Compositio Mathematica **106:** 211–233, 1997.

Complex monodromy and the topology of real algebraic sets

CLINT McCRORY[1⋆] and ADAM PARUSIŃSKI[2⋆]
[1]*University of Georgia, Department of Mathematics, Athens, GA30602, USA*
[2]*Département de Mathématiques, Université d'Angers, 2 bd. Lavoisier, F-49045 Angers Cedex, France; School of Mathematics and Statistics, University of Sydney, Sydney NSW 2006, Australia*

Received 4 July 1995; accepted in final form 26 February 1996

Abstract. A relation between the Euler characteristics of the Milnor fibres of a real analytic function is derived from a simple identity involving complex monodromy and complex conjugation. A corollary is the result of Coste and Kurdyka that the Euler characteristic of the local link of an irreducible algebraic subset of a real algebraic set is generically constant modulo 4. A similar relation for iterated Milnor fibres of ordered sets of functions is used to define topological invariants of ordered collections of algebraic subsets.

Mathematics Subject Classifications (1991): 14P25, 32S50.

Key words: Milnor fibre, monodromy, Euler characteristic, real algebraic set, link.

1. Introduction

Let X be a real analytic variety, let $x \in X$, and let $f : X \to \mathbf{R}$ be a real analytic function with $f(x) = 0$. Let F be the Milnor fibre at x of a complexification $f_{\mathbf{C}}$ of f. We construct a geometric monodromy homeomorphism $h : F \to F$ such that $chch = 1$, where c is complex conjugation (Section 2.1). A special case of this monodromy relation – for weighted homogeneous polynomials – was first discovered by Dimca and Paunescu [DP]. The equation $chch = 1$ has implications for the action of c on eigenspaces of the algebraic monodromy h_*. As a consequence, the difference between the Euler characteristics of the *real* Milnor fibres of f over $+\delta$ and $-\delta$ can be expressed in terms of the dimensions of generalized eigenspaces of h_* (Section 2.2). Applying this result to a nonnegative function f which vanishes only at the point x, we obtain Sullivan's theorem [Su] that the link of x in X has even Euler characteristic,

$$\chi(\mathrm{lk}(x; X)) \equiv 0 \pmod 2.$$

Applying our result to a nonnegative defining function f for the subvariety Y of X, we recover (Section 2.4) a theorem of Coste and Kurdyka [CK]:

⋆ Research partially supported by a University of Sydney Research Grant. First author also partially supported by NSF grant DMS-9403887. We thank Laurentiu Paunescu for discussions which led to the results presented here.

If X is a real algebraic set and Y is an irreducible algebraic subset, then there exists a proper algebraic subset Z of Y such that for all points x and x' in $Y \setminus Z$,

$$\chi(\mathrm{lk}_x(Y;X)) \equiv \chi(\mathrm{lk}_{x'}(Y;X)) \pmod 4.$$

Here $\mathrm{lk}_x(Y;X)$ denotes the link at x of Y in X (Section 2.3), and $\chi(\mathrm{lk}_x(Y;X))$ is even by Sullivan's theorem.

Akbulut–King and Benedetti–Dedò have proved that a compact triangulable space of dimension less than or equal to 2 is homeomorphic to a real algebraic set if and only if the link of every point has even Euler characteristic (see [AK, p. 192]). Akbulut and King introduced further topological invariants with which they have characterized 3-dimensional algebraic sets [AK, Ch. VII]. Coste and Kurdyka showed that the mod 4 invariance theorem for the local link of a subvariety $Y \subset X$, and an analogous mod 8 invariance theorem for subvarieties $Z \subset Y \subset X$, are sufficient to recapture the 3-dimensional Akbulut–King invariants [CK]. Using a nice stratification of X, Coste and Kurdyka also defined combinatorial invariants for longer chains of algebraic subsets, and they asked whether these invariants can be expressed in terms of Euler characteristics of links. We give a new definition of their invariants in a more general setting, and we express these invariants in terms of Euler characteristics of iterated links, which we define using iterated Milnor fibres. In particular we prove that the Coste–Kurdyka invariants depend only on topological properties of the given chain of algebraic subsets.

We generalize the Coste–Kurdyka invariants as follows. The relation between complex conjugation and the monodromy of the complex Milnor fibre of an ordered family of functions $\{f_1, \dots, f_k\}$ (Section 3.1) gives a relation between the Euler characteristics of the real Milnor fibres over the points $(\pm\delta_1, \dots, \pm\delta_k)$ (Section 3.2). This in turn gives information about the Euler characteristic of the iterated link $\mathrm{lk}_x(X_1, \dots, X_k; X)$ of an ordered family $\{X_1, \dots, X_k\}$ of algebraic subsets of X. We show how to compute this Euler characteristic in terms of the topology of the family $\{X_1, \dots, X_k\}$ (Section 3.3). We prove that $\chi(\mathrm{lk}_x(X_1, \dots, X_k; X))$ is divisible by 2^k, and that, as x varies along an irreducible algebraic subset Y of X, $\chi(\mathrm{lk}_x(X_1, \dots, X_k; X))$ is generically constant mod 2^{k+1} (Sections 3.4 and 3.5).

Akbulut and King defined their invariants using resolution towers; Coste and Kurdyka used valuation-theoretic methods and stratifying families of polynomials. Our definition of invariants using complex monodromy introduces a new technique for the topological characterization of real algebraic sets.

2. The Euler characteristic of real Milnor fibres and complex monodromy

2.1. Complex conjugation and monodromy

Let X be an analytic subset of $\mathbf{R}^n$ and let $f : X \to \mathbf{R}$ be a real analytic function defined in a neighbourhood of $x_0 \in X$ such that $f(x_0) = 0$. Let $X_{\mathbf{C}} \subset \mathbf{C}^n$,

$f_{\mathbf{C}}\colon X_{\mathbf{C}} \to \mathbf{C}$ be complexifications of X and f, respectively. The Milnor fibration of $f_{\mathbf{C}}$ at x_0 (*cf.* [Lê], [Mi]) is the map

$$\Psi\colon B(x_0,\varepsilon) \cap f_{\mathbf{C}}^{-1}(S_\delta) \to S_\delta$$

induced by $f_{\mathbf{C}}$, where $B(x_0,\varepsilon)$ is the ball in $\mathbf{C}^n$ centered at x_0 with radius ε, S_δ is the circle in $\mathbf{C}$ with radius δ, and $0 < \delta \ll \varepsilon \ll 1$. The fibre of Ψ is called the Milnor fibre of $f_{\mathbf{C}}$ at x_0. We are particularly interested in the fibres over the real numbers,

$$F = \Psi^{-1}(\delta), \qquad F' = \Psi^{-1}(-\delta).$$

Let $h: F \to F$ be the geometric monodromy homeomorphism determined (up to homotopy) by Ψ. The automorphism induced at the homology level $h_*: H_*(F;\mathbf{C}) \to H_*(F;\mathbf{C})$ is called the algebraic monodromy.

Since $f_{\mathbf{C}}$ is a complexification of a real analytic function, complex conjugation acts on the total space of the Milnor fibration Ψ of $f_{\mathbf{C}}$ at x_0, fixing F and F' as sets. We denote the restriction of this action to F and F' by c and c', respectively. It was observed by Dimca and Paunescu [DP] that, for weighted homogeneous f, the monodromy homeomorphism h and the complex conjugation c on F satisfy $chch = 1$. We shall show that, in general, one may always choose h such that this relation holds. This will allow us to describe, for arbitrary f, the relation between the induced automorphisms h_* and c_* of the homology of the Milnor fibre F.

To this end, we construct a special geometric monodromy h compatible with complex conjugation. A trivialization of Ψ over the upper semicircle $S_\delta^+ = \{z \in S_\delta \mid \mathrm{Im}(z) \geqslant 0\}$ induces a homeomorphism

$$g\colon F \to F'.$$

Then

$$\bar{g} = c'gc\colon F \to F' \tag{1}$$

comes from the conjugate trivialization of Ψ over the lower semicircle, and

$$h = \bar{g}^{-1}g$$

is a monodromy homeomorphism associated to Ψ.

Let $h_*, c_*: H_*(F,\mathbf{C}) \to H_*(F,\mathbf{C})$ be the homomorphisms induced by h, c, respectively, on the homology of F with complex coefficients. Let $E_{\lambda,m} = \ker(h_* - \lambda\mathbf{I})^m$. Then $E_{\lambda,1}$ is the eigenspace of h_* corresponding to the eigenvalue λ, and $E_{\lambda,N}$, for N large enough, is the generalized eigenspace corresponding to λ. Note also that c, and hence c_*, is an involution; that is, $c^2 = 1$.

