 .$$

Let

$$K^*_m(\sigma) = M/N .$$

It is easy to verify directly that $K^*_m(\sigma)$ is indecomposable, that the $K^*_m(\sigma)$ are non-isomorphic for distinct $\sigma$, and that $K^*_m(\sigma)$ remains indecomposable on restriction to $A_4$. Further direct verification shows that $K^*_m(\sigma)_{A_4}$ has a submodule isomorphic to $K_1(\sigma)$, which implies that $f(K^*_m(\sigma)) = K_m(\sigma)$.

## 7. Classification of modules for $q > 5$

LEMMA. *Let $A$ be a $k$-algebra of finite dimension as a vector space. Let $U$ be a direct sum of projective indecomposable $A$-modules, with exactly one summand of each isomorphism type, and let $B$ be the algebra* $\operatorname{end}_A(U)$. *Then the functor $M \mapsto \hom_A(U, M)$ sets up an equivalence between the category of finitely generated left $A$-modules and the category of finitely generated right $B$-modules.*

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Proof. This is a special case of theorems of Gabriel, [4]. The required quasi-inverse is $N \mapsto \hom_B(V, N)$ , where $V = \hom_A(U, A)$ .

Thus the isomorphism types of the indecomposable modules of the principal 2-block algebra of PSL(2, $q$) $(q \equiv 3 \pmod 8)$ are in one-to-one correspondence with the types of indecomposable modules of $kA_4$ , and can be constructed by the method described in Sections 4 and 5. Likewise the isomorphism types of indecomposable modules of the principal 2-block algebra of PSL(2, $q$) $(q \equiv 5 \pmod 8)$ are in one-to-one correspondence with the types of indecomposable modules of the principal 2-block algebra of $A_5$ , and can be constructed by the method described in Sections 4 and 6.

This classification theorem extends to other block algebras, such as the non principal 2-block algebra of $A_7$ , whenever the lemma can be invoked.

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