PROPOSITION 1. *Let $g\colon F \to F'$, $\bar{g}\colon F \to F'$ and $h = \bar{g}^{-1}g$ be as above. Then*

(i) *ch corresponds via g to c'; that is, $g^{-1}c'g = ch$. In particular, $chch = 1$, and hence*

$$c_* h_* c_* h_* = 1; \tag{2}$$

(ii) *c_* interchanges the eigenspaces of h_* corresponding to conjugate eigenvalues; that is,*

$$c_* E_{\lambda,m} = E_{\bar{\lambda},m}.$$

Proof. Part (i) follows directly from the definition of h. Indeed, by (1), $\bar{g}^{-1} = cg^{-1}c'$, which gives

$$ch = c\bar{g}^{-1}g = c(cg^{-1}c')g = g^{-1}c'g,$$

as required. By induction on m, (2) gives

$$c_* h_*^m c_* = h_*^{-m},$$

which implies

$$h_*^m c_*(h_* - \lambda \mathbf{I})^m c_* = h_*^m (h_*^{-1} - \lambda \mathbf{I})^m = (\mathbf{I} - \lambda h_*)^m = \lambda^m (\lambda^{-1}\mathbf{I} - h_*)^m.$$

Taking the kernels of both sides of the above equality we get $c_* E_{\lambda,m} = E_{\lambda^{-1},m}$. Then (ii) follows from the monodromy theorem [Lê], which says that all eigenvalues of h_* are roots of unity; in particular $\lambda^{-1} = \bar{\lambda}$. □

2.2. Real Milnor fibres

Let $f : X \to \mathbf{R}$ be, as above, a real analytic function defined in a neighbourhood of x_0 with $f(x_0) = 0$. There is no real analogue of the Milnor fibration. Nevertheless, we may define the positive and the negative Milnor fibres of f at x_0 by

$$\begin{aligned} F_+ &= B(x_0, \varepsilon) \cap f^{-1}(\delta), \\ F_- &= B(x_0, \varepsilon) \cap f^{-1}(-\delta), \end{aligned}$$

where $0 < \delta \ll \varepsilon \ll 1$, and $B(x_0, \varepsilon)$ is now the ball of radius ϵ about x_0 in $\mathbf{R}^n$. In general, F_+ and F_- are not homeomorphic.

Let $f_{\mathbf{C}}$ be a complexification of f. Consider the associated Milnor fibration Ψ described in Section 2.1. The positive real Milnor fibre F_+ is the fixed point set of the action of complex conjugation c on the complex Milnor fibre $F = \Psi^{-1}(\delta)$. In particular, by the Lefschetz fixed point theorem, the Euler characteristic of F_+ equals the Lefschetz number of c; that is,

$$\chi(F_+) = L(c) = \sum_i (-1)^i \operatorname{Tr}[c_i : H_i(F; \mathbf{C}) \to H_i(F; \mathbf{C})].$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

Similarly,

$$\chi(F_-) = L(c') = \sum_i (-1)^i \operatorname{Tr}[c'_i : H_i(F';\mathbf{C}) \to H_i(F';\mathbf{C})].$$

The following observation establishes a link between the complex monodromy h and the real Milnor fibres of f. It plays a crucial role in our interpretation and generalizations of the Coste–Kurdyka results.

PROPOSITION 2. $\chi(F_+) - \chi(F_-)$ *is always even, and*

$$\chi(F_+) - \chi(F_-) \equiv 2\, l(h;-1) \pmod 4,$$

where, for the eigenvalue λ,

$$l(h;\lambda) = \sum_i (-1)^i \dim E_\lambda(h_i),$$

and $E_\lambda(h_i) = \ker(h_*|_{H_i(F,\mathbf{C})} - \lambda\mathbf{I})^N$, *for* N *large enough.*

Proof. By Proposition 1(i),

$$\chi(F_+) - \chi(F_-) = L(c) - L(ch).$$

By Proposition 1(ii), for $\lambda \neq -1, 1$, c_* interchanges $E_\lambda(h_i)$ and $E_{\bar\lambda}(h_i)$. Hence, the trace of c_* on $E_\lambda(h_i) \oplus E_{\bar\lambda}(h_i)$ is 0. Consequently, in the calculation of $L(c)$ only the eigenvalues -1 and 1 matter. Both $E_{-1}(h_i)$ and $E_1(h_i)$ are preserved by c_i and $h_i = h_*|_{H_i(F,\mathbf{C})}$. By Proposition 1(ii), c_i preserves the filtration

$$E_1(h_i) = \ker(h_i - \mathbf{I})^N \supset \cdots \supset \ker(h_i - \mathbf{I})^1 \supset \{0\}.$$

On the quotient spaces of this filtration, h_i acts as the identity, and hence $c_i \equiv (ch)_i$. This shows, by additivity of trace,

$$\operatorname{Tr}(c_i|E_1(h_i)) - \operatorname{Tr}((ch)_i|E_1(h_i)) = 0.$$

Hence the eigenvalue $\lambda = 1$ does not contribute to $L(c) - L(ch)$. By a similar argument

$$\operatorname{Tr}(c_i|E_{-1}(h_i)) = -\operatorname{Tr}((ch)_i|E_{-1}(h_i)).$$

This gives

$$\chi(F_+) - \chi(F_-) = 2\sum_i (-1)^i \operatorname{Tr}(c_i|E_{-1}(h_i)).$$

Since c is an involution, it can have only 1 and -1 as eigenvalues. This implies

$$\operatorname{Tr}(c_i|E_{-1}(h_i)) \equiv \dim E_{-1}(h_i) \pmod 2,$$

which completes the proof. □

By a similar argument, $\chi(F_+) + \chi(F_-)$ is even, and

$$\chi(F_+) + \chi(F_-) \equiv 2l(h;1) \pmod 4.$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

Remark 1. In the proof of Proposition 2 we used only the relation between c_* and the semisimple part of h_* given by (2). But from (2) we also get a relation between c_* and the nilpotent part of the monodromy. The nilpotent part of h_* defines the weight filtration on $H^*(F)$ (*cf.* [AGV]). Clearly, by Proposition 1(ii), c_* preserves this filtration. Fix an eigenvalue λ of h_*. Let $N = h_* - \lambda\mathbf{I}$, respectively $\bar{N} = h_* - \bar{\lambda}\mathbf{I}$, denote the nilpotent part acting on E_λ, respectively $E_{\bar{\lambda}}$. Then on E_λ,

$$h_* c_* N = \lambda(-\bar{N})c_*,$$

which gives

$$\bar{\lambda} c_* N + \lambda \bar{N} c_* = -\bar{N} c_* N,$$

and hence, on the associated graded $\mathrm{Gr}(E_\lambda)$ with respect to the weight filtration,

$$\bar{\lambda} c_* N + \lambda \bar{N} c_* = 0.$$

2.3. Tubular neighbourhoods and links

Fix an algebraic set $X \subset \mathbf{R}^n$, and let Y be a compact algebraic subset of X. We can always find a nonnegative proper polynomial function $f\colon X \to \mathbf{R}$ defining Y; that is, $Y = f^{-1}(0)$. Then, for $\delta > 0$ sufficiently small,

$$T(Y, X) = f^{-1}[0, \delta]$$

is a tubular neighbourhood of Y in X. By $\mathrm{lk}(Y; X)$, the *link* of Y in X, we mean the boundary of $T(Y, X)$, that is

$$\mathrm{lk}(Y; X) = f^{-1}(\delta).$$

If $Y = \{x_0\}$, then $\mathrm{lk}(Y; X)$ is called the link of x_0 in X and denoted by $\mathrm{lk}(x_0; X)$. (For the dependence of $\mathrm{lk}(Y; X)$ on f and δ see Remark 2 below.)

Let Y be smooth at x_0 and let N_{x_0} be the normal space in $\mathbf{R}^n$ to Y at x_0. Following [CK] we define the link of Y in X at x_0 as $\mathrm{lk}(x_0; X \cap N_{x_0})$ and we denote it by $\mathrm{lk}_{x_0}(Y; X)$. Note that $\mathrm{lk}_{x_0}(Y; X)$ is defined only at a generic point of Y; that is x_0 has to be a nonsingular point of Y, and X has to be sufficiently equisingular along Y at x_0 so that $\mathrm{lk}(x_0; X \cap N_{x_0})$ does not depend on the choice of N_{x_0}. We give a different definition which makes sense at any point of Y. Let f be a non-negative polynomial defining Y. For $x_0 \in X$ define the *localization* at x_0 of $\mathrm{lk}(Y; X)$ as the positive Milnor fibre of f at x_0; that is,

$$\widetilde{\mathrm{lk}}_{x_0}(Y; X) = B(x_0, \varepsilon) \cap f^{-1}(\delta),$$

where $0 < \delta \ll \varepsilon \ll 1$.

Let x_0 be a generic point of Y. In particular we assume that, near x_0, Y is nonsingular and is a stratum of a Whitney stratification which satisfies the Thom condition a_f. Then, by Thom's isotopy lemmas (cf. [CG]), there is a stratified homeomorphism

$$\widetilde{\mathrm{lk}}_{x_0}(Y;X) \cong \mathrm{lk}_{x_0}(Y;X) \times B^d, \tag{3}$$

where d is the dimension of Y at x_0 and B^d denotes the ball of dimension d. Then, in particular, $\widetilde{\mathrm{lk}}_{x_0}(Y;X)$ and $\mathrm{lk}_{x_0}(Y;X)$ are homotopy equivalent. Note that for the localized link $\widetilde{\mathrm{lk}}_{x_0}(Y;X)$ we do not need the compactness of Y; the assumption that Y is closed in X is sufficient.

Note that the notion of the link of a subset can be reduced to that of the link of a point, since any real algebraic subset Y of X can be contracted to a point so that X/Y naturally has the structure of a real algebraic set (*cf.* [BCR, Prop. 3.5.5]).

All the above definitions and remarks make perfect sense if X and Y are closed semialgebraic subsets of $\mathbf{R}^n$, and f is continuous and semialgebraic. In particular, choosing a semialgebraic triangulation of (X, Y), which exists and is unique up to isotopy by [SY], we see that links in the PL category are special cases of semialgebraic links.

Similarly one defines the tubular neighbourhood and the link of Y in a closed semialgebraic subset of X, or in a finite family of closed semialgebraic subsets of X, and so in any semialgebraic stratification of X.

Finally, let U be an arbitrary (not necessarily closed) semialgebraic subset of X, and let $Y \subset X$ be compact and semialgebraic. Then the tubular neighbourhood and link of Y in U can be defined by

$$T(Y,U) = f^{-1}[0,\delta] \cap U, \quad \mathrm{lk}(Y;U) = f^{-1}(\delta) \cap U,$$

for $\delta > 0$ sufficiently small. Every semialgebraic subset U of X is a union of strata of a semialgebraic stratification of X. This allows us to use stratifications to study the properties of such links and tubular neighbourhoods, in particular to show that they are well-defined up to homeomorphism.

LEMMA 1. *Let Y be a compact semialgebraic subset of X, and let U be another (not necessarily closed) semialgebraic subset of X. Then the following spaces are homotopy equivalent:*

$$Y \simeq Y \cup T(Y,U), \qquad U \setminus Y \simeq \overline{U \setminus T(Y,U)},$$

where the closure is taken in U. In particular,

$$\chi(\mathrm{lk}(Y;U)) = \chi(Y) + \chi(U \setminus Y) - \chi(U \cup Y). \tag{4}$$

Proof. For U closed in X the statement follows, for instance, from the triangulability of the pair (Y, U). If U is not closed, then we can find a semialgebraic stratification compatible with Y and U, and then simultaneously triangulate the closures of the strata.

The last statement follows from

$$\mathrm{lk}(Y;U) = T(Y,U) \cap \overline{U \setminus T(Y,U)}.$$

The details are left to the reader. □

Remark 2. (Uniqueness of links and tubular neighbourhoods)

(a) For real algebraic (or even closed semialgebraic) $X \subset \mathbf{R}^n$, the link at a point is well-defined up to semialgebraic homeomorphism [CK, Prop. 1]. A similar result holds for the link of Y in X at x_0 [*loc. cit.*].

(b) Uniqueness up to stratified homeomorphism was established also in [Du, Prop. 1.7, Prop. 3.5].

(c) In [DS] the authors define a functor which allows one to study the sheaf cohomology of links without referring to the actual construction of the link. Let $\mathcal{F}$ be a (semialgebraically) constructible bounded complex of sheaves on $U = X \setminus Y$. Denote by $i\colon Y \hookrightarrow X$, $j\colon U \hookrightarrow X$ the embeddings. Then the local link cohomology functor Λ_Y of Y in X is defined by

$$\Lambda_Y \mathcal{F} = i^* R j_* \mathcal{F}.$$

In particular, it is shown in [*loc. cit.*] that

$$H^*(\mathrm{lk}(Y;X);\mathbf{Q}) = \mathbf{H}^*(Y;\Lambda_Y \mathbf{Q}_U), \tag{5}$$

where $\mathbf{Q}_U$ is the constant sheaf on U and $\mathbf{H}$ denotes hypercohomology. Clearly the right-hand side of (5) does not depend on the choice of the link. Let $x_0 \in Y$ and denote by i_{x_0} the embedding of x_0 in Y. Using arguments similar to [DS], one may show that the cohomology of $\widetilde{\mathrm{lk}}_{x_0}(Y;X)$ equals the stalk cohomology of $\Lambda_Y \mathbf{Q}_U = i^* R j_* \mathbf{Q}_U$; that is,

$$H^*(\widetilde{\mathrm{lk}}_{x_0}(Y;X);\mathbf{Q}) = H^*(\Lambda_Y \mathbf{Q}_U)_{x_0}.$$

2.4. The Coste–Kurdyka theorem

Let X be an algebraic subset of $\mathbf{R}^n$ and let Y be an algebraic subset of X. By a theorem of Sullivan [Su], for any $x \in X$, the Euler characteristic $\chi(\mathrm{lk}(x;X))$ of the link of X at x is even. Hence the same is true for $\chi(\mathrm{lk}_x(Y;X))$ for x generic in Y, since then $\mathrm{lk}_x(Y;X) = \mathrm{lk}(x; X \cap N_x)$.

Let $f : X \to \mathbf{R}$ be a non-negative polynomial such that $f^{-1}(0) = Y$ and let $X_{\mathbf{C}}, Y_{\mathbf{C}}$ and $f_{\mathbf{C}} : X_{\mathbf{C}} \to \mathbf{C}$ be complexifications of X, Y and f respectively (*cf.* [BR, Section 3.3]). Consider the real and complex Milnor fibres of f and $f_{\mathbf{C}}$, respectively, at $x \in Y$. The negative real Milnor fibre F_- is empty, and the positive real Milnor fibre F_+ is the localized link $\widetilde{\mathrm{lk}}_x(Y;X)$. Consequently, by Proposition 2, $\chi(\widetilde{\mathrm{lk}}_x(Y;X))$ is always even and

$$\chi(\widetilde{\mathrm{lk}}_x(Y;X)) \equiv 2\,l(h_x;-1) \pmod 4, \tag{6}$$

where h_x denotes the complex monodromy induced by the Milnor fibration at x. Thus we obtain Sullivan's theorem as a corollary of Proposition 2.

Assume now that Y is an irreducible algebraic set. Then $Y_{\mathbf{C}}$ is also irreducible (*cf.* [BR, Section 3.3]). Note that the left hand side of (6) is constant on strata of some semialgebraic stratification of Y. Nevertheless, the generic value of the Euler characteristic $\chi(\mathrm{lk}_x(Y, X))$ along Y may not be well-defined. Indeed, even for irreducible Y, a semialgebraic stratification of Y may have more than one open stratum.

The right-hand side of (6) makes sense for any $x \in Y_{\mathbf{C}}$ and is constant on strata of some algebraic stratification of $Y_{\mathbf{C}}$. For instance, it suffices to take the restriction to $Y_{\mathbf{C}}$ of a Whitney stratification of $X_{\mathbf{C}}$ which satisfies the Thom conditon $a_{f_{\mathbf{C}}}$. Now if $Y_{\mathbf{C}}$ is irreducible, then there is only one open (and dense) stratum S_0 in $Y_{\mathbf{C}}$. Thus it makes sense to talk about $l(h_x; -1)$ for generic $x \in Y_{\mathbf{C}}$. Since $\dim_{\mathbf{C}} (Y_{\mathbf{C}} \setminus S_0) < \dim_{\mathbf{C}} Y_{\mathbf{C}}$, we have $\dim_{\mathbf{R}} (Y \setminus S_0) < \dim_{\mathbf{R}} Y$. Hence (6) implies the following result of Coste and Kurdyka [CK, Thm. 1′].

THEOREM 1. *Let Y be an irreducible real algebraic subset of the algebraic set X. The Euler characteristic $\chi(\mathrm{lk}_x(Y; X))$ of the link of Y in X at x is generically constant modulo 4; that is, there exists a real algebraic subset $Z \subset Y$, with* $\dim Z < \dim Y$, *such that for all $x, x' \in Y \setminus Z$,*

$$\chi(\mathrm{lk}_x(Y; X)) \equiv \chi(\mathrm{lk}_{x'}(Y; X)) \pmod 4. \qquad \square$$

Remark 3. Theorem 1 can also be proved using Akbulut and King's *resolution towers* [AK, Exercise, p. 192].

Remark 4. Theorem 1 is equivalent to the constancy along Y of $\chi(\mathrm{lk}(x; X))$ mod 4. Indeed, let x be a generic point of Y. Then Y is nonsingular of dimension $d = \dim Y$ at x, and there is a homeomorphism

$$\mathrm{lk}(x; X) \cong \mathrm{lk}_x(Y; X) * S^{d-1},$$

where $*$ denotes the join. This, together with (3), implies

$$\chi(\mathrm{lk}(x; X)) = \begin{cases} 2 - \chi(\widetilde{\mathrm{lk}}_x(Y; X)), & d \text{ odd} \\ \chi(\widetilde{\mathrm{lk}}_x(Y; X)), & d \text{ even.} \end{cases}$$

At special points of Y the relation between $\widetilde{\mathrm{lk}}_x(Y; X)$ and $\mathrm{lk}(x; X)$ is more delicate. Using arguments similar to the proof of Lemma 1, the interested reader may check that at an arbitrary point x of Y,

$$\chi(\mathrm{lk}(x; X)) = \chi(\widetilde{\mathrm{lk}}_x(Y; X)) + \chi(\mathrm{lk}(x; Y)) - \chi(\mathrm{lk}(\{\{x\}, Y\}; X)),$$

where $\mathrm{lk}(\{\{x\}, Y\}; X)$ is the iterated link defined in section 3.3 below. In particular, by Theorem 2

$$\chi(\mathrm{lk}(x; X)) \equiv \chi(\widetilde{\mathrm{lk}}_x(Y; X)) + \chi(\mathrm{lk}(x; Y)) \pmod 4.$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

3. Iterated Milnor fibres and Coste–Kurdyka invariants

3.1. Monodromies induced by a finite ordered set of functions

Let X be a complex analytic subset of $\mathbf{C}^n$, and let $f = (f_1, \dots, f_k) : X \to \mathbf{C}^k$ be a complex analytic map defined in a neighbourhood of $x_0 \in X$ such that $f(x_0) = 0$. If $k > 1$ then, in general, it is not possible to define the Milnor fibre of f at x_0 unambiguously (see, for instance, [Sa]). But if we consider $\{f_1, \dots, f_k\}$ as an ordered set of complex-valued functions, then we will show that the notions of Milnor fibre and Milnor fibration make sense.

LEMMA 2. *Let $f = (f_1, \dots, f_k) \colon X \to \mathbf{C}^k$ be a complex analytic map and let $x_0 \in Y = f^{-1}(0)$. Then f induces a locally trivial topological fibration*

$$\Psi \colon B(x_0, \varepsilon) \cap f^{-1}(T^k_\delta) \to T^k_\delta,$$

where $\delta = (\delta_1, \dots, \delta_k)$ and ε are chosen such that $0 < \delta_k \ll \cdots \ll \delta_1 \ll \varepsilon \ll 1$, and T^k_δ is the torus $\{(z_1, \dots, z_k) \in \mathbf{C}^k \mid |z_i| = \delta_i,\ i = 1, \dots, k\}$.

Moreover,

(i) *up to a fibred homeomorphism, the map $\Psi = \Psi(f, x_0)$ does not depend on the choice of δ and ε, and*
(ii) *there exists a stratification $\mathcal{S}$ of Y such that, as x_0 varies in a stratum of $\mathcal{S}$, the type of the map $\Psi(f, x_0)$ is locally constant up to fibred homeomorphism.*

Proof. The statement is well known if f is *sans éclatement*, in particular if there exist Whitney stratifications of X and $\mathbf{C}^k$ which stratify f with the Thom condition a_f. Such stratifications always exists if $k = 1$. The general case can be reduced to the *sans éclatement* case by Théorème 1 of [Sa], or derived directly from Lagrangian specialization. We present the latter argument.

First recall briefly the proof for $k = 1$. Choose a Whitney stratification of X compatible with $Y = f^{-1}(0)$. Define the projectivized relative conormal space to f as

$$C_f = \bigcup_S \overline{\{(x, H) \in \mathbf{C}^n \times \check{\mathbf{P}}^{n-1} \mid x \in S, H \supset T_x f|_S\}},$$

where $T_x f|_S$ denotes the tangent space to the level of $f|_S$ through x, and the union is taken over all strata $S \subset X \setminus Y$. Let $\pi : C_f \to \mathbf{C}$ be the composition of f with the standard projection $C_f \to X$. Then, by construction, $\pi^{-1}(\lambda)$, for $\lambda \neq 0$ and sufficiently small, is a Lagrangian subvariety of $\mathbf{C}^n \times \check{\mathbf{P}}^{n-1} = \mathbf{P}T^*\mathbf{C}^n$. By Lagrangian specialization (see, for instance, [HMS, Cor. 4.2.1] or [LT]) the same is true for $\lambda = 0$, and then

$$\pi^{-1}(0) = \bigcup C_{Y_\alpha},$$

where Y_α are analytic subsets of Y and C_{Y_α} are their (absolute) conormal spaces. Then any stratification compatible with $\{Y_\alpha\}$ satisfies the Thom condition a_f.

For $k > 1$ this argument fails only if the dimension of $\pi^{-1}(0)$ is bigger than that of $\pi^{-1}(\lambda)$, $\lambda > 0$; that is, $\dim_{\mathbf{C}} \pi^{-1}(0) > n - 1$. But as we show below in Lemma 3, the dimension of the set of limits $\pi^{-1}(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_k) \to 0$, along a cuspidal neighbourhood $0 < |\lambda_k| \ll \cdots \ll |\lambda_1| \ll 1$, cannot jump, and so in our case it stays constant. Hence the limit is a Lagrangian subvariety of $\mathbf{C}^n \times \check{\mathbf{P}}^{n-1}$. Now using a standard argument we may refine any Whitney stratification of X compatible with Y to a stratification which satisfies the Thom condition a_f for all limits $x \to x_0 \in Y$ such that $0 < |f_k(x)| \ll \cdots \ll |f_1(x)| \ll 1$. Then the statement follows by standard arguments of stratification theory as in the case $k = 1$.

To complete the proof of Lemma 2, we have to show that taking limits along a cuspidal neighbourhood does not increase the dimension of the fibre. This is a general fact which holds also in the real analytic case. We present a proof based on standard properties of subanalytic sets and the Łojasiewicz inequality (see [BM]).

LEMMA 3. *Let Z be a compact subanalytic subset of $\mathbf{R}^N$, and let $\varphi\colon Z \to \mathbf{R}^k$ be a continuous subanalytic mapping. Given positive real numbers $m_1, \ldots, m_k$, consider the space of limits of $\varphi^{-1}(\lambda_1, \ldots, \lambda_k)$, $(\lambda_1, \ldots, \lambda_k) \to 0$, over a cuspidal neighbourhood*

$$\Gamma = \{\lambda \in \mathbf{R}^k \mid 0 < |\lambda_k|^{m_k} < \cdots < |\lambda_1|^{m_1}\},$$

that is,

$$Z_{\Gamma,0} = \varphi^{-1}(0) \cap \overline{\varphi^{-1}(\Gamma)}.$$

Then, if $\frac{m_1}{m_2}, \ldots, \frac{m_{k-1}}{m_k}$ are sufficiently large,

$$\dim Z_{\Gamma,0} \leqslant \dim Z - k.$$

Proof. The proof is by induction on k, the case $k = 1$ being obvious.

Let $Z' = \varphi_k^{-1}(0)$. Without loss of generality we may assume that $Z = \overline{Z \setminus Z'}$. Hence $\dim Z' \leqslant \dim Z - 1$. Let $\varphi' = (\varphi_1, \ldots, \varphi_{k-1})\colon Z' \to \mathbf{R}^{k-1}$. By inductive assumption, for $\frac{m_1}{m_2}, \ldots, \frac{m_{k-2}}{m_{k-1}}$ sufficiently large and $\Gamma' = \{\lambda' \in \mathbf{R}^{k-1} \mid 0 < |\lambda_{k-1}|^{m_{k-1}} < \cdots < |\lambda_1|^{m_1}\}$, the set

$$Z'_{\Gamma',0} = \varphi'^{-1}(0) \cap \overline{\varphi'^{-1}(\Gamma')}$$

is of dimension not greater than $\dim Z - k$. Thus to complete the inductive step we show that for m_k small enough

$$Z'_{\Gamma',0} = Z_{\Gamma,0}. \tag{7}$$

The inclusion $\subset$ of (7) is obvious. To show $\supset$ we replace Z by

$$\tilde{Z} = \overline{\{x \in Z \mid 0 < |\varphi_{k-1}(x)|^{m_{k-1}} < \cdots < |\varphi_1(x)|^{m_1}\}}$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

and in what follows we shall work on $\tilde{Z}$. Since dist(x, Z') and φ_k are continuous subanalytic functions with the same zero sets, by the Łojasiewicz inequality, for m sufficiently large,

$$[\mathrm{dist}(x, Z')]^m \leqslant |\varphi_k(x)|.$$

Also by the Łojasiewicz inequality, for M sufficiently large,

$$\mathrm{dist}(x, \varphi^{-1}(0)) \geqslant |\varphi_{k-1}(x)|^M.$$

We claim that (7) is satisfied provided $m_k < \frac{m_{k-1}}{mM}$. Indeed, $|\varphi_k(x)|^{m_k} < |\varphi_{k-1}(x)|^{m_{k-1}}$ then implies

$$\begin{aligned} \mathrm{dist}(x, Z') \leq |\varphi_k(x)|^{1/m} &< |\varphi_{k-1}(x)|^{\frac{m_{k-1}}{mm_k}} \\ &< |\varphi_{k-1}(x)|^M \leqslant \mathrm{dist}(x, \varphi^{-1}(0)), \end{aligned}$$

which implies $\supset$ in (7). This completes the proofs of Lemmas 2 and 3. $\square$

We call the map Ψ defined in Lemma 2 the *Milnor fibration* and its fibre $F = B(x, \epsilon) \cap f^{-1}(\delta)$ the *Milnor fibre* of the ordered family of functions $\{f_1, \ldots, f_k\}$ at x_0. Such a fibration defines, up to homotopy, homeomorphisms $h_i : F \to F$, $i = 1, \ldots, k$, called the geometric monodromy homeomorphisms. Since the fundamental group of T_δ^k is commutative, the induced homomorphisms on homology (the algebraic monodromies) commute.

The sheaf cohomology of the Milnor fibre of $\{f_1, \ldots, f_k\}$ can be defined in terms of neighbouring cycles. Recall that for $f : X \to \mathbf{C}$, and a constructible bounded complex of sheaves $\mathcal{F}$ on X, the sheaf of neighbouring cycles $\psi_f \mathcal{F}$ (in fact, again a complex of sheaves) on $X_0 = f^{-1}(0)$ is defined as follows [KS, p. 350]. Let

$$\psi_f \mathcal{F} = i^* R(j \circ \tilde{\pi})_* (j \circ \tilde{\pi})^* \mathcal{F},$$

where $i \colon X_0 \hookrightarrow X$, $j \colon X \setminus X_0 \hookrightarrow X$ denote the embeddings, and $\tilde{\pi} \colon \tilde{X} \to X \setminus X_0$ is the cyclic covering of $X \setminus X_0$ induced from the universal covering of $\mathbf{C}^*$ by the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\tilde{\pi}"'] & C \arrow[d, "\pi = \exp"] \\ X \setminus X_0 \arrow[r, "f"] & C^* \end{tikzcd}$$

Then, for $x \in X_0$ and the Milnor fibre $F = B(x, \varepsilon) \cap f^{-1}(\delta)$,

$$H^i(F; \mathcal{F}) = H^i(\psi_f \mathcal{F})_x.$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

In general, if F is the Milnor fibre of $\{f_1, \dots, f_k\}$ at $x \in f^{-1}(0)$, then

$$H^i(F;\mathcal{F}) = H^i(\psi_{f_1}\psi_{f_2}\cdots\psi_{f_k}\mathcal{F})_x. \tag{8}$$

We show this by induction on k. Choose a Whitney stratification $\mathcal{S}$ of $F'' = f_3^{-1}(0) \cap \ldots \cap f_k^{-1}(0)$ such that $\psi_{f_2}\cdots\psi_{f_k}\mathcal{F}$ is constructible with respect to $\mathcal{S}$, and satisfies the Thom condition a_{f_2}. Then for a regular value δ_1 of f_1 restricted to all strata of $\mathcal{S}$,

$$(\psi_{f_2}\psi_{f_3}\cdots\psi_{f_k}\mathcal{F})|_{f_1^{-1}(\delta_1)\cap f_2^{-1}(0)\cap F''} = \psi_{f_2}\big((\psi_{f_3}\cdots\psi_{f_k}\mathcal{F})|_{f_1^{-1}(\delta_1)\cap F''}\big).$$

In particular, we may take $\delta_1 \neq 0$ and sufficiently small. Repeating this procedure, we show

$$(\psi_{f_2}\cdots\psi_{f_k}\mathcal{F})|_{f_1^{-1}(\delta_1)\cap f_2^{-1}(0)\cap F''} = \psi_{f_2}\psi_{f_3}\cdots\psi_{f_k}(\mathcal{F}|_{f_1^{-1}(\delta_1)}).$$

Hence, for $F' = B(x,\varepsilon) \cap f_1^{-1}(\delta_1) \cap f_2^{-1}(0) \cap \ldots \cap f_k^{-1}(0)$,

$$\begin{aligned} H^i(\psi_{f_1}\psi_{f_2}\cdots\psi_{f_k}\mathcal{F})_x &= H^i(F';\psi_{f_2}\cdots\psi_{f_k}\mathcal{F}) \\ &= H^i(F';\psi_{f_2}\cdots\psi_{f_k}(\mathcal{F}|_{f_1^{-1}(\delta_1)})) \\ &= H^i(F;\mathcal{F}). \end{aligned}$$

The last equation follows from the inductive assumption applied to the sheaf $\mathcal{F}|_{f_1^{-1}(\delta_1)}$ on $X \cap f_1^{-1}(\delta_1)$ and the set of functions $\{f_2, \dots, f_k\}$.

3.2. Real Milnor fibres of a finite ordered set of functions

Let X be a real analytic subset of $\mathbf{R}^n$, and let $f = (f_1, \dots, f_k)\colon X \to \mathbf{R}^k$ be a real analytic map defined in a neighbourhood of $x_0 \in X$, $f(x_0) = 0$. Let $X_{\mathbf{C}}$ and $f_{\mathbf{C}}$ be complexifications of X and f, respectively. In particular, by Lemma 2, $\{f_{1\mathbf{C}}, \dots, f_{k\mathbf{C}}\}$, as an ordered set, induces the Milnor fibration Ψ at x_0.

Similarly, for each $\gamma = (\gamma_1, \dots, \gamma_k) \in \{0,1\}^k$, we may define the real Milnor fibre

$$F_\gamma = B(x_0,\varepsilon) \cap f^{-1}((-1)^{\gamma_1}\delta_1, \dots, (-1)^{\gamma_k}\delta_k),$$

where $0 < \delta_k \ll \cdots \ll \delta_1 \ll \varepsilon \ll 1$ and $B(x_0,\varepsilon)$ now denotes the ball in $\mathbf{R}^n$.

Complex conjugation acts on Ψ and preserves each fibre

$$F_{\mathbf{C},\gamma} = \Psi^{-1}((-1)^{\gamma_1}\delta_1, \dots, (-1)^{\gamma_k}\delta_k).$$

Denote the restriction of this action to $F_{\mathbf{C},\gamma}$ by c_γ. Then F_γ is the fixed point set of c_γ.

As in Section 2.1, we let $F = F_{\mathbf{C},(0,\dots,0)}$, and we construct complex monodromies $h_\gamma\colon F \to F$ compatible with complex conjugation; that is, satisfying

$$g_\gamma^{-1} c_\gamma g_\gamma = c h_\gamma,$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

where $g_\gamma : F \to F_{\mathbf{C},\gamma}$ is a homeomorphism. Since the fundamental group of the base space T_δ^k of Ψ is commutative, all the h_γ commute up to homotopy. In particular the induced automorphisms $h_{\gamma,*}$ on homology are generated by those which come from $\gamma(j) = (0,\dots,0,1,0,\dots,0)$, 1 in the j-th place. Denote $h_{\gamma(j)}$ by $h_{(j)}$ for short. Thus for $\gamma = (\gamma_1,\dots,\gamma_k)$,

$$h_{\gamma,*} = \prod_{j=1}^{k} h_{(j),*}^{\gamma_j}.$$

For the set of eigenvalues $\lambda = (\lambda_1,\dots,\lambda_k)$ and multiplicities $m = (m_1,\dots,m_k)$, we let $E_{\lambda,m} = \bigcap_j \ker(h_{(j),*} - \lambda_j \mathbf{I})^{m_j}$. Then the argument of the proof of Proposition 1 generalizes, and we have the following:

PROPOSITION 3. *Let g_γ and h_γ and c_γ be as above. Then*

(i) *via g_γ, complex conjugation c_γ on $F_{\mathbf{C},\gamma}$ corresponds to ch_γ on F; in particular, $ch_\gamma ch_\gamma = 1$;*

(ii) *c_* interchanges the common eigenspaces of $h_{(j),*}$ corresponding to conjugate eigenvalues; that is,*

$$c_* E_{\lambda,m} = E_{\bar\lambda,m},$$

where $\bar\lambda = (\bar\lambda_1,\dots,\bar\lambda_k)$. □

By Proposition 3(i), the Euler characteristic of the real Milnor fibre F_γ is given by the Lefschetz number

$$\chi(F_\gamma) = L(ch_\gamma).$$

For the eigenvalues $\lambda = (\lambda_1,\dots,\lambda_k)$, let

$$E_\lambda(h_i) = \bigcap_j \ker(h_{(j0),*}|_{H_i(F,\mathbf{C})} - \lambda_j \mathbf{I})^N,$$

for N sufficiently large. Let

$$l(h;\lambda) = \sum_i (-1)^i \dim E_\lambda(h_i).$$

For $\gamma = (\gamma_1,\dots,\gamma_k)$ we let $|\gamma| = \sum_{j=1}^k \gamma_j$. Then the argument of the proof of Proposition 2 generalizes and gives:

PROPOSITION 4. $\sum_\gamma (-1)^{|\gamma|}\chi(F_\gamma)$ *is divisible by* 2^k *and*

$$\sum_\gamma (-1)^{|\gamma|}\chi(F_\gamma) \equiv 2^k l(h;(-1,\dots,-1)) \pmod{2^{k+1}}. \tag{9}$$

□

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

3.3. ITERATED LINKS

Let X be a compact algebraic subset of $\mathbf{R}^n$, and let $\mathcal{X} = \{X_i\}_{i=1}^k$ be an ordered family of algebraic (or closed semialgebraic) subsets of X. Then we define the *link* of $\mathcal{X}$ in X as

$$\mathrm{lk}(\mathcal{X}; X) = f_1^{-1}(\delta_1) \cap \cdots \cap f_k^{-1}(\delta_k),$$

where $f_i : X \to \mathbf{R}$ are nonnegative polynomials (or continuous semialgebraic functions) with zero sets X_i, and the δ_i's are chosen such that $0 < \delta_k \ll \cdots \ll \delta_1 \ll 1$. (Note that $\mathrm{lk}(\mathcal{X}; X) = \emptyset$ if $\bigcap_i X_i = \emptyset$.) Similarly, we define the localized link $\widetilde{\mathrm{lk}}_{x_0}(\mathcal{X}; X)$. Note that $\mathrm{lk}(\mathcal{X}; X)$ depends on the ordering of the X_i's, but it does not depend on the choice of the f_i's and δ_i's; we show this in Remark 6 below.

For a given family $\{X_i\}_{i=1}^k$ of subsets of X, we let $X_{k+1} = X$ and $X_0 = \emptyset$.

LEMMA 4. *Let $\mathcal{X} = \{X_i\}_{i=1}^k$ be an ordered family of closed semialgebraic subsets of X, and let $U_i = X_i \setminus \bigcup_{j=1}^{i-1} X_j$, $i = 1, \ldots, k+1$. Then*

$$\chi(\mathrm{lk}(\mathcal{X}; X)) = \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant k+1} \chi(U_{i_1} \cup \cdots \cup U_{i_j}) \tag{10}$$

and locally at any $x \in \mathbf{R}^n$

$$\chi(\widetilde{\mathrm{lk}}_x(\mathcal{X}; X)) = \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant k+1} \chi(\mathrm{lk}(x; U_{i_1} \cup \cdots \cup U_{i_j})). \tag{11}$$

Proof. For $k = 1$, (10) follows from (4) of Lemma 1,

$$\chi(\mathrm{lk}(X_1; X)) = \chi(X_1) + \chi(X \setminus X_1) - \chi(X),$$

since $X_1 = U_1$, $X \setminus X_1 = U_2$ and $X = U_1 \cup U_2$.

Let $f_1 : X \to \mathbf{R}$ be a non-negative polynomial function defining X_1. For the inductive step choose δ_1 so small that Lemma 1 holds for $Y = X_1$ and for all $U = U_{i_1} \cup \cdots \cup U_{i_j} \cup Y$, $1 < i_1 < \cdots < i_j \leqslant k+1$. Since $U_1 = X_1$, this gives

$$\begin{aligned} &\chi((U_{i_1} \cup \cdots \cup U_{i_j}) \cap f_1^{-1}(\delta_1)) \\ &\quad = \chi(U_{i_1} \cup \cdots \cup U_{i_j}) + \chi(U_1) - \chi(U_1 \cup U_{i_1} \cup \cdots \cup U_{i_j}) \end{aligned} \tag{12}$$

for $2 \leqslant i_1 < \cdots < i_j \leqslant k+1$.

Apply the inductive hypothesis to the set $\mathcal{X}' = \{X_i'\}_{i=2}^k$ given by $X_i' = X_i \cap f_1^{-1}(\delta_1)$. By construction, $\mathrm{lk}(\mathcal{X}; X) = \mathrm{lk}(\mathcal{X}'; X \cap f_1^{-1}(\delta_1))$. The inductive hypothesis gives

$$\begin{aligned} &\chi(\mathrm{lk}(\mathcal{X}'; X \cap f_1^{-1}(\delta_1))) \\ &\quad = \sum_{j=1}^{k} (-1)^{j+1} \sum_{2 \leqslant i_1 < \cdots < i_j \leqslant k+1} \chi((U_{i_1} \cup \cdots \cup U_{i_j}) \cap f_1^{-1}(\delta_1)). \end{aligned}$$

Now the result follows from (12), since the coefficient of $\chi(U_1)$ is

$$\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j} = 1.$$

The proof of the local case is similar. $\square$

For $k = 2$, if $X_1 \subset X_2$ the right hand side of (10) equals

$$\begin{aligned}&\chi(X_1) + \chi(X_2 \setminus X_1) + \chi(X \setminus X_2)\\ &\quad -\chi(X_2) - \chi(X \setminus X_1) - \chi((X \setminus X_2) \cup X_1)) + \chi(X).\end{aligned}$$

This expression appears in [C, Section 8] in a context which we explain in the next section.

Remark 5. The sets U_i in Lemma 4 do not change if the ordered family $\mathcal{X} = \{X_i\}_{i=1}^k$ is replaced by the nested family $\mathcal{Y} = \{Y_i\}_{i=1}^k$, $Y_i = \bigcup_{j=1}^i X_j$. Furthermore $\mathrm{lk}(\mathcal{X}; X) = \mathrm{lk}(\mathcal{Y}; X)$. For if $f_i : X \to \mathbf{R}$, $i = 1, \ldots, k$, are nonnegative continuous semialgebraic functions with $f_i^{-1}(0) = X_i$, then $g_i = \min(f_1, \ldots, f_i)$ is a nonnegative continuous semialgebraic function with $g_i^{-1}(0) = Y_i$, and if $0 < \delta_k < \cdots < \delta_1$, then

$$f_1^{-1}(\delta_1) \cap \cdots \cap f_k^{-1}(\delta_k) = g_1^{-1}(\delta_1) \cap \cdots \cap g_k^{-1}(\delta_k).$$

Similarly, for the nested family $\mathcal{Z} = \{Z_i\}_{i=1}^k$, $Z_i = \bigcap_{j=i}^k X_j$, we have $\mathrm{lk}(\mathcal{X}; X) = \mathrm{lk}(\mathcal{Z}; \mathcal{X})$. For if $h_i = \max(f_i, \ldots, f_k)$, then $h^{-1}(0) = Z_i$, and if $0 < \delta_k < \cdots < \delta_1$, then

$$f_1^{-1}(\delta_1) \cap \cdots \cap f_k^{-1}(\delta_k) = h_1^{-1}(\delta_1) \cap \cdots \cap h_k^{-1}(\delta_k).$$

It follows that Lemma 4 is true if the sets U_i are replaced by $U_i' = \bigcap_{j=i}^{k+1}(X_j \setminus X_{i-1})$.

Remark 6. (a) Using Proposition 1 of [CK] we show that iterated links are well-defined up to semialgebraic homeomorphism.

First consider the case of the link $\mathrm{lk}(\mathcal{X}; X)$ of a family $\mathcal{X} = \{X_i\}_{i=1}^k$ in X. It is more convenient to work in the semialgebraic category, so we assume that X is semialgebraic and X_i are compact semialgebraic subsets of X. We show uniqueness by induction on k.

For $k = 1$ we may collapse X_1 to a point x_1, and there is a unique semialgebraic structure on X/X_1 such that the projection $X \to X/X_1$ is semialgebraic. Then $\mathrm{lk}(X_1; X) = \mathrm{lk}(x_1; X/X_1)$ has a unique semialgebraic structure by [CK, Prop. 1]. Moreover, if we let $X_i' = X_i/(X_i \cap X_1)$, $i = 2, \ldots, k$, be the induced semialgebraic subsets of X/X_1, then, by [*loc. cit.*], $\mathrm{lk}(x_1; X_i/(X_i \cap X_1))$ are well-defined

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

semialgebraic subsets of $\mathrm{lk}(x_1; X/X_1)$. Let $\tilde{\mathcal{X}} = \{\mathrm{lk}(x_1; X_i/(X_i \cap X_1))\}_{i=2}^k$. It follows from the inductive hypothesis that

$$\mathrm{lk}(\mathcal{X}; X) = \mathrm{lk}(\tilde{\mathcal{X}}; X/X_1)$$

is unique up to semialgebraic homeomorphism.

To show the uniqueness of localized links $\widetilde{\mathrm{lk}}_x(\mathcal{X}; X)$ we may argue as follows. First note that [*loc. cit.*] gives, in fact, the semialgebraic invariance of semialgebraic tubular neighbourhoods. In particular, we may take $X \cap B(x, \varepsilon)$ as a representative of a neighbourhood of x in X. Then we apply the above argument to $X \cap B(x, \varepsilon)$ and the family $\{X_i \cap B(x, \varepsilon)\}$.

(b) For iterated links to be well-defined, we do not actually need the compactness of X; the compactness of $\bigcap_i X_i$ is sufficient. For localized links it suffices that the X_i are closed in X.

(c) In Section 3.5 below we show that the iteration of the link operator of [DS] allows us to study the sheaf cohomology of iterated links. This shows independently that the sheaf cohomology of an iterated link is well-defined.

3.4. Coste–Kurdyka invariants

We use iterated links to generalize Coste and Kurdyka's invariants of chains of algebraic subsets to invariants of ordered families of algebraic subsets of arbitrary codimensions.

Let $\mathcal{X} = \{X_i\}_{i=1}^k$ be an ordered family of closed algebraic subsets of the algebraic set $X \subset \mathbf{R}^n$; we do not assume X to be compact. Let

$$\Delta(\mathcal{X}; X) = \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leqslant k+1} \chi(U_{i_1} \cup \cdots \cup U_{i_j})$$

and similarly for $x \in X$

$$\Delta_x(\mathcal{X}; X) = \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant k+1} \chi(\mathrm{lk}(x; U_{i_1} \cup \cdots \cup U_{i_j})),$$

where $U_i = X_i \setminus \bigcup_{j=1}^{i-1} X_j$, $i = 1, \ldots, k+1$, $X_0 = \emptyset$, $X_{k+1} = X$.

Remark 7. If X is compact, then by Lemma 4,

$$\Delta(\mathcal{X}; X) = \chi(\mathrm{lk}(\mathcal{X}; X)), \qquad \Delta_x(\mathcal{X}; X) = \chi(\widetilde{\mathrm{lk}}_x(\mathcal{X}; X)).$$

If X is not compact, we may compactify it by choosing an algebraic one-point compactification S of $\mathbf{R}^n$, such that $S \subset \mathbf{R}^N$ (*cf.* [BCR, Prop. 3.5.3]). Let ∞ denote the point at infinity. Then $\tilde{\mathcal{X}} = \{\tilde{X}_i = X_i \cup \{\infty\}\}$ is an ordered family of

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

algebraic subsets of $\tilde{X} = X \cup \{\infty\}$. Note that $\tilde{U}_i = \tilde{X}_i \setminus \bigcup_{j=1}^{i-1} \tilde{X}_j = U_i$, except $\tilde{U}_1 = U_1 \cup \{\infty\}$. Thus, if $i_1 > 1$,

$$\chi(\tilde{U}_{i_1} \cup \cdots \cup \tilde{U}_{i_j}) = \chi(U_{i_1} \cup \cdots \cup U_{i_j}).$$

If $i_1 = 1$, by Lemma 1,

$$\begin{aligned} \chi(\tilde{U}_{i_1} \cup \cdots \cup \tilde{U}_{i_j}) &= \chi(\{\infty\} \cup U_{i_1} \cup \cdots \cup U_{i_j}) \\ &= \chi(U_{i_1} \cup \cdots \cup U_{i_j}) + 1 - \chi(\mathrm{lk}(\infty; U_{i_1} \cup \cdots \cup U_{i_j})). \end{aligned}$$

Summing up, $\Delta(\mathcal{X}; X) = \Delta(\tilde{\mathcal{X}}; \tilde{X}) - \Delta(\mathcal{X}'; X')$, where $\mathcal{X}' = \{\mathrm{lk}(\infty; \tilde{X}_i)\}$ is the family of links at infinity of $\mathcal{X}$, and $X' = \mathrm{lk}(\infty; \tilde{X})$. Thus the study of $\Delta(\mathcal{X}; X)$ can be reduced to the case of compact X.

We will also consider parametrized families. For the standard projection $\pi : \mathbf{R}^n \to \mathbf{R}^m$ ($n \geqslant m$) and $t \in \mathbf{R}^m$, we denote by $\mathcal{X}_t$ the induced ordered family of algebraic subsets of $X_t = \pi^{-1}(t)$. Let t vary in an algebraic set $T \subset \mathbf{R}^m$. Then $\{\mathcal{X}_t\}$ is an ordered algebraic family of algebraic subsets of $X \cap \pi^{-1}(T)$ parametrized by $t \in T$.

THEOREM 2. *Let $\mathcal{X} = \{X_i\}_{i=1}^k$ be an ordered family of algebraic subsets of the algebraic set $X \subset \mathbf{R}^n$. Then*

(i) *For any $x \in X$, $\Delta(\mathcal{X}; X)$ and $\Delta_x(\mathcal{X}; X)$ are divisible by 2^k;*
(ii) *Let x vary along an irreducible algebraic subset Y of X. Then $\Delta_x(\mathcal{X}; X)$ is generically constant modulo 2^{k+1}; that is, there exists a real algebraic subset $Z \subset Y$, with $\dim Z < \dim Y$, such that for any $x, x' \in Y \setminus Z$,*

$$\Delta_{x'}(\mathcal{X}; X) \equiv \Delta_x(\mathcal{X}; X) \pmod{2^{k+1}}.$$

(iii) *Let t vary in an irreducible algebraic subset T of $\mathbf{R}^m$. Then $\Delta(\mathcal{X}_t; X_t)$ is generically constant modulo 2^{k+1}; that is, there exists a real algebraic subset $Z \subset T$, with $\dim Z < \dim T$, such that for any $t, t' \in T \setminus Z$,*

$$\Delta(\mathcal{X}_{t'}; X_{t'}) \equiv \Delta(\mathcal{X}_t; X_t) \pmod{2^{k+1}}.$$

This also holds for $k = 0$ for $\Delta(X_t) = \chi(X_t)$.

Proof. To show (i) for $\Delta_x(\mathcal{X}; X)$ and (ii), we follow the proof of Theorem 1. By Lemma 4, $\Delta_x(\mathcal{X}; X) = \chi(\widetilde{\mathrm{lk}}_x(\mathcal{X}; X))$. Let $f_i : X \to \mathbf{R}$ be nonnegative polynomials such that $f_i^{-1}(0) = X_i$, and let $X_{\mathbf{C}}$, $(X_i)_{\mathbf{C}}$ and $(f_i)_{\mathbf{C}} : X_{\mathbf{C}} \to \mathbf{C}$ be complexifications of X, X_i and f_i, respectively. Consider the real and complex Milnor fibres of $(f_1, \ldots, f_k)$ and $((f_1)_{\mathbf{C}}, \ldots, (f_k)_{\mathbf{C}})$, respectively, at $x \in X$. For $\lambda \neq (0, \ldots, 0)$, the real Milnor fibre F_λ is empty, and $F_{(0,\ldots,0)}$ is the localized link $\widetilde{\mathrm{lk}}_x(\mathcal{X}; X)$. Consequently, by Proposition 4, $\chi(\widetilde{\mathrm{lk}}_x(\mathcal{X}; X))$ is divisible by 2^k, and

$$\chi(\widetilde{\mathrm{lk}}_x(\mathcal{X}; X)) \equiv 2^k l(h_x; (-1, \ldots, -1)) \pmod{2^{k+1}}. \tag{13}$$

https://doi.org/10.1023/A:1000126025773 Published online by Cambridge University Press

If Y is an irreducible algebraic subset of X, then $Y_{\mathbf{C}}$ is also irreducible, and the right hand side of (13) is constant on strata of some algebraic stratification of $Y_{\mathbf{C}}$. Hence the left-hand side of (13) is constant modulo 2^{k+1} for generic $x \in Y$.

To show (i) for $\Delta(\mathcal{X}; X)$ we argue as follows. Let $f(x_1, \dots, x_n)$ be a polynomial defining X. Then $f^2 - x_{n+1}^2$ defines the double cone $\tilde{X}$ over X in $\mathbf{R}^{n+1}$. The link of the origin p_0 in $\tilde{X}$ consists just of two copies of X. Similarly we define the family $\tilde{\mathcal{X}}$. Let $X'_{i+1} = \tilde{X}_i$ for $i = 1, \dots, k$, and $X'_1 = \{p_0\}$. Then $\Delta_{p_0}(\mathcal{X}'; \tilde{X}) = 2\,\Delta(\mathcal{X}; X)$, and the global case follows from the local case.

To show (iii) we may assume that $\pi(X) = T$ and that X is compact. Apply the above double cone construction to the fibres of π. Let Y be the zero section of this family of cones. Since Y is isomorphic to T, it is also irreducible. Then, by an argument similar to the above, (iii) follows from (ii). $\square$

Theorem 2 generalizes Théorème 5 and Proposition 4 of [C] and the corollary after Lemma 3 of [CK]. Indeed, in the notation of Lemma 3 of [CK], for X_1, X_2 real algebraic sets, S a nonsingular semialgebraic set such that $S \subset X_1 \subset X_2$, and x generic in S,

$$\begin{aligned}
\Phi(S, X_1, X_2) &= \chi(\mathrm{lk}_x(S; X_2) \setminus \mathrm{lk}_x(S; X_1)) \\
&\quad -\chi(\mathrm{lk}_x(S; X_2)) + \chi(\mathrm{lk}_x(S; X_1)) \\
&= \chi(\mathrm{lk}_x(N_x \cap X_1; N_x \cap X_2)) \\
&= \Delta_x(S, X_1; X_2) \\
&= \chi(\widetilde{\mathrm{lk}}_x(S, X_1; X_2)),
\end{aligned}$$

where N_x is the normal space at x to S in X_2. Similarly, in the notation of Théorème 5 of [C], for algebraic subsets $Y_2 \subset Y_1$ of X and $S \subset Y_2$ a nonsingular semialgebraic set,

$$\Delta_3(S, Y_2, Y_1; X) = \chi(\widetilde{\mathrm{lk}}_x(\mathcal{Y}; X)),$$

where $\mathcal{Y} = \{S, Y_2, Y_1\}$.

Theorem 2 seems to be the result anticipated in section 8 of [C]. Moreover, Theorem 2 shows that the dimensional assumptions of Lemma 3 of [CK] and Théorème 5 of [C] ($\dim X_2 = \dim X_1 + 1 = \dim S + 2$ and $\dim X = \dim Y_1 + 1 = \dim Y_2 + 2 = \dim S + 3$) can be dropped, answering positively the question stated in part (c) of the last remark of [CK].

3.5. The iterated link cohomology functor

By reformulating Theorem 2 in terms of the local link cohomology functor Λ_Y of [DS], we show that our invariant Δ generalizes Coste and Kurdyka's invariant Φ for a chain of algebraic subsets of X. We obtain Coste and Kurdyka's fundamental result about Φ [CK, Thm. 4] as an immediate corollary of Theorem 3 below.

Let Y be a closed semialgebraic subset of the algebraic set X. We do not assume that X or Y is compact. We slightly modify the definition of Remark 2(c), so that now Λ_Y is defined only for bounded constructible sheaves on X; that is,

$$\Lambda_Y \mathcal{F} = i^* R j_* j^* \mathcal{F},$$

where, as before, $i\colon Y \hookrightarrow X$ and $j\colon X \setminus Y \hookrightarrow X$ denote the embeddings.

Suppose that we have a finite ordered family $\mathcal{X} = \{X_i\}_{i=1}^k$ of closed semialgebraic subsets of X. Then, for any $x \in X$,

$$H^*(\widetilde{\mathrm{lk}}_x(\mathcal{X}; X); \mathbf{Q}) = H^*(\Lambda_{X_1} \ldots \Lambda_{X_k} \mathbf{Q}_X)_x. \tag{14}$$

Indeed, (14) can be considered as a real analogue of (8), with the same proof.

Moreover, suppose that $\bigcap_i X_i$ is compact. Then

$$H^*(\mathrm{lk}(\mathcal{X}; X); \mathbf{Q}) = \mathbf{H}^*(Y; \Lambda_{X_1} \ldots \Lambda_{X_k} \mathbf{Q}_X),$$

where $Y = \bigcap_i X_i$ and $\mathbf{H}$ denotes hypercohomology.

A function $\varphi\colon X \to \mathbf{Z}$ is called *semialgebraically constructible* if there exists a locally finite family $\{X_i\}$ of closed semialgebraic subsets of X, and integers c_i such that

$$\varphi = \sum_i c_i \mathbf{1}_{X_i}. \tag{15}$$

We refer the reader to [Sch] for the main properties of constructible functions. The theory of subanalytically constructible functions developed there holds also, of course, for semialgebraically constructible functions.

Similarly one may define algebraically constructible functions on a real algebraic set X by demanding all the X_i in (15) to be algebraic. With this definition, one can easily check that the basic operations on semialgebraically constructible functions given by duality D_X and push-forward f_*, for a polynomial map $f\colon X \to Y$, do *not* preserve the family of algebraically constructible functions.

The following theorem is equivalent to Theorem 2.

THEOREM 3. *Let $\mathcal{X} = \{X_i\}_{i=1}^k$ be an ordered family of algebraic subsets of the algebraic set $X \subset \mathbf{R}^n$. Then the stalk Euler characteristic of $\Lambda_{X_1} \ldots \Lambda_{X_k} \mathbf{Q}_X$,*

$$\varphi(x) = \chi(H^*(\Lambda_{X_1} \ldots \Lambda_{X_k} \mathbf{Q}_X)_x),$$

is always divisible by 2^k and algebraically constructible mod 2^{k+1}. □

Define the *link operator* Λ on the constructible function $\varphi = \sum_i c_i \mathbf{1}_{X_i}$ by

$$\Lambda\varphi(x) = \sum_i c_i \chi(\mathrm{lk}(x; X_i));$$

for $Y \subset X$ we let

$$\Lambda_Y \varphi(x) = \sum_i c_i \chi(\mathrm{lk}_x(Y; X_i)).$$

Then the duality operator of [Sch] satisfies

$$D_X \varphi(x) = \varphi(x) - \Lambda\varphi(x).$$

The following corollary is equivalent to Theorem 1.

COROLLARY 1. *If φ is an algebraically constructible function, then $\Lambda\varphi$ always has even values, and both $\Lambda\varphi$ and $D_X\varphi$ are algebraically constructible* mod 4.

Finally, we show that Theorem 4 of [CK] is a corollary of Theorem 3. Let $X_1 \subset \cdots \subset X_k = X$ be real algebraic sets and let $\dim X_i = d + i$. Suppose that we have a stratification of X given by a stratifying family of polynomials (see [*loc. cit.* Section 1]) compatible with each X_i. For a given d-dimensional stratum $S \subset X_1$, Coste and Kurdyka's invariant $\Phi(S, X_1, \ldots, X_k)$ is the number of flags of strata

$$T_k \xrightarrow{1} T_{k-1} \xrightarrow{1} \cdots \xrightarrow{1} T_1 \xrightarrow{1} T_0 = S;$$

that is, $T_i \subset \overline{T}_{i+1}$, $\dim T_{i+1} = \dim T_i + 1$, and $T_i \subset X_i$. Theorem 4 of [CK] states that $\Phi(S, X_1, \ldots, X_k)$ is divisible by 2^k, and

$$\Phi(S, X_1, \ldots, X_k) \equiv \Phi(S', X_1, \ldots, X_k) \pmod{2^{k+1}}$$

if S and S' have for Zariski closure the same irreducible algebraic set. This result follows immediately from Theorem 3 provided we show that

$$\Phi(S, X_1, \ldots, X_k) = \Lambda_S \Lambda_{X_1} \ldots \Lambda_{X_{k-1}} \mathbf{1}_X(x), \tag{16}$$

for an arbitrary point x of S.

The strata of a stratification given by a stratifying family of polynomials form a very regular cell decomposition, and (cf. [CK, end of Section 1]) we may calculate the Euler characteristic of the link just by counting the number of strata; that is, for any $x \in S$,

$$\chi(\mathrm{lk}_x(S; X)) = \sum_{i=d+1}^{n} (-1)^{i-d-1} m_i(S),$$

where $d = \dim S$ and $m_i(S)$ is the number of i-dimensional strata T contained in X with $S \subset \overline{T}$. Clearly this gives, for any constructible function φ compatible with the given stratification (i.e. φ constant on the strata),

$$\Lambda_S \varphi(x) = \sum_T (-1)^{\dim T - \dim S - 1} \varphi(T), \tag{17}$$

where $\varphi(T)$ denotes the value of φ on T.

Now we show (16) by induction on k. Consider on X_1 a constructible function φ which on strata $T \subset X_1$, $\dim T = d+1$, satisfies

$$\varphi(T) = \Phi(T, X_2, \ldots, X_k) = \Lambda_T \Lambda_{X_2} \ldots \Lambda_{X_{k-1}} \mathbf{1}_X.$$

(The values of φ on smaller dimensional strata do not matter, thanks to the assumption on the dimensions of S and X_1). Hence by (17), for $x \in S$,

$$\Phi(S, X_1, \ldots, X_k) = \sum_T \varphi(T) = \Lambda_S \varphi(x),$$

as required.

3.6. The Analytic Category

All local statements of this paper hold in the real analytic category, by virtually the same proofs, where – whenever necessary – we replace semialgebraic sets by subanalytic sets.

References

[AK] Akbulut, S. and King, H.: *Topology of Real Algebraic Sets*, MSRI Publ., Vol. 25, Springer-Verlag, New York, 1992.

[AGV] Arnold, V. I., Gusein-Zade, S. M. and Varchenko, A. N.: *Singularities of Differentiable Maps*, vol. 2, Birkhäuser, Boston, 1988.

[BR] Benedetti, R. and Risler, J-J.: *Real Algebraic Geometry*, Hermann, Paris, 1990.

[BD] Bierstone, E. and Milman, P. D.: Semianalytic and subanalytic sets, *Publ. Math. IHES* 67 (1988), 5–42.

[BCR] Bochnak, J., Coste, M. and Roy, M-F.: *Géométrie Algébrique Réelle*, Springer-Verlag, Berlin, 1987.

[C] Coste, M.: Sous-ensembles algébriques réels de codimension 2, *Real Analytic and Algebraic Geometry*, Lecture Notes in Math., vol. 1420, Springer-Verlag, 1990, pp. 111–120.

[CK] Coste, M. and Kurdyka, K.: On the link of a stratum in a real algebraic set, *Topology* 31 (1992), 323–336.

[PD] Dimca, A. and Paunescu, L.: *Real Singularities and Dihedral Representations*, University of Sydney Pure Maths, Research Reports 92-4, (1992).

[D] Durfee, A.: Neighborhoods of algebraic sets, *Trans. Amer. Math. Soc.* 276 (1983), 517–530.

[DS] Durfee, A. and Saito, M.: Mixed Hodge structures on the intersection cohomology of links, *Compositio Math.* 76 (1990), 49–67.

[G] Gibson, C. G. et al.: *Topological Stability of Smooth Mappings*, Lecture Notes in Math., Vol. 552, Springer-Verlag, 1976.

[HMS] Henry, J-P., Merle, M. and Sabbah, C.: Sur la condition de Thom stricte pour un morphisme analytique complexe, *Ann. Sci. Éc. Norm. Sup.* 114 (1984), 227–268.

[KS] Kashiwara, M. and Schapira, P.: *Sheaves on Manifolds*, Springer-Verlag, Berlin, 1990.

[Lê] Lê D. T.: Some remarks on relative monodromy, *Real and Complex Singularities (Oslo 1976)*, Sijthoff and Noordhoff, Amsterdam, 1977, pp. 397–403.

[Lê] Lê D. T. and Teissier, B.: Limites d'espaces tangents en géométrie analytique, *Comm. Math. Helvetici* 63 (1988), 540–578.

[M] Milnor, J.: *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies, 61, Princeton Univ. Press, Princeton, 1968.

[S] Sabbah, C.: Morphismes analytiques stratifiés sans éclatement et cycles évanescents, *Astérisque* 101-102, 286–319.

[Sch] Schapira, P.: Operations on constructible functions, *J. Pure Appl. Algebra* 72 (1991), 83–93.

[SY] Shiota, M. and Yokoi, M.: Triangulations of subanalytic sets and locally subanalytic manifolds, *Trans. Amer. Math. Soc.* 286 (1984), 727–750.

[Su] Sullivan, D.: Combinatorial invariants of analytic spaces, *Proc. Liverpool Singularities Symposium I*, Lecture Notes in Math., vol. 192, Springer-Verlag, 1971, pp. 165–169